Reinterpretation of the Robertson-Walker metric, the Shapes of the Cosmos, and the Cosmological Constant

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Abstract

The Robertson-Walker metric has been recognized for describing the global space-time Universe which could be one of the three models: flat ($k = 0$), closed ($k = 0$) or open ($k = -0$). This paper reinterprets the Robertson-Walker metric, which actually describe the geometrical shapes of the global space-time Universe and two local space-time. The global space-time is the "infinite" hyper-sphere of the Universe without boundary (open sphere). The two forms of local space-time caused by the agglomeration of matter into stars and stellar systems, which appeared as sphere (or ellipsoid) and flat. The shapes of spheres range from small elementary particles in quantum physics to planets, stars and giant objects such as globular nebulae in cosmological physics. The flat shapes are disk like galaxies, and the solar system in its infancy, etc. In last chapter, the cosmological constant was derived from the volume of the five dimension hyper-sphere. The cosmological constant is relevant with the Gaussian curvature $\frac{1}{R^2}$ of the Universe. The value of the cosmological constant is: $\Lambda = \frac{5{H_0}^2}{2\pi c^2}$.

Keywords: Robertson-Walker metric, observable Universe, spherical shape, flat shape, "infinite" hyper-sphere shape, five dimensional Universe, cosmological constant, Gauss curvature.

1. Introduction

When we look around the Universe, there are two types of geometric shapes of distribution forms for observable objects: spheres and thin cylindrical flat disks. In addition, scientists discovered the global space-time Universe has been expanding for 14 billion years, like a continuously been inflated balloon. The global space-time Universe might be as a hyper-sphere, a non-Euclidean space, might also be an open sphere without boundary. Why the observable objects in the Universe take shapes of spheres and flat disks?

First, we investigate the objects of spherical forms in the Universe. The smallest spherical objects which we can observe are elementary particles, such as an electron, it appears as a sphere cloud. Atoms and molecules also appear having the spherical shapes. A DNA is the combination of a series of spherical molecules. A water drop and a tiny flame in the Space appear as spheres. The Earth and planets, the Moon and the Sun, all stars are in the spherical shapes. The largest spherical objects in the Universe are the globular nebulae. There are 150 to 158 globular currently discovered in the Milky Way galaxy. Globular clusters can contain a high density of stars; on average about 0.4 stars per cubic parsec, increasing to 100 or 1000 stars per cubic parsec in the core of the cluster, see Figure 1. The typical distance between stars in a globular cluster is about 1 light year, but at its core, the separation is comparable to the size of the Solar System. Almost all globular clusters have a half-light radius of less than 10 pc, although there are well-established globular clusters with very large radii (i.e. NGC 2419 (R_h = 18 pc) and Palomar 14 (R_h = 25 pc)). [1]

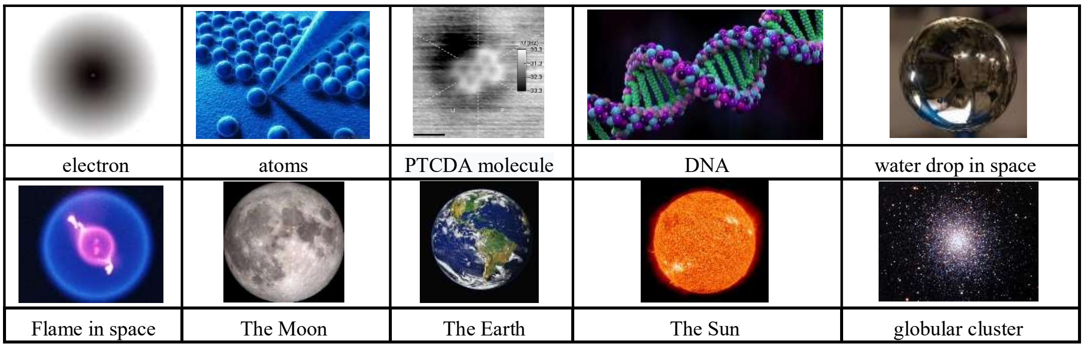

Figure 1: Type of spheres in the Universe
from: https://en.wikipedia.org/

Next, we investigate the flat objects in the Universe. The infancy of the Solar system formed as a flat disk cloud with a spherical Sun at its center. Planets like the Saturn is a sphere with a huge flat disk ring. The largest objects formed as flat disk in the Universe are galaxies. The shape of a galaxy is a giant flat disk with a spherical shape of bulge, and a black hole at its center, see Figure 2. There are approximately 2 trillion galaxies in the Universe. The largest galaxy ever discovered is the galaxy IC 1101, its diameter could be 2,800,000 light years. [2]

