Classical Unified Field Theory of Gravitation and Electromagnetism

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Abstract

According to Noether's theorem, for every continuous symmetry of the action, there exists corresponding conserved quantity. If we assume the stationary condition as a role of symmetry, there is a conserved quantity. By using the definition of the Komar mass, one can calculate the mass in a curved spacetime. However, it fails to give the mass for the well-known charged 4D black hole such as Kerr-Newman black hole. In this paper, to solve this problem, we consider an extension to 5-dimensions by using results of ADM formalism. In terms of (4+1) decomposition, with alternative surface integral, we find the rotating and charged 5D metric solution and check whether it gives the mass, charge and angular momentum exactly.

1 Introduction

The Kaluza-Klein theory is a classical unified field theory of gravitation and electromagnetism in terms of the general relativity extended to 5D [1]. In this paper, we use a different but basically equivalent metric ansatz based on ADM formalism, which decomposes spacetime into the time component and 3 spatial components, and use the concept of extrinsic curvature [2–5]. It plays mathematically an important role in the unified field theory to be discussed in this paper. The main assumption is that as the gravitation can be described geometrically in 4D, the electromagnetism also can be described geometrically in 5D [6]. The main goal of this paper is to find the 5D metric form for the spherical or rotating black hole solutions which have the mass and charge. In addition, we would like to deal with the Lorentz force as well which is known as a main shortfall of Kaluza-Klein theory. In 4D, there are black hole solutions, which have the charge and mass, simultaneously known as the Reisner-Nordstrom(RN), Kerr-Newman(KN) black holes [7]. To get 5D solution, we assume two symmetries of the metric. One is cylindrical condition, and the other is the stationary condition.

Under the assumption that the electromagnetism is an effect of the pure geometry in 5D, the expected results will be similar to that of Kaluza-Klein theory [8–10]. After we obtain the 5D metric, which is exactly the 5D vacuum solution, we also obtain the mass, charge and angular momentum of the object.

2 Metric Ansatz

In the following, Greek indices refer to spacetime components (0,123) and the index 5 refers to the fifth dimension. Roman indices (a,b) span (5,0,123). Similar to Kaluza's ansatz, we consider the ansatz,

$$ {}^5g_{\mu\nu} = {}^4g_{\mu\nu}, \quad {}^5g_{5\mu} = \beta_\mu, \quad {}^5g_{55} = -N^2 + \beta^\lambda \beta_\lambda. \tag{1} $$

Then the inverse metric is given by

$$ {}^5g^{\mu\nu} = {}^4g^{\mu\nu} - \frac{\beta^\mu \beta^\nu}{N^2}, \quad {}^5g^{5\mu} = \frac{\beta^\mu}{N^2}, \quad {}^5g^{55} = -\frac{1}{N^2}. \tag{2} $$

In other words,

$$ {}^5g_{ab} = \begin{bmatrix} -N^2 + \beta^\lambda \beta_\lambda & \beta_\nu \\ \\ \beta_\mu & {}^4g_{\mu\nu} \end{bmatrix}, \quad {}^5g^{ab} = \begin{bmatrix} -\frac{1}{N^2} & \frac{\beta^\nu}{N^2} \\ \\ \frac{\beta^\mu}{N^2} & {}^4g^{\mu\nu} - \frac{\beta^\mu \beta^\nu}{N^2} \end{bmatrix}, \tag{3} $$

where ${}^5g_{ab}$ is the 5D metric and ${}^4g_{\mu\nu}$ is the standard 4D metric with the Lorentzian signature, (−,+++). The noticeable point is ${}^5g_{55} = -N^2 + \beta^\lambda \beta_\lambda$. We set the fifth dimension to timelike and there is no problem at this stage, see the first item in the discussion section.

3 5D Christoffel Symbol and 5D Ricci Tensor

We assumed two symmetries of metric. One is cylindrical condition, and the other is the stationary condition,

$$\frac{\partial}{\partial \omega}\, {}^5g_{ab} = 0, \quad \frac{\partial}{\partial t}\, {}^5g_{ab} = 0, \tag{4}$$

where ω is fifth coordinate. Under these conditions, the 5D Christoffel symbols are given by

$${}^5\Gamma^5_{\mu\nu} = -\frac{K_{\mu\nu}}{N}, \tag{5}$$

$${}^5\Gamma^5_{\mu 5} = -\frac{1}{N}K_{5\mu} + \frac{1}{N}\partial_\mu N, \tag{6}$$

$${}^5\Gamma^5_{55} = -\frac{\beta^\sigma}{2N^2}\partial_\sigma(-N^2 + \beta^\lambda\beta_\lambda), \tag{7}$$

