Note on Mathematical Inequality

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Abstract: This is very fundamental concept of mathematical inequality .I have focused in some preliminary concept of mathematical inequality. Here I am dealing with real number and their properties.

The property of trichotomy: Any two real number a, b must be satisfy one and only one of the following relations–

(i) a is equal to b $(a = b)$,

(ii) a is greater than b $(a > b)$,

(iii) a is less than b $(a < b)$.

The last two relations are inequality relations.

Properties: If a, b, c be real numbers, then

(i) $a > b$ and $b > c \Rightarrow a > c$,

(ii) $a > b \Rightarrow a + c > b + c$,

(iii) $a > b$ and $c > 0 \Rightarrow ac > bc$,

(iv) $a > b$ and $c < 0 \Rightarrow ac < bc$,

(v) $a > b$ and $c = 0 \Rightarrow ac = bc$,

Corollary: (i) $a \geq b$ and $b \geq c \Rightarrow a \geq c$,

(ii) $a \geq b$ and $b > c \Rightarrow a > c$,

(iii) $a \geq b \Rightarrow a + c \geq b + c$,

(iv) $a \geq b$ and $c > 0 \Rightarrow ac \geq bc$,

(v) $a \geq b$ and $c < 0 \Rightarrow ac \leq bc$.

Theorem: If $a_1, a_2, \cdots, a_n;\ b_1, b_2, \cdots, b_n$ be all real number such that $a_i > b_i$ for $i = 1,2, \cdots, n$, then

$$a_1 + a_2 + \cdots + a_n > b_1 + b_2 + \cdots + b_n.$$

$proof.$ $(a_1 + a_2 + \cdots + a_n) - (b_1 + b_2 + \cdots + b_n)$

$= (a_1 - b_1) + (a_2 - b_2) + \cdots (a_3 - b_3) > 0, since\ a_i - b_i > 0$ for $i = 1,2, \cdots n$.

Therefore $a_1 + a_2 + \cdots + a_n > b_1 + b_2 + \cdots + b_n$.

Theorem: If $a_1, a_2, \cdots, a_n;\ b_1, b_2, \cdots, b_n$ be all positive real number such that $a_i > b_i$ for $i = 1,2, \cdots, n$, then $a_1 a_2 \cdots a_n > b_1 b_2 \cdots b_n$.

$proof.$ $a_1 a_2 - b_1 b_2 = a_1(a_2 - b_2) + b_2(a_1 - b_1) > 0$, since each term is positive.

Therefore $a_1 a_2 > b_1 b_2$. Thus $a_1 > b_1$ and $a_2 > b_2 \Rightarrow a_1 a_2 > b_1 b_2$.

Similarly, $a_1 a_2 > b_1 b_2$ and $a_3 > b_3 \Rightarrow a_1 a_2 a_3 > b_1 b_2 b_3$. successive application give $a_1 a_2 \cdots a_n > b_1 b_2 \cdots b_n$.

Means: Let x be a real number. The most basic inequalities are

$$x^2 \geq 0, \quad \cdots (1)$$

$$\sum_{i=1}^{n} x_i^2 \geq 0. \quad \cdots (2)$$

We have equality only if $x = 0$ in (1) or $x_i = 0$ for all i in (2). one strategy for proving inequalities is to transform them into the form (1) or (2). This is usually a long road. So we derive some consequences equivalent to (1). With $x = a - b$, $a > 0, b > 0$, We get the following equivalent inequalities:

$$a^2 + b^2 \geq 2ab \Leftrightarrow 2(a^2 + b^2) \geq (a + b)^2 \Leftrightarrow \frac{a}{b} + \frac{b}{a} \geq 2 \Leftrightarrow \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}.$$

Replacing a, b by $\sqrt{a}, \sqrt{b}$, we get $a + b \geq 2\sqrt{ab} \Leftrightarrow \frac{a+b}{2} \geq 2\sqrt{ab} \Leftrightarrow \sqrt{ab} \geq \frac{2ab}{a+b}$.