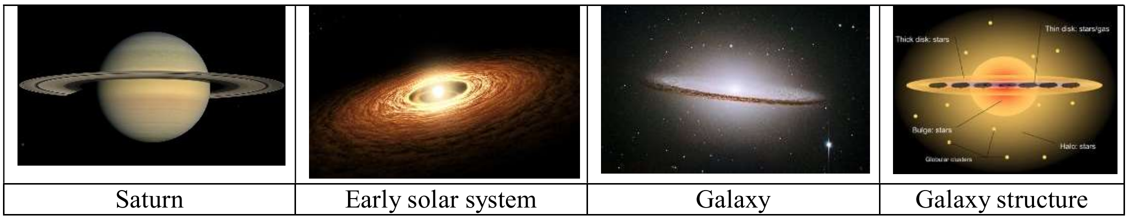

Figure 2: Cosmic flat disk systems
from: https://en.wikipedia.org/

Finally, the global space-time Universe might be an “infinite” hyper-sphere without boundary, because the Universe is still expanding, it might be an opened sphere. Some cosmologists believe the Universe begin with a Big Bang 13 billion ago. The Universe may be shaped as a Lobachevsky pseudosphere. The Poincare ball may also be a model of the Universe, see Figure 3. Along with the Klein model and the Poincaré half-space model, it was proposed by Eugenio Beltrami who used these models to show that hyperbolic geometry was equiconsistent with Euclidean geometry. [3] Scientists are still debating what shape our Universe could be and what dimensions it could have.

Why the cosmological objects appears the geometrical shapes as spherical and flat discs? Is there some physics or math theory explains why the Galaxy turns out to be thin cylindrical-flat disk like instead of spherical-like?

In next chapter, for answer of above questions, we are going to reinterpret the Robertson-Walker metric for the models of sphere, flat and opened sphere.

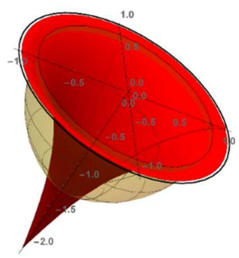	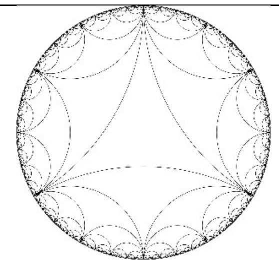
Lobachevsky pseudosphere	Poincaré disk model

Figure 3: hyperbolic geometry
from: https://en.wikipedia.org/

2. Reinterpretation of the Robertson-Walker metric

H.P. Robertson wrote: “The general theory of relativity attributes the particular metrical properties of the space-time Universe, considered as a 4-dimensional Riemannian manifold, directly to the distribution of matter within it, and has naturally led to speculations concerning the structure of the Universe as a whole, in which the local irregularities caused by the agglomeration of matter into stars and stellar systems are disregarded. Chief among the resulting relativistic cosmologies are those based on the cylindrical world of Einstein' and the spherical world of de Sitter;”[4]

The 4-dimensional Riemannian manifold of space-time is isometric and homogeneous, however the matter which distributed within it are unnecessarily to be isometric and homogeneous, the distribution forms of matter could be described with the Robertson-Walker metric. The chief among the forms would be cylindrical and spherical shape, as described by Robertson. I remind the reader for remember of the key word “cylindrical” and “spherical”, which related with the subjects of this paper.