where $K_{\mu\nu}$ is the extrinsic curvature tensor defined as

$$K_{ab} = -\nabla_b n_a - n_b n^c \nabla_c n_a \tag{8}$$

with the unit normal vector $\mathbf{n}$ [5]. This extrinsic curvature comes from (4+1) decomposition, foliates 5-dimension with respect to fifth coordinate ω [4]. Note that the original ADM-formalism is (3+1) decomposition which foliates spacetime with respect to time t. Then the extrinsic curvature tensor $K_{\mu\nu}$ is given by

$$K_{\mu\nu} = \frac{1}{2N}\left({}^4\nabla_\mu\beta_\nu + {}^4\nabla_\nu\beta_\mu - \frac{\partial}{\partial\omega}\,{}^4g_{\mu\nu}\right), \; {}^4\nabla_\mu\beta_\nu \equiv \partial_\mu\beta_\nu - {}^4\Gamma^\lambda_{\mu\nu}\beta_\lambda. \tag{9}$$

From now on, where the covariant derivative is related with ${}^4g_{\mu\nu}$ and emit ^{4} index for covariant derivative. In this paper, we assume that the charge identified with $\frac{dx^5}{ds} = \frac{q}{m}$ is not changed along the geodesic curve. The fifth component of the geodesic equation is as follows

$$\frac{d}{ds}\left(\frac{dx^5}{ds}\right) + {}^5\Gamma^5_{ab}\frac{dx^a}{ds}\frac{dx^b}{ds} = 0. \tag{10}$$

Assuming that the charge does not change along the geodesic curve for any particle, one can set ${}^5\Gamma^5_{ab} = 0$. Then from eqns(4),(5),(6),(7),(9), we obtain

$$N = constant, \quad \beta_\mu = \alpha\, {}^4g_{0\mu}, \tag{11}$$

where α is constant. Then with eqn(11), ${}^5\Gamma^\mu_{ab}$ is given by

$${}^5\Gamma^\mu_{\lambda\rho} = {}^4\Gamma^\mu_{\lambda\rho}, \tag{12}$$

$${}^5\Gamma^\mu_{\lambda 5} = {}^4g^{\mu\nu}F_{\lambda\nu}, \tag{13}$$

$${}^5\Gamma^\mu_{55} = -\,{}^4g^{\mu\nu}\partial_\nu\phi = -\nabla^\mu\phi, \tag{14}$$

where we define

$$F_{\mu\nu} = \partial_\mu \left(\frac{\beta_\nu}{2}\right) - \partial_\nu \left(\frac{\beta_\mu}{2}\right), \tag{15}$$

$$\phi = \frac{1}{2}\, {}^5g_{55}. \tag{16}$$

Now, we have all components of Christoffel symbols. Then Ricci tensor is given by

$${}^5R_{ab} = \begin{bmatrix} -\Box\phi + F_{\lambda\rho}F^{\lambda\rho} & \nabla^\rho F_{\nu\rho} \\ \nabla^\rho F_{\mu\rho} & {}^4R_{\mu\nu} \end{bmatrix}. \tag{17}$$

4 Lorentz force

The Lorentz force can be derived from variation of the 5D geodesic equations. In Kaluza's hypothesis, however, the problem with this is that there is quadratic term of $\frac{dx^5}{ds}$ [11]. For clarity, we use the relation $-c^2d\tilde{\tau}^2 = {}^5g_{ab}dx^a dx^b$ rather than $ds^2 = {}^5g_{ab}dx^a dx^b$, where $\tilde{\tau}$ is supertime [19] and may be null even though test particle has a mass [18]. Now, because of $-c^2d\tau^2 = {}^4g_{\mu\nu}dx^\mu dx^\nu$, $\frac{dx^5}{d\tau}$ equal to 0 then $\frac{d\tilde{\tau}}{d\tau} = 1$. From now on, $\frac{dx^5}{d\tilde{\tau}} \equiv \frac{q}{m}$. The relations are

$$\frac{dx^\mu}{d\tau} = \frac{d\tilde{\tau}}{d\tau}\frac{dx^\mu}{d\tilde{\tau}} = \frac{dx^\mu}{d\tilde{\tau}}, \tag{18}$$

$$\frac{dx^5}{d\tau} \neq \frac{d\tilde{\tau}}{d\tau}\frac{dx^5}{d\tilde{\tau}} = \frac{dx^5}{d\tilde{\tau}}. \tag{19}$$

