"*Proof that Space is Equivalent to Energy, and that Compactification Necessarily Leads to Changes in Volume, Energy and Entropy*"

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ABSTRACT:

Using simple box quantization, we demonstrate explicitly that space is equivalent to energy, and that compactification releases latent heat with an attendant change in volume and entropy. Increasing spatial dimension costs energy while decreasing dimensions releases energy, which can be quantified, using a generalized version of the Clausius-Clapyeron relation. We show this for a massive particle trapped in a box. Compactification from N-dimensional space to (N-1) spatial dimensions is also simply demonstrated and the correct limit to achieve a lower energy result is to take the limit where $L_w \rightarrow 0$, where L_w is the compactification length parameter. Higher dimensional space has more energy and more entropy, all other things being equal, for a given cutoff in energy.

Introduction

Recently [1], we generalized the Clausius-Clapeyron (CC) relation to allow for a phase transition involving a change in spatial dimension. Using radiative photons as our substance, this generalization allowed us to quantify the latent heat given off in transitioning from N-spatial dimensions to (N-1) dimensions, and the latent heat absorbed in going from (N-1) spatial dimensions to N dimensions. Moreover, expressions for the changes in entropy and volume were derived when undergoing this type of first order phase transition.

We gave specific numerical examples considering temperatures which applied to the very early universe. It was conjectured that, based on the explosive release of heat energy, and the attendant changes in entropy and volume, that the N=4 to (N-1) =3 transition may have a connection to inflation. The universe may have underwent a phase transition from 4-space to 3-space within the earliest times [2]. We argued that this would bypass the need for the inflaton field, as well as do away with a-causal expansion. If it is the spatial dimension itself, which is changing, i.e., undergoing a phase transition, a-causality becomes a secondary issue. Finally, quantum mechanical fluctuations about a mean temperature were also discussed when transitioning between spatial dimensions. It was shown that if the phase transition involves the same thermodynamic process on both sides of the co-existence curve, such as from adiabatic expansion to adiabatic expansion, then the temperature fluctuations, $\delta T/T$, carry through unperturbed from one space to the neighboring space. If, on the other hand, the phase transition involves a difference in thermodynamic process on either side of the coexistence curve, such as from isothermal expansion to adiabatic expansion, then it was discovered that the thermal fluctuations can be created within the transition itself.

In this short note, we carry on the analysis and prove that space is equivalent to energy another way. We focus on box quantization of a massive particle and consider what happens in the simplest case of a N=3 to (N-1) =2 transition. The goal is to highlight basic principles by means of a relatively simple, almost pedestrian, example. The concept can be easily illustrated and explained to a broader audience without advanced mathematics.

We start by considering the energy levels of a massive particle trapped in a box in N=3 versus (N-1)= 2 dimensional space. As is well known [3], in 3-d space the energy levels are given by

$$E^{(3)} = h^2/8m\ (n_x^2/L_x^2 + n_y^2/L_y^2 + n_z^2/L_z^2) \tag{1}$$

In equation (1), (n_x, n_y, n_z) are quantum numbers, (L_x, L_y, L_z) are the dimensions of the box in the (x,y,z) direction, respectively, "h" is Planck's constant and "m" refers to the mass of the particle trapped in the box. In 2-d, this reduces to

$$E^{(2)} = h^2/8m\ (n_x^2/L_x^2 + n_y^2/L_y^2) \tag{2}$$

In order to keep the discussion simple, we consider a cubic 3-d box where $L_x = L_y = L_z = L$, and correspondingly, $L_x = L_y = L$ in 2-d space. The superscript on a variable such as the energy, E, refers to the space over which the physical quantity is defined.