Following is one deriving method of the Robertson-Walker metric: [5]

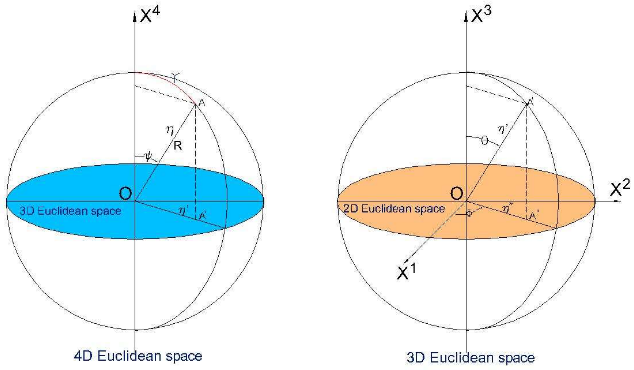

Figure 4: The generalized spherical coordinates η, ψ,θ,φ of 4-dimensional Euclidean space explained. Left panel: the radial coordinate η, and “polar” angle ψ. The blue plane represents the 3-dimensional hyper-plane normal to the x4 axis. The other two coordinates of the point A are the same as the coordinates of projected point A' in this hyper-plane. Right panel: the spherical coordinates of point A' in the hyper-plane. (Diagram modified from Figure3.1 in [5])

A four dimension hyper-sphere with four Cartesian coordinates of the four dimensional Euclidean space, centered on center of the hyper-sphere of radius R, then its equation is

$$(x^1)^2+(x^2)^2+(x^3)^2+(x^4)^2=Rl_{\Phi}^2 \quad (1)$$

By construction, $\psi, \theta \in [0, \pi]$, and $\Phi \in [0, 2\pi)$. The transformation law between the generalized spherical and Cartesian coordinates is

$$x^4 = \eta cos\psi,$$
$$x^3 = \eta' cos\theta = \eta sin\psi sin\theta,$$
$$x^2 = \eta'' sin\Phi = \eta sin\psi sin\theta sin\Phi,$$
$$x^1 = \eta'' cos\Phi = \eta sin\psi sin\theta cos\Phi,$$

Next we find the metric form of the 4-dimensional Euclidean space in the generalized spherical coordinates. In fact, these coordinates are orthogonal, and hence the infinitesimally small parallelepiped whose sides are made of the coordinate lines is rectangular. Its diagonal has the length given by the equation

$$dl^2 = dl_\eta^2 + dl_\psi^2 + dl_\theta^2 + dl_\Phi^2 \quad (2)$$

where dl_i are the lengths of the sides. It is easy to see that

$$dl_\eta = d\eta,$$
$$dl_\psi = \eta d\psi,$$
$$dl_\theta = \eta' d\theta = \eta sin\psi d\theta,$$
$$dl_\Phi = \eta'' d\Phi = \eta sin\psi sin\theta d\Phi.$$

thus, the metric form is

$$dl^2 = d\eta^2 + \eta^2 d\psi^2 + \eta^2 sin^2\psi(d\theta^2 + sin^2\theta d\Phi^2). \quad (3)$$

Now we can find the metric form of the hyper-sphere of radius *R*. Since on the hyper-surface $\eta = R$ and $d\eta = 0$, we have

$$dl^2 = R^2\left(d\Psi^2 + sin^2\Psi(d\theta^2 + sin^2\theta d\Phi^2)\right) \quad (4)$$

Are there any other types of uniformly curved 3-dimensional manifolds? Yes. First, the trivial case of 3-dimensional Euclidean space, which has zero curvature everywhere. Its metric form can also be written as

$$dl^2 = R^2\left(d\Psi^2 + \Psi^2(d\theta^2 + sin^2\theta d\Phi^2)\right) \quad (5)$$

where $\Psi = \frac{r}{R}$ varies from 0 to $+\infty$.

Then, there is the so-called hyperbolic space with the metric

$$dl^2 = R^2\left(d\Psi^2 + sinh^2\Psi(d\theta^2 + sin^2\theta d\Phi^2)\right) \quad (6)$$

Where R is the imaginary radius of hyper-sphere, and Ψ varies from 0 to $+\infty$. This also corresponds to a hyper-surface but now in the Minkowskian spacetime (also known as pseudo-Euclidean space). Its geometric properties are similar to those of a saddle-like surface in Euclidean space.