With equation

$$\frac{D}{d\tau}\left(\frac{dx^\mu}{d\tilde{\tau}}\right) = \frac{d\tilde{\tau}}{d\tau}\frac{D}{d\tilde{\tau}}\left(\frac{dx^\mu}{d\tilde{\tau}}\right) \tag{20}$$

and eqn(19), we obtain

$$\frac{d}{d\tau}\left(\frac{dx^\mu}{d\tilde{\tau}}\right) \neq \frac{d\tilde{\tau}}{d\tau}\frac{d}{d\tilde{\tau}}\left(\frac{dx^\mu}{d\tilde{\tau}}\right). \tag{21}$$

Therefore, we consider eqn(21) carefully and only use eqn(20). From eqn(20),

$$\frac{D}{d\tau}\left(\frac{dx^\mu}{d\tilde{\tau}}\right) = \frac{d}{d\tau}\left(\frac{dx^\mu}{d\tilde{\tau}}\right) + {}^5\Gamma^\mu_{ab}\frac{dx^a}{d\tilde{\tau}}\frac{dx^b}{d\tau} = 0. \tag{22}$$

With eqn(18), $\frac{dx^5}{d\tau} = 0$, the eqn(22) becomes

$$\frac{d}{d\tau}\left(\frac{dx^\mu}{d\tau}\right) + {}^5\Gamma^\mu_{\lambda\rho}\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau} = -{}^5\Gamma^\mu_{5\rho}\frac{dx^5}{d\tilde{\tau}}\frac{dx^\rho}{d\tau} \tag{23}$$

with eqns(12),(13), the eqn(23) becomes

$$\frac{d}{d\tau}\left(\frac{dx^{\mu}}{d\tau}\right) + {}^{4}\Gamma^{\mu}_{\lambda\rho}\frac{dx^{\lambda}}{d\tau}\frac{dx^{\rho}}{d\tau} = \frac{q}{m}\,{}^{4}g^{\mu\nu}F_{\nu\rho}\frac{dx^{\rho}}{d\tau}. \tag{24}$$

It is an important point that eqns(12),(13) just comes from the conservation of charge along the geodesic curve and the symmetric condition. At this stage, we have identified $F_{\mu\nu}$ of eqn(17) as the field strength tensor. Then by separate several components of eqn(17), ${}^{5}R_{5\nu}$ describe the Maxwell's equations and ${}^{4}R_{\mu\nu}$ denote the Einstein equations [10], see the third item in the discussion section.

5 Correspondence with Classical Dynamics

Unexpectedly, the results of our analysis for the Ricci tensor in 5D say, ${}^{4}R_{\mu\nu} = 0$. The expected metric, ${}^{4}g_{\mu\nu}$, is well-known charged 4D black hole such as RN or KN black hole, but our results say that those are not solutions in our formalism. In this paper, we check why the 4D metric should be vacuum solution as like Schwarzschild or Kerr black hole, although it is charged black hole. As we identified $\beta_{\mu} = \alpha^{4}g_{0\mu} = 2A_{\mu}$, we can fix α as $\alpha = -\frac{Qc}{4\pi\varepsilon_0 GM}$. Then for rotating charged black hole, the 5D metric solution is given by

$${}^{5}g_{ab} = \begin{bmatrix} -N^2 - (\frac{Qc}{4\pi\varepsilon_0 GM})^2(1-\frac{2GMr}{\Sigma c^2}) & \frac{Qc}{4\pi\varepsilon_0 GM}(1-\frac{2GMr}{\Sigma c^2}) & 0 & 0 & \frac{2Qr}{4\pi\varepsilon_0 c\Sigma}asin^2(\theta) \\ \frac{Qc}{4\pi\varepsilon_0 GM}(1-\frac{2GMr}{\Sigma c^2}) & -(1-\frac{2GMr}{\Sigma c^2}) & 0 & 0 & -\frac{2GMr}{\Sigma c^2}asin^2(\theta) \\ 0 & 0 & \frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & 0 & \Sigma & 0 \\ \frac{2Qr}{4\pi\varepsilon_0 c\Sigma}asin^2(\theta) & -\frac{2GMr}{\Sigma c^2}asin^2(\theta) & 0 & 0 & (r^2+a^2)sin^2(\theta)+\frac{2GMr}{\Sigma c^2}a^2 sin^4(\theta) \end{bmatrix}.(*)$$

Where $\Sigma = r^2 + a^2\cos^2\theta$ and we used Boyer-Lindquist coordinates. Since we just used condition ${}^{4}R_{\mu\nu} = 0$ of eqn(17), we cannot assure it is vacuum solution. In fact, condition of ${}^{4}R_{\mu\nu} = 0$ is just only condition of (17). So, this ${}^{5}g_{ab}$ is vacuum solution. This point will be discussed later. Now, to see the dynamics, we assume the spherical vacuum solution and just beginning of free falling. Then we are considering $a \equiv \frac{J}{mc} = 0$ and $\frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}, \frac{dx^3}{d\tau} = 0$. Note that $\frac{dx^0}{d\tau} = c\ {}^{-\frac{1}{2}}\!\!\sqrt{-{}^{4}g_{00}} = c\ {}^{-\frac{1}{2}}\!\!\sqrt{1-\frac{2GM}{rc^2}}$. From eqn(24), we obtain

$$m\frac{d}{d\tau}\left(\frac{dx^1}{d\tau}\right) = \sqrt{1-\frac{2GM}{rc^2}}\frac{Qq}{4\pi\varepsilon_0 r^2} - \frac{GMm}{r^2}. \tag{25}$$

