Confirmation of the Vickrey auction theorem

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated *proof* value, F as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal.

LET p, q, r, s: value, bidder, i, j; & And; + Or; - Not Or; > Imply, greater than; < Not Imply, lesser than; = Equivalent; @ Not Equivalent; (p@p) ordinal zero, 0; (p=p) T, *proof*.

From: en.wikipedia.org/wiki/Vickrey_auction

Proof of dominance of truthful bidding: The dominant strategy in a Vickrey auction with a single, indivisible item is for each bidder to bid their true value of the item.

Let v_i be bidder i's value for the item. Let b_i be bidder i's bid for the item. (10.1)

p&r ; FFFF FTFT FFFF FTFT (10.2)

Let v_i be bidder i's value for the item. Let b_i be bidder i's bid for the item. (11.1)

q&r ; FFFF FFTT FFFF FFTT (11.2)

The payoff for bidder i is $v_i - \max_{j \neq i} b_j$ if $b_i > \max_{j \neq i} b_j$, or 0 otherwise. (12.0)

We rewrite Eq. 12.0 because it does not take into account the third state of payoff as a negative amount, also amounting as a net no payoff:

The payoff for bidder i is $v_i - \max_{j \neq i} b_j$ if $b_i > \max_{j \neq i} b_j$, or [≤] 0 otherwise.(12.1)

payoff for (q&r): ((((q&r)>((r@s)&(q&s)))>((p&r)-((r@s)>(q&s))))>(p@p))
+~((~((q&r)>((r@s)&(q&s)))>((p&r)-((r@s)>(q&s))))>(p@p)) ;
TTTT TTTF TTTT TTFF (12.2)

The strategy of overbidding is dominated by bidding truthfully.

Assume that bidder i bids $b_i > v_i$. (20.1)

(q&r)>(p&r) ; TTTT TTFT TTTT TTFT (20.2)

If $\max_{j \neq i} b_j < v_i$ then the bidder would win the item with a truthful bid as well as an overbid. (21.1)

(((q&r)>(p&r))&(((r@s)>(q&s))<(p&r))) > (((((q&r)>((r@s)&(q&s)))>((p&r)-
((r@s)>(q&s))))>(p@p))+~((~((q&r)>((r@s)&(q&s)))>((p&r)-
((r@s)>(q&s))))>(p@p))) ; TTTT TTTF TTTT TTTT (21.2)

The bid's amount does not change the payoff so the two strategies have equal payoffs in this case.

If $\max_{j \neq i} b_j > b_i$ then the bidder would lose the item either way so the strategies have equal payoffs in this case. (22.1)

(((q&r)>(p&r))>(((r@s)&(q&s))>(q&r))) > (((((q&r)>((r@s)&(q&s)))>((p&r)-((r@s)>(q&s))))>(p@p))+~((~((q&r)>((r@s)&(q&s)))>((p&r)-((r@s)>(q&s))))>(p@p))) ; TTTT TTTF TTTT TTFF (22.2)

If $v_i < \max_{j \neq i} b_j < b_i$ then only the strategy of overbidding would win the auction. (23.1)

(((q&r)>(p&r))>((p&r)<(((r@s)&(q&s))<(q&r)))) > (((((q&r)>((r@s)&(q&s)))>((p&r)-((r@s)>(q&s))))>(p@p))+~((~((q&r)>((r@s)&(q&s)))>((p&r)-((r@s)>(q&s))))>(p@p))) ; TTTT TTTF TTTT TTFF (23.2)

The payoff would be negative for the strategy of overbidding because they paid more than their value of the item, while the payoff for a truthful bid would be zero. Thus the strategy of bidding higher than one's true valuation is dominated by the strategy of truthfully bidding. The strategy of underbidding is dominated by bidding truthfully.