We now probe three metric forms of (4), (5) and (6). We want to find out what kind of geometrical shapes, that they would represented. The simplest way is find out the volume formulas of those three metrics.

For a non-Euclidean space, the volume formula shall be

$$V = \iiint \sqrt{g}\, d\Psi d\theta d\Phi \;\; [6] \quad (7)$$

First, we probe metric form (4), we got the volume as

$$V_1 = 4\pi R^3 \int_0^{\Psi} sin^2\Psi \, d\Psi \tag{8}$$

$$= 4\pi R^3 \left(\frac{1}{2}\Psi - \frac{1}{4}sin2\Psi\right)$$

here $\Psi = \frac{r}{R}$, therefore

$$V_1 = 4\pi R^3 \left(\frac{r}{2R} - \frac{1}{4}sin\left(\frac{2r}{R}\right)\right) \tag{9}$$

$$= 2r\pi R^2 - \pi R^3 sin\left(\frac{2r}{R}\right) \tag{10}$$

The equation (10) appears two items of Euclidean shapes, here $r \ll R$ varies from 0 to πR. When 0< $r < \frac{\pi R}{2}$, the second item in equation (10) is negative values. When $\frac{\pi R}{2} < r < \pi R$ the values of the second item in equation (10) becomes positive:

$$V_1 = 2r\pi R^2 + \pi R^3 sin\left(\frac{2r}{R}\right) \tag{11}$$

There must exist some small positive values $\varepsilon > 0$, which lead $sin\left(\frac{2r}{R}\right) \to \frac{4}{3}\varepsilon$, then combining (10) and (11), we got following equations:

$$V_1 = 2r\pi R^2 \pm \varepsilon\frac{4}{3}\pi R^3 \qquad (2r \ll R) \tag{12}$$

We shall recognize the first item in (12) is a thin cylinder with volume of $2r\pi R^2$, and the second item is a sphere with the volume of $\varepsilon\frac{4}{3}\pi R^3$. The matter distributed within this manifold would be look like a thin cylinder disc of galaxy in zero curvature, plus a spherical bulge in positive curvature and a black hole with a negative curvature at its center. The black holes with negative curvature could be the black holes with negative mass. [7] The galaxies are composed by the agglomeration matter distributed in the geometric shapes with the multiple curvatures of zero, positive and negative.

Next, we probe metric form (5), then we got the volume as

$$V_2 = 4\pi R^3 \int_0^{\Psi} \Psi^2 \, d\Psi \tag{13}$$

$$= \frac{4}{3}\pi R^3 \Psi^3$$

here $\Psi = \frac{r}{R}$, therefore

$$V_2 = \frac{4}{3}\pi r^3 \tag{14}$$

This is a volume of a 3D Euclidean sphere, the radius r range from as small radius of elementary particles to as big as the radius of the globular nebulae. For an electron, its radius of r could be as small as $r_e = 2.8194 \times 10^{-15} m$, however most globular nebulae have radius about 25pc, which is about $7.7142 \times 10^{17} m$, compare with the electron, the radius of globular nebulae could be considered as "infinity".

Finally, we probe metric form (6), we got the volume as

$$V_3 = 4\pi R^3 \int_0^{\Psi} sinh^2\, \Psi d\Psi \tag{15}$$

$$= 4\pi R^3 \left(\frac{1}{2} sinh\Psi cosh\Psi - \frac{\Psi}{2}\right)$$

$$= 2\pi R^3 \sinh\Psi\, cosh\Psi - 2\pi R^3\Psi \tag{16}$$

here, Ψ varies from 0 to ∞ .

However in (16), both $\sinh\Psi$ and $cosh\Psi$ exhibit in exponential grow faster than Ψ, therefore the second item in (16) could be ignored.

$$V_3 = 2\pi R^3 \sinh\Psi\, cosh\Psi \tag{17}$$

What is dimensions of the volume V_3 in (17) which $2\pi R^3 \sinh\Psi\, cosh\Psi \rightarrow 2\pi R^3 \infty^2$? It looks like a five dimensional object.