This describes the known classical dynamics well. Now for the RN black hole, from eqn(24), we obtain

$$m\frac{d}{d\tau}\left(\frac{dx^1}{d\tau}\right) = -\frac{GMm}{r^2} + \frac{GQ^2 m}{4\pi\varepsilon_0 c^4 r^3} \tag{26}$$

$$+\sqrt{1-\frac{2GM}{rc^2}+\frac{GQ^2}{4\pi\varepsilon_0 c^4 r^2}}\left(\frac{Qq}{4\pi\varepsilon_0 r^2} - \frac{Q^3 q}{(4\pi\varepsilon_0 c^2)^2 M r^3}\right).$$

It is a strange result because there are $\frac{1}{r^3}$ terms. Furthermore, well-known charged 4D black hole cannot be solution under 5D vaccum solution. Now, we would like to explain the reason that the 5D vacuum condition gives 4D vacuum solution rather than the known charged 4D black hole. Then let us check what was neglected in the well-known 4D charged black hole,

$$S = \int {}^5R\sqrt{{}^5g}\, d^5x. \tag{27}$$

Noting $\det(A) = adj(A_{ij})\frac{1}{A_{ij}^{-1}}$, where A_{ij}^{-1} is a component of i-th row j-th column of inverse metric of A and $adj(A_{ij})$ is adjoint of A_{ij}, then we get from eqn(3), $\sqrt{{}^5g} = N\sqrt{-{}^4g}$. Then the action of integrand can be rewritten as

$$S = \int {}^5R\sqrt{-{}^4g}\, d^4x\, Ndx^5. \tag{28}$$

Since N is a constant and the integrand is independent of x^5, we can ignore the overall quantity Ndx^5. From eqns(3),(17), we obtain

$${}^5R = ({}^4R - \frac{F_{\lambda\rho}F^{\lambda\rho}}{N^2}) + \frac{1}{N^2}(\Box\phi - {}^4R_{\lambda\rho}\beta^\lambda\beta^\rho + 2\beta^\lambda\nabla^\rho F_{\lambda\rho}). \tag{29}$$

From eqns(9),(11), β_μ follows killing equation, it's divergence equals to 0, that is

$$\nabla^\rho F_{\lambda\rho} = \nabla^\rho\nabla_\lambda\beta_\rho = (\nabla_\rho\nabla_\lambda - \nabla_\lambda\nabla_\rho)\beta^\rho = {}^4R_{\rho\lambda}\beta^\rho. \tag{30}$$

Here we notice that $\nabla_\lambda\nabla\rho\beta^\rho$ is a hypothetical representation of zero terms. With eqns(29),(30), The second parentheses in eqn(29) can be rewritten as

$$\frac{1}{N^2}(\Box\phi + \beta^\lambda\nabla^\rho F_{\lambda\rho}). \tag{31}$$

Now we put $\nabla^\rho F_{\lambda\rho} = \mu_0 J_\lambda$. By neglecting total divergence of eqn(31), from eqn(29) we obtain

$${}^5R = {}^4R - (\frac{F_{\lambda\rho}F^{\lambda\rho}}{N^2} - \frac{\mu_0}{N^2}J_\lambda\beta^\lambda). \tag{32}$$

Since we are considering a source free region, $J_\mu = 0$, one can think that by excluding current term, eqn(32) will give the well-known charged 4D black hole solutions. But in fact, we have relation $\mu_0 J^\lambda\beta_\lambda = -\Box\phi + F_{\lambda\rho}F^{\lambda\rho}$. Now, with eqn(32) and by neglecting total divergence, matter Lagrangian effectively equals to 0. Until now, we just neglected J_λ since we are considering source free region. But this result says that this is not the case. Resultantly, our action is equivalent to ${}^4R$. Now we try to get a real solution without using variation principle. In fact, to obtain the solution, it is more reasonable to solve eqn(17) rather than solving eqn(32) with the variation principle. From eqns(16),(17), we obtain

$$ {}^5R_{55} = -(\nabla^\rho\beta^\lambda\nabla_\rho\beta_\lambda + \beta^\lambda\nabla^\rho\nabla_\rho\beta_\lambda) + F_{\lambda\rho}F^{\lambda\rho} \tag{33} $$