Assume that bidder i bids $b_i < v_i$. (30.1)

(q&r)<(p&r) ; FFFF FFTF FFFF FFTF (30.2)

If $\max_{j \neq i} b_j > v_i$ then the bidder would lose the item with a truthful bid as well as an underbid, so the strategies have equal payoffs for this case. (31.1)

(((q&r)<(p&r))>(((r@s)>(q&s))>(p&r))) > (((((q&r)>((r@s)&(q&s)))>((p&r)-((r@s)>(q&s))))>(p@p)) +~((~((q&r)>((r@s)&(q&s)))>((p&r)-((r@s)>(q&s))))>(p@p))) ; TTTT TTTF TTTT TTTF (31.2)

If $\max_{j \neq i} b_j < b_i$ then then the bidder would win the item either way so the strategies have equal payoffs in this case. (32.1)

(((q&r)<(p&r))>(((r@s)>(q&s))<(q&r))) > (((((q&r)>((r@s)&(q&s)))>((p&r)-((r@s)>(q&s))))>(p@p)) +~((~((q&r)>((r@s)&(q&s)))>((p&r)-((r@s)>(q&s))))>(p@p))) ; TTTT TTTF TTTT TTTF (32.2)

If $b_i < \max_{j \neq i} b_j < v_i$ then only the strategy of truthfully bidding would win the auction. (33.1)

(((q&r)>(p&r))>((q&r)<(((r@s)&(q&s))<(p&r)))) > (((((q&r)>((r@s)&(q&s)))>((p&r)-((r@s)>(q&s))))>(p@p)) +~((~((q&r)>((r@s)&(q&s)))>((p&r)-((r@s)>(q&s))))>(p@p))) ;

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TTTT TTTF TTTT TTFF    (33.2)
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The payoff for the truthful strategy would be positive as they paid less than their value of the item, while the payoff for an underbid bid would be zero. Thus the strategy of underbidding is dominated by the strategy of truthfully bidding. Truthful bidding dominates the other possible strategies (underbidding and overbidding) so it is an optimal strategy. (40.0)

We write as: Eqs. (((21.1 and 22.1) and 23.1) or ((31.1 and 32.1) and 33.1)) imply 12.1. (40.1)

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( ((((((q&r)>(p&r))&(((r@s)>(q&s))<(p&r))) > (((((q&r)>((r@s)&(q&s)))>
((p&r)-((r@s)>(q&s))))>(p@p))+~((~((q&r)>((r@s)&(q&s)))>((p&r)-
((r@s)>(q&s))))>(p@p))))&((((q&r)>(p&r))>(((r@s)&(q&s))>(q&r))) >
(((((q&r)>((r@s)&(q&s)))>((p&r)-((r@s)>(q&s))))>(p@p))
+~((~((q&r)>((r@s)&(q&s)))>((p&r)-
((r@s)>(q&s))))>(p@p)))))&((((q&r)>(p&r))>((p&r)<(((r@s)&(q&s))<
(q&r)))) > (((((q&r)>((r@s)&(q&s)))>((p&r)-((r@s)>(q&s))))>(p@p))
+~((~((q&r)>((r@s)&(q&s)))>((p&r)-((r@s)>(q&s))))>(p@p)))))
+
((((((q&r)<(p&r))>(((r@s)>(q&s))>(p&r))) >
(((((q&r)>((r@s)&(q&s)))>((p&r)-((r@s)>(q&s))))>(p@p))
+~((~((q&r)>((r@s)&(q&s)))>((p&r)-
((r@s)>(q&s))))>(p@p))))&((((q&r)<(p&r))>(((r@s)>(q&s))<(q&r))) >
(((((q&r)>((r@s)&(q&s)))>((p&r)-((r@s)>(q&s))))>(p@p))
+~((~((q&r)>((r@s)&(q&s)))>((p&r)-
((r@s)>(q&s))))>(p@p)))))&((((q&r)>(p&r))>((q&r)<(((r@s)&(q&s))<
(p&r)))) > (((((q&r)>((r@s)&(q&s)))>((p&r)-((r@s)>(q&s))))>(p@p))
+~((~((q&r)>((r@s)&(q&s)))>((p&r)-((r@s)>(q&s))))>(p@p)))))) )
>
(((((q&r)>((r@s)&(q&s)))>((p&r)-((r@s)>(q&s))))>(p@p))
+~((~((q&r)>((r@s)&(q&s)))>((p&r)-((r@s)>(q&s))))>(p@p))) ;
                                        TTTT TTTT TTTT TTTT    (40.2)
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Eq. 40.2 as rendered is tautologous. This means the Vickrey auction theorem is confirmed.

Remark: Processing Eq. 40.2 required 519 steps.