The Kaluza-Klein theory was one of the first five-dimensional theory which attempts to create an unified field theory developed in 1921.[8] Latterly in 1938, Einstein and Bergmann were among the first to introduce the modern viewpoint in which a four-dimensional theory that coincides with Einstein-Maxwell theory at long distances is derived from a five-dimensional theory with complete symmetry among all five dimensions.[9]

We would assume the Universe is a five dimension space. The hyper-sphere is a five dimension sphere, then the volume of the five dimension sphere could also be written more nicely form as

$$V_5 = \frac{8}{15}\pi^2 R^5 \quad [10] \quad \text{here } R \rightarrow \infty \tag{18}$$

Summary of above analysis, we have a conclusion that Robertson-Walker metric represent the three types of geometrical shapes of space-time in the Universe:

1. The spherical objects range from as small as electron to as large objects as globular clusters, they have positive curvature $k > 0$. In the Robertson-Walker metric formula, the Ψ range from $0 \rightarrow \infty$. The Robertson-Walker metric has been widely studied in quantum physics.[11]

2. The most typical disk shape-like objects in the Universe are the galaxies. A galaxy has a huge spiral disc like stellar cloud, with a spherical shape of central bulge and a black holes at its center. A galaxy has three kinds of geometrical curvatures; the flat disc consist of zero curvature $k = 0$ on the axial direction; the bulge with positive curvature $k > 0$ on the radial direction, and the black hole with negative curvature $k < 0$. The infancy of the solar system also formed a flat dust cloud disk ($k = 0$) with a hot ball ($k > 0$). The Saturn like plants have a sphere ($k > 0$) and a thin ring ($k = 0$).

3. The space-time Universe is an open boundary hyper-sphere with negative curvature $k < 0$ and the infinite radius.

Following table is the summary of the three type of geometry space-times described by the Robertson-Walker metric:

Geometry shapes	Curvature	$S_k(\Psi)$	Scale	Examples of Cosmological Objects
Sphere (closed)	$k > 0$	Ψ	$0 \rightarrow 25pc$	Electron, atoms, moons, planets, stars, globular clusters
cylindrical Flat disc$\pm$Spheres	$k = 0, k > 0, k < 0$	$sin\ \Psi$	$0 \rightarrow 130kpc$	Saturn, early solar system, galaxies with bulge and black holes
Hyper-sphere (open)	$k < 0$	$sinh\ \Psi$	$0 \rightarrow \infty$	The space-time of the Universe

Table 1: Types of geometries in the Universe; $1pc = 3.0857 \times 10^{16} m$

3. The Cosmological Constant

In this chapter, we are going to discuss the cosmological constant and derive its theoretical value.

The Gravitational field equation was a theoretical equation established by Einstein in 1915, the Universe and the Milky Way (our Galaxy) were considered as the same thing, and the Milky Way appeared to be very much static. Einstein analyzed the models of static Universe and concluded that such Universe cannot be infinite. Instead, it must be finite but without boundaries, the space must be wrapped onto itself like a sphere in Euclidean geometry. The original field equation did not include the cosmological constant Λ.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \quad [12] \tag{19}$$

In 1916, after Einstein published his general relativity theory, Karl Schwarzschild soon obtained the exact solution to Einstein's field equations for the gravitational field outside a non-rotating, static spherically symmetric body.

In 1922 Friedmann solved the Einstein gravitational field equation and obtained non static cosmological solutions presenting an expanding Universe. Einstein thought that the Universe should be static and unchanged forever, then he modified his original field equations by adding the so-called cosmological term Λ which can stop the expansion.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \ [13] \tag{20}$$

here:
$R_{\mu\nu}$ is the Ricci tensor.
R is the curvature scalar.
Λ is the cosmological constant.
$g_{\mu\nu}$ is the metric tensor.
G is the gravitational constant.
$T_{\mu\nu}$ is the energy–momentum tensor.