From eqn(15),

$$ \nabla^\rho\beta^\lambda\nabla_\rho\beta_\lambda = F^{\rho\lambda}F_{\rho\lambda}. \tag{34} $$

With eqn(30),

$$ \beta^\lambda\nabla^\rho\nabla_\rho\beta_\lambda = -\,{}^4R_{\rho\lambda}\beta^\rho\beta^\lambda. \tag{35} $$

Then the eqn(17) becomes as follows,

$$ {}^5R_{ab} = \begin{bmatrix} {}^4R_{\lambda\rho}\beta^\lambda\beta^\rho & {}^4R_{\nu\rho}\beta^\rho \\ {}^4R_{\mu\rho}\beta^\rho & {}^4R_{\mu\nu}. \end{bmatrix}. \tag{36} $$

Then we can easily get the condition which satisfy ${}^5R_{ab} = 0$ under the conditions of symmetry and the conservation of charge along the geodesic curve. It is only ${}^4R_{\mu\nu} = 0$ condition. In this step, we can say that our ${}^5g_{ab}$ in (*) is a 5D vacuum solution and we checked this with MATLAB.
ln summary, from the 5D perspective RN, KN black hole solutions are incompatible with $J_\mu = 0$ and our dynamic result. In contrast, 4D vacuum solution is a solution of 5D vacuum solution with relations describing the classical dynamics for the Lorentz force. The result, ${}^4R_{\mu\nu} = 0$, is same with [15], although assumption is different.

6 5D Energy-Momentum Tensor

In Kaluza's hypothesis,

$$ {}^5T_{ab} = \begin{bmatrix} \gamma_s c_5^2\rho_s & \gamma_e c J_\nu \\ \gamma_e c J_\mu & {}^4T_{\mu\nu} \end{bmatrix}, \tag{37} $$

see eqn(84) of [10]. We will induce energy momentum tensor which is consistent with our result, by assuming perfect fluid source. In 4D, energy-momentum tensor ${}^4T_{\mu\nu}$ for the perfect fluid is given by [12]

$$ {}^4T_{\mu\nu} = (\rho + \frac{P}{c^2})\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau}\,{}^4g_{\mu\lambda}\,{}^4g_{\nu\rho} + P\,{}^4g^{\lambda\rho}\,{}^4g_{\mu\lambda}\,{}^4g_{\nu\rho}. \tag{38} $$

From Einstein equations, we have the relation,

$$ {}^4R_{\mu\nu} = \kappa\left[(\rho + \frac{P}{c^2})\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau}\,{}^4g_{\mu\lambda}\,{}^4g_{\nu\rho} + \frac{1}{2}(\rho c^2 - P)\,{}^4g^{\lambda\rho}\,{}^4g_{\mu\lambda}\,{}^4g_{\nu\rho}\right]. \tag{39} $$

Now we try to induce ${}^5R_{ab}$. With eqn(36):

$$ {}^5R_{\mu\nu} = {}^4R_{\mu\nu}, \tag{40} $$

$$ {}^5R_{\mu 5} = {}^4R_{\mu\lambda}\beta^\lambda, \tag{41} $$

$$ {}^5R_{55} = {}^4R_{\lambda\rho}\beta^\lambda\beta^\rho, \tag{42} $$

we obtain

$$ {}^5R_{ab} = \kappa[(\rho + \frac{P}{c^2})\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau}\,{}^5g_{a\lambda}\,{}^5g_{b\rho} + \frac{1}{2}(\rho c^2 - P)\,{}^4g^{\lambda\rho}\,{}^5g_{a\lambda}\,{}^5g_{b\rho}]. \tag{43} $$

Then ${}^5T_{ab}$ is

$$ {}^5T_{ab} = (\rho + \frac{P}{c^2})\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau}\,{}^5g_{a\lambda}\,{}^5g_{b\rho} + P\,{}^4g^{\lambda\rho}\,{}^5g_{a\lambda}\,{}^5g_{b\rho}. \tag{44} $$

Note that from eqn(37), ${}^4T_{\mu\nu}$ is independent of charge. Then by solving some equation, see the third or fourth item in the discussion section, ${}^4g_{\mu\nu}$ is independent of charge. Then it is consistent with our opinion of section 5.

7 No-hair theorem

We would like to get the mass, charge and angular momentum exactly in 5D. Remind that we assumed the cylindrical condition and stationary condition. And we consider zero cosmological constant.