In particular, we have the inequality chain

$$\min(a,b) \leq \frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}} \leq \max(a,b).$$

This is the $harmonic - geometric - arithmetic - quadratic\ mean\ inequality$.

Example: If $a_i > 0$ for $i = 1, \cdots, n$ and $a_1 a_2 \cdots a_n = 1$, then $(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq 2^n$

Solution: we know that $\frac{(1+a_1)}{2} \geq \sqrt{a_1}, \frac{(1+a_2)}{2} \geq \sqrt{a_2}, \cdots, \frac{(1+a_n)}{2} \geq \sqrt{a_n}$.

Multiplying all the inequalities we get,

$$\frac{(1+a_1)}{2} \cdot \frac{(1+a_2)}{2} \cdots \frac{(1+a_n)}{2} \geq \sqrt{a_1 a_2 a_3 \cdots a_n}.$$

$$\Rightarrow \frac{(1+a_1)}{2} \cdot \frac{(1+a_2)}{2} \cdots \frac{(1+a_n)}{2} \geq 1 \Rightarrow (1 + a_1) \cdot (1 + a_2) \cdots (1 + a_n) \geq 2^n.$$

Generally, for n positive number a_i, we have the following inequalities:

$$\min(a_i) \leq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \leq \sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n} \leq \sqrt{\frac{a_1 + a_2 + \cdots + a_n}{n}} \leq \max(a_i).$$

The equality sign is valid only if $a_1 = \cdots = a_n$.

EXERCISE

1. For positive $a, b\ and\ c$, prove that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$
2. If $a_1, a_2, \cdots, a_n$ are positive real numbers prove that $(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2$.
3. If $a > 0, b > 0$ and $c > 0$, prove that $\frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} \geq 6$.
4. If a, b, c are positive, prove that $(a + b + c)(ab + bc + ca) \geq 9abc$.

Cauchy-Schwarz Inequality: Let $a_1, a_2, \cdots a_n$ and $b_1, b_2, \cdots b_n$ be two sets of real numbers. Then

$(\sum_{i=1}^{n} a_i b_i)^2 \geq (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2)$.

And equality holds iff

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots \frac{a_n}{b_n}.$$

$proof$: Let us put $A = \sum_{i=1}^{n} a_i^2, B = \sum_{i=1}^{n} b_i^2, C = \sum_{i=1}^{n} a_i b_i$.

We have to prove $C^2 \leq AB$. If $B = 0$ then $b_i = 0$ for all $i = 1,2, \cdots, n$. Hence $C = 0$.Therefore it is sufficient to consider the case $B \neq 0$. This implies that $B > 0$. We now have $0 \leq \sum_{i=1}^{n}(Ba_i - Cb_i)^2$

$= B^2 \sum_{i=1}^{n} a_i^2 -$

$2BC \sum_{i=1}^{n} a_i b_i + C^2 \sum_{i=1}^{n} b_i^2$.

$= B(AB - C^2)$.

Since $B > 0$, we get $AB - C^2 \geq 0$. Which is the required inequality. Moreover, equality holds iff

$$\sum_{i=1}^{n}(Ba_i - Cb_i)^2 = 0.$$

This is equivalent to $\frac{a_i}{b_i} = \frac{C}{B}$, $i = 1,2, \cdots, n$.

Example: If $a_1, a_2, \cdots, a_n$ are real numbers such that $a_1 + a_2 + \cdots + a_n = 1$, prove that

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq \frac{1}{n}.$$

Solution: we have, $(1 + 1 + \cdots n\ times)(a_1^2 + a_2^2 + \cdots + a_n^2) \geq (a_1 \cdot 1 + a_2 \cdot 1 + a_3 \cdot 1 + \cdots + a_n \cdot 1)^2$.

$$\Rightarrow n \cdot (a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2) \geq 1$$

$$\Rightarrow (a_1^2 + a_2^2 + \cdots + a_n^2) \geq \frac{1}{n}$$

REFERENCE

1. Problem-solving strategies-Arthur Engel.

2. Classical Algebra-S.K.MAPA