In 1929 Hubble found experimentally that the distant galaxies are receding from us, and the farther the galaxy the bigger its velocity as determined by its redshift. Hubble's initial value for the expansion rate is the Hubble constant H_0. This observation caused Einstein to abandon the cosmological constant, and calling it the '"biggest blunder" of his life.

The Hubble's Law is

$$\dot{D} = H_0 D \quad [14] \tag{21}$$

here:
$\dot{D}$ is the recessional velocity, typically expressed in km/s.
H_0 is the Hubble constant. $H_0 = 67.6^{+0.7}_{-0.6} km\ s^{-1} Mpc^{-1}$ [15]
D is the proper distance, measured in mega parsecs (Mpc).

However, recent discover indicate that the expansion of the Universe has been accelerating. Einstein's modified equation may be the apposite form after all. Scientists have revived Einstein's cosmological constant to explain a mysterious force called dark energy that seems to be counteracting gravity, and causing the Universe to expand at an accelerating speed.

Without cosmological constant Λ, the Universe is static sphere; adding cosmological constant Λ, then the Universe is an expanding hyper-sphere. We could guess $\Lambda \propto H_0$. Although the Universe is a five dimension hyper-sphere, it appears as a three dimension of Euclidean sphere. How could a five-dimensional hyper-sphere observed as a 3D Euclidean sphere? What is the relation between the hyper-sphere and the Euclidean sphere?

We use Λ as a differential operator acting on the volume of the hyper-sphere (18), and assume it would equate to the volume of the Euclidean sphere, then see what result would come out.

$$\Lambda \frac{8}{15}\pi^2 R^5 = \frac{4}{3}\pi R^3 \tag{22}$$

$$\Lambda \frac{2}{5}\pi R^2 = 1 \tag{23}$$

$$\Lambda = \frac{5}{2\pi R^2} = \frac{5}{2\pi}\left(\frac{1}{R^2}\right) \tag{24}$$

We got the value of the cosmological constant, recall that R is the radius of the observable Universe, and $R = \frac{c}{H_0}$, then the theoretical value of the cosmological constant will be:

$$\Lambda = \frac{5{H_0}^2}{2\pi c^2} \tag{25}$$

$$\Lambda = 5.5336 \times 10^{-5}\ m^{-2} \text{ or}$$

$$\Lambda = 1.4455 \times 10^{-122} \tag{26}$$

This value of the cosmological constant Λ very close to the values in many research papers. Such as $\Lambda = 2.90 \times 10^{-122}$ shown on Wikipedia, the free encyclopedia [16]

There is an item of $\frac{1}{R^2}$ in equation (24), it is the Gaussian curvature of the Universe, and R is the curvature radius of the hyper-sphere. The cosmological constant relates to the Gaussian curvature of the hyper-sphere.

3. Conclusion

The Robertson-Walker metric describes three types of geometry objects in our Universe:

When $k < 0$, it is the globe space-time itself, it is an isometric and homogeneous five dimensions open hyper-sphere without boundary. The scale range from 0 to infinity.

When $k > 0$, it is a local space-time, the agglomeration of matter appear as spherical objects. The scale of the spherical objects range from 0 to 25 pc.

The metric which formed with the combination of the curvatures of $k = 0,\ k > 0$, and $k < 0$ describes the larger scale of local space-time. The typical example is a galaxy in which the agglomeration of matters form a giant cylindrical like disc ($k = 0$), a giant sphere like bulge ($k > 0$) and black holes with the negative mass ($k < 0$). A cylinder and a sphere are homeomorphism, they are topological invariants. The scale range from 0 to 125 kpc.

The Robertson-Walker metric describes the isometric and homogeneous space-time which is a five dimensions of hyper-sphere. When the cosmological constant Λ acts on five dimensions of hyper-sphere, the five dimensions of hyper-sphere converted to the three dimensions of Euclidean sphere. The theoretical value of the cosmological constant is:

$$\Lambda = \frac{5{H_0}^2}{2\pi c^2}$$

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