There are mass quantity in 4D which called as the Komar mass. Komar mass is defined as

$$ M_K \equiv -\frac{1}{8\pi}\oint_{\mathbf{S_t}} \nabla^\mu\zeta^\nu dS_{\mu\nu} \quad dS_{\mu\nu} = (s_\mu n_\nu - s_\nu n_\mu)\sqrt{q}d^2y \tag{45} $$

where vector $\mathbf{n}$ is the unit normal to Σ_t, vector $\mathbf{s}$ is the unit normal to S_t, within Σ_t oriented towards the exterior of S_t [5, 13]. But as we introduced the concept of gravitational and electromagnetic vector potential, we would to like define M, Q as like surface integral of gravity field and electric field. Before applying the surface integral, we want to consider integrating in a different way from Komar mass [14]. Let us consider the following integral

$$ \oint \mathbf{E}\cdot d\mathbf{S}, \tag{46} $$

where $d\mathbf{S}$ is

$$ d\mathbf{S} = \frac{\nabla x^i}{\sqrt{\nabla x^i \cdot \nabla x^i}}\sqrt{-adj({}^4g_{ii})}dS_i, \tag{47} $$

and $dS_1 = dx^2dx^3$, $dS_2 = dx^1dx^3$, $dS_3 = dx^1dx^2$. Resultantly, It is equivalent with Koma mass expression. But it is more easy to calculate. Notice,

$\frac{1}{\sqrt{\nabla x^i \cdot \nabla x^i}} \sqrt{-adj({}^4 g_{ii})} = \sqrt{-{}^4 g}$ for all i. Now we are ready to calculate M, Q and J. First, we would like to caluate Q. As the E-field is related with $F^{0i} \simeq \frac{\mathbf{E}}{c}$,

$$Q \equiv \varepsilon_0 c \oint F^{0i} \sqrt{-{}^4 g} dS_i. \tag{48}$$

We want to make sure this is exact Q. Before the calculation, we want to make sure that it doesn't matter whatever surface we choose. From eqn(48),

$$Q \Rightarrow \oint F^{00} \sqrt{-{}^4 g} dx^1 dx^2 dx^3 + F^{01} \sqrt{-{}^4 g} dx^0 dx^2 dx^3 \tag{49}$$

$$+ F^{02} \sqrt{-{}^4 g} dx^0 dx^1 dx^3 + F^{03} \sqrt{-{}^4 g} dx^0 dx^1 dx^2.$$

In eqn(49), the quantity $F^{00} \sqrt{-{}^4 g} dx^1 dx^2 dx^3$ is a hypothetical 0 term, dx^0 is a virtual integration. Then eqn(49) is as follows

$$\int \nabla_\nu F^{0\nu} \sqrt{-{}^4 g} \, d^3 x \, dx^0. \tag{50}$$

Now we obtain

$$Q \equiv \varepsilon_0 c \int \nabla_\nu F^{0\nu} \sqrt{-{}^4 g} \, d^3 x. \tag{51}$$

For exterior region, $\nabla_\nu F^{0\nu} = 0$. This guarantees what we want to know. Now we calculate eqn(56) with r=constant surface. Under our 5D metric in (*), we obtain

$$\varepsilon_0 c \oint F^{0i} \sqrt{-{}^4 g} \, dS_i \tag{52}$$

$$= \varepsilon_0 c \int_0^{2\pi} \int_0^{\pi} \left[\frac{Q}{4\pi \varepsilon_0 c} \frac{(r^2 + a^2)(-r^2 + a^2 \cos^2 \theta)}{(r^2 + a^2 \cos^2 \theta)^3} \right] (r^2 + a^2 \cos^2 \theta) \sin \theta d\theta d\phi$$

$$= \varepsilon_0 c \int_0^{2\pi} \left[\frac{Q}{4\pi \varepsilon_0 c} \frac{-(r^2 + a^2) \cos \theta}{r^2 + a^2 \cos^2 \theta} \right] |_0^\pi d\phi = Q.$$

Now we calculate the mass. Gravitational vector potential is $A_\mu^{GM} = \frac{c}{2} {}^4 g_{0\mu}$ and let $F_{\mu\nu}^{GM} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and then we obtain

$$-\frac{c}{4\pi G} \oint F_{GM}^{0i} \sqrt{-{}^4 g} \, dS_i \tag{53}$$

$$= -\frac{c}{4\pi G} \int_0^{2\pi} \int_0^{\pi} \frac{c}{2} \left[\frac{-2GM(r^2 + a^2)(-r^2 + a^2 \cos \theta)}{c^2 (r^2 + a^2 \cos^2 \theta)^3} \right] (r^2 + a^2 \cos^2 \theta) \sin \theta d\theta d\phi$$

$$= -\frac{c}{4\pi G} \int_0^{2\pi} \frac{c}{2} \left[\frac{2GM(r^2 + a^2) \cos \theta}{c^2 (r^2 + a^2 \cos^2 \theta)} \right] |_0^\pi d\phi = M.$$

