Question 471: The Ramanujan Identity

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Abstract: In this note we recall an identity given by Ramanujan.

1. Introduction: Identity

If $ab = \pi$, then

$$\sqrt{a}\int_0^\infty \frac{e^{-x^2}}{\cosh(ax)}dx = \sqrt{b}\int_0^\infty \frac{e^{-x^2}}{\cosh(bx)}dx \tag{1}$$

2. Alternative formula:

If $ab = \pi, u > 0, v > 0$, then

$$\begin{aligned} &\sqrt{a}\left\{\int_0^u \frac{e^{-x^2}}{\cosh(ax)}dx - \frac{(2u+a)e^{-u^2}}{2\cosh(au)} + \frac{a e^{-au-u^2}}{2(\cosh(au))^2}\right\} + \\ &\qquad + 2\sqrt{a}\sum_{n=0}^\infty (-1)^n e^{(2n+1)^2a^2/4}\,\Gamma\left(\frac{3}{2}, \frac{(2n+1)^2a^2}{4} + (2n+1)au + u^2\right) = \\ &= \sqrt{b}\left\{\int_0^v \frac{e^{-x^2}}{\cosh(bx)}dx - \frac{(2v+b)e^{-v^2}}{2\cosh(bv)} + \frac{b e^{-bv-v^2}}{2(\cosh(bv))^2}\right\} + \\ &\qquad + 2\sqrt{b}\sum_{n=0}^\infty (-1)^n e^{(2n+1)^2b^2/4}\,\Gamma\left(\frac{3}{2}, \frac{(2n+1)^2b^2}{4} + (2n+1)bv + v^2\right) \end{aligned} \tag{2}$$

where $\Gamma(\alpha, x) = \int_x^\infty e^{-t}t^{\alpha-1}dt$, is the incomplete gamma function.

$$\Gamma\left(\frac{3}{2}, x\right) = \int_x^\infty e^{-t}\sqrt{t}\,dt \tag{3}$$

$$\Gamma\left(\frac{3}{2}, x\right) = \sqrt{x}\,e^{-x} + \frac{1}{2}\Gamma\left(\frac{1}{2}, x\right) \tag{4}$$

$$\Gamma\left(\frac{3}{2}, x\right) = \sqrt{x}e^{-x} + \frac{1}{2}\sqrt{x}e^{-x}\left\{\frac{x^{-1}}{1+}\,\frac{(1/2)x^{-1}}{1+}\,\frac{x^{-1}}{1+}\,\frac{(3/2)x^{-1}}{1+}\,\frac{2x^{-1}}{1+}\,\frac{(5/2)x^{-1}}{1+}\cdots\right\} \tag{5}$$

References

[1] Bruce C. Berndt: Ramanujan’s Notebooks Part II. Springer-Verlag , 1989.