Finally, we define the angular momentum. The mass and charge are derived by surface integral of E-field. However, as far as we know, there is no physical quantity to obtain the angular momentum through the surface integral. So, let's just do the same procedure with the quantities, $\Phi_\mu = c^4 g_{3\mu}$, $\Omega_{\mu\nu} = \partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu$.

$$\frac{c^2}{16\pi G} \oint \Omega^{0i} \sqrt{-{}^4g}\, dS_i \tag{54}$$

$$= \tfrac{c^2}{16\pi G} \int_0^{2\pi} \int_0^{\pi} c \left[\tfrac{2GMa \sin^2\theta(-a^4\cos^2\theta + 3r^4 + 2a^2r^2 - a^2r^2\sin\theta)}{c^2(r^2+a^2\cos^2\theta)^3} \right] (r^2+a^2\cos^2\theta)\sin\theta d\theta d\phi$$

$$= \frac{c^2}{16\pi G} \int_0^{2\pi} c \left[\frac{2GMa\cos\theta((r^2-a^2)\cos^2\theta - 3r^2 - a^2)}{c^2(r^2+a^2\cos^2\theta)} \right] |_0^\pi d\phi = Mac = J.$$

It is already known from the Komar integral that mass is related to staionary symmetry and angular momentum is related to axial symmetry. In this paper, as we start with cylindrical symmetry we can expect there is conserved quantity. From eqn(52), it can be seen that the conserved quantity corresponding to the cylindrical symmetry is the charge.

8 Discussion

First, we did not care whether the fifth dimension is timelike or spacelike. We set it to timelike to follow the way of ADM formalism easily, and if it is spacelike, we can do substitution. But it seems timelike is correct from eqn(32) [17, 19–21]. In the case of spacelike, in other words, ${}^5g_{55} = N^2 + \beta_\lambda\beta^\lambda$, eqn(32) becomes as follows: ${}^5R = {}^4R + (\frac{F_{\lambda\rho}F^{\lambda\rho}}{N^2} - \frac{\mu_0}{N^2}J_\lambda\beta^\lambda)$. Since these world have two timelike, ${}^5g_{00} < 0$, ${}^5g_{55} < 0$, someone can't imagine that we live in these world. But even if ${}^5g_{55} < 0$, we don't have to consider about this because of $dx^5 = 0$ for electrically-neutral body [15].
Second, until now, we hadn't been mentioned about N. With eqn(32), ${}^5R = {}^4R - (\frac{F_{\lambda\rho}F^{\lambda\rho}}{N^2} - \frac{\mu_0}{N^2}J_\lambda\beta^\lambda)$. It seems like $\frac{1}{N^2} = \frac{\kappa}{2\mu_0}$. where $\kappa = \frac{8\pi G}{c^4}$. By dividing eqn(32) by 2κ,

$$\frac{{}^5R}{2\kappa} = \frac{{}^4R}{2\kappa} - (\frac{1}{4\mu_0}F_{\lambda\rho}F^{\lambda\rho} - \frac{1}{2}J_\lambda A^\lambda). \tag{55}$$

In eqn(55), we used $\beta_\mu = 2A_\mu$. Now we get $\mathfrak{L}_{EH} = \frac{{}^4R}{2\kappa}$, $\mathfrak{L}_{EM} = -\frac{1}{4\mu_0}F_{\lambda\rho}F^{\lambda\rho} + \frac{1}{2}J_\lambda A^\lambda$. And for $\mathfrak{L}_{EM}$, it satisfies

$$\frac{\partial \mathfrak{L}_{EM}}{\partial A^\mu} - \nabla^\nu \frac{\partial \mathfrak{L}_{EM}}{\partial \nabla^\nu A^\mu} = J_\mu + \frac{1}{\mu_0}\nabla^\nu F_{\nu\mu} = 0. \tag{56}$$

Also eqn(56) gives $\nabla^\nu F_{\mu\nu} = \mu_0 J_\mu$. Noteworthy is that these contents were naturally induced from the 5-dimensions. Note that $J_\lambda = \frac{2}{\mu_0}R_{\rho\lambda}A^\rho$.

Third, for weakly perturbed system, ${}^5g_{ab} = {}^5\eta_{ab} + {}^5h_{ab}$, the linearized equation is given by

$$ {}^5\Box\left({}^5h_{ab} - \frac{1}{2}\,{}^5\eta_{ab}\,{}^5h\right) = -2\kappa\,{}^5T_{ab}, \qquad {}^5\Box \equiv {}^5\eta^{ab}\partial_a\partial_b. \tag{57} $$

Note that ${}^5\eta$ is expressed in Cartesian coordinate. By imposing the cylindrical and stationary conditions to ${}^5h_{ab}$, we obtain

$$ \frac{{}^5h_{ab}(\mathbf{X})}{2} = \frac{1}{4\pi}\int \frac{\kappa(\,{}^5T_{ab}(\mathbf{Y}) - \frac{1}{3}\,{}^5\eta_{ab}\,{}^5T(\mathbf{Y}))}{|\mathbf{X}-\mathbf{Y}|}d^3Y, \tag{58} $$

where $\mathbf{X}, \mathbf{Y}$ is spartial components. The details, see [16]. If ${}^5h_{5\nu}$ is proportional to electromagnetic vector potential then ${}^5R_{5\nu}$ should be related with the charge current, $\mu_0 J_\nu$. In section 3 and 4, we identified ${}^5h_{5\nu} = 2A_\nu$ then ${}^5R_{5\nu}$ should be related with the charge current. In fact, we got ${}^5R_{5\nu} = \nabla^\rho F_{\nu\rho}$. Note that ${}^5R_{ab} = \kappa({}^5T_{ab} - \frac{1}{3}\,{}^5\eta_{ab}\,{}^5T)$ for 5D.
Fourth, under the stationary condition, ${}^4R_{00} = -\Box\phi_t + \frac{1}{c^2}F^{GM}_{\lambda\rho}F^{\lambda\rho}_{GM}$, ${}^4R_{0i} = \frac{1}{c}\nabla^\rho F^{GM}_{i\rho}$, where $\phi_t \equiv \frac{1}{2}{}^4g_{00}$. From Following equation

$$ \frac{{}^4h_{\mu\nu}(\mathbf{X})}{2} = \frac{1}{4\pi}\int \frac{\kappa(\,{}^4T_{\mu\nu}(\mathbf{Y}) - \frac{1}{2}\,{}^4\eta_{\mu\nu}\,{}^4T(\mathbf{Y}))}{|\mathbf{X}-\mathbf{Y}|}d^3Y, \tag{59} $$

we obtain

$$ A_i^{GM} = \frac{1}{4\pi}\int \frac{\nabla^\rho F^{GM}_{i\rho}}{|\mathbf{X}-\mathbf{Y}|}d^3Y, \tag{60} $$

where $\nabla^\rho F^{GM}_{i\rho} = -\frac{4\pi G}{c^2}J_i$ and J_i is matter current, ρU_i. In this step, as we said at third, if ${}^4g_{0\mu}$ is identified as gravitational vector potential, then ${}^4R_{0i}$ is identified as matter current. In this way, we can develop a theory of gravitomagnetism [10, 18].
Under azimuthal symmetry condition, ${}^4R_{33} = -\Box\phi_\phi + \frac{1}{4c^2}\Omega_{\lambda\rho}\Omega^{\lambda\rho}$, ${}^4R_{3\delta} = \frac{1}{2c}\nabla^\rho\Omega_{\delta\rho}$, where $\phi_\phi \equiv \frac{1}{2}\,{}^4g_{33}$, δ span 0,1,2. Then we obtain

$$ \Phi_\delta = \frac{1}{4\pi}\int \frac{\nabla^\rho\Omega_{\delta\rho}}{|\mathbf{X}-\mathbf{Y}|}d^3Y. \tag{61} $$

Fifth, in eqn(28), since N is constant and the integrand is independent of x^5, we ignored the overall quantity Ndx^5. Note that this is one of the results of the conservation of charge along the geodesic curve.
Sixth, in section 3, we assumed conservation of charge along geodesic curve. We want to tell if there is a contradiction in this assumtion.

$$ U_5 = {}^5g_{5b}U^b = (-N^2 + \beta^\lambda\beta_\lambda)\frac{q}{m} + \beta_\nu U^\nu, \tag{62} $$

where $U^a = \frac{dx^a}{d\tilde{\tau}}$. It is reasonable to assume that uncharged particles are not charged along the geodesic curve. Assume that at this stage, only uncharged particles are not changed. Then with $\beta^\mu = (\alpha, 0, 0, 0)$, $\beta_\nu U^\nu + \beta_\nu \beta^\nu U^5 = \alpha g_{0\nu} U^\nu + \alpha g_{05} U^5 = \alpha U_0 = conserved$. Then eqn(62) is as follows:

$$U_5 = -N^2 \frac{q}{m} + \alpha U_0. \tag{63}$$

Under the condition that uncharged particle is not charged along geodesic curve, If N is a constant, $\frac{q}{m}$ is a conservative, so $N^2 \frac{q}{m}$ is a conservative. Therefore, it can be assumed without contradiction that charged particles also are not changed along the geodesic curve.

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