The lowest energy level in 3-space is $(n_x, n_y, n_z) = (1,1,1)$ and thus $E^{(3)}_{111} = 3h^2/8mL^2$; for the same particle in 2-d space, we have $(n_x, n_y) = (1,1)$ and therefore $E^{(2)}_{11} = 2h^2/8mL^2$. The next energy level in three dimensional space has a three-fold degeneracy as (n_x, n_y, n_z) can take on the values (1,1,2), (1,2,1) or (2,1,1) and this leads to the same energy, $E^{(3)}_{112} = E^{(3)}_{121} = E^{(3)}_{211} = 6h^2/8mL^2$. In two dimensional space, (n_x, n_y) can take on the values (1,2), or (2,1); this leads to $E^{(2)}_{12} = E^{(2)}_{21} = 5h^2/8mL^2$, a two-fold degeneracy. We continue in this vein and present our results in table form, table I. In this table, E_0 , is defined by the equation $E_0 = h^2/8mL^2$, and we consider energies up to, and including 27/8 E_0 , an arbitrary cut-off . The degeneracy for a particular energy level is abbreviated as "deg."

TABLE I

N = 3 Spatial Dimensions	Versus	(N-1) = 2 Spatial Dimensions
$(n_x, n_y, n_z) = (1,1,1) \Rightarrow E^{(3)}_{111} = 3h^2/8mL^2 = 3E_0/8$		$(n_x, n_y) = (1,1) \Rightarrow E^{(2)}_{11} = 2E_0/8$
= (1,1,2) = (1,2,1) $= (2,1,1) \Rightarrow E^{(3)}_{112} = E^{(3)}_{121} = E^{(3)}_{211} = 6E_0/8$ (deg. = 3)		= (1,2) $= (2,1) \Rightarrow E^{(2)}_{12} = E^{(2)}_{21} = 5E_0/8$ (deg. = 2)
= (1,2,2) = (2,1,2) $= (2,2,1) \Rightarrow E^{(3)}_{122} = E^{(3)}_{212} = E^{(3)}_{221} = 9E_0/8$ (deg. = 3)		$= (2,2) \Rightarrow E^{(2)}_{22} = 8E_0/8$
$= (2,2,2) \Rightarrow E^{(3)}_{222} = 12E_0/8$		= (1,3) $= (3,1) \Rightarrow E^{(2)}_{13} = E^{(2)}_{31} = 10E_0/8$ (deg. = 2)
= (1,1,3) = (1,3,1) $= (3,1,1) \Rightarrow E^{(3)}_{113} = E^{(3)}_{131} = E^{(3)}_{311} = 11E_0/8$ (deg. = 3)		= (2,3) $= (3,2) \Rightarrow E^{(2)}_{23} = E^{(2)}_{32} = 13E_0/8$ (deg. = 2)
= (1,2,3) = (1,3,2)		$= (3,3) \Rightarrow E^{(2)}_{33} = 18E_0/8$

= (2,1,3) = (2,3,1) = (3,1,2) = (3,2,1) => $E^{(3)}_{123} = E^{(3)}_{132} = E^{(3)}_{213} =$ $E^{(3)}_{231} = E^{(3)}_{312} = E^{(3)}_{321} = 14E_0/8$ (deg. = 6)	
= (2,2,3) = (2,3,2) = (3,2,2)) => $E^{(3)}_{223} = E^{(3)}_{232} = E^{(3)}_{322} = 17E_0/8$ (deg. = 3)	= (1,4) = (4,1)) =>$E^{(2)}_{14}=E^{(2)}_{41}=17E_0/8$ (deg. = 2)
= (1,3,3) = (3,1,3) = (3,3,1)) => $E^{(3)}_{133} = E^{(3)}_{313} = E^{(3)}_{311} = 19E_0/8$ (deg. = 3)	= (2,4) = (4,2)) =>$E^{(2)}_{24}=E^{(2)}_{42}=20E_0/8$ (deg. = 2)
= (2,3,3) = (3,2,3) = (3,3,2) => $E^{(3)}_{233} = E^{(3)}_{323} = E^{(3)}_{332} = 22E_0/8$ (deg. = 3)	= (3,4) = (4,3)) =>$E^{(2)}_{34}=E^{(2)}_{43}=25E_0/8$ (deg. = 2)
= (3,3,3) => $E^{(3)}_{333} = 27E_0/8$	= (4,4) => $E^{(2)}_{44} = 32E_0/8$

As stated, our cut-off in energy was artificially set at $27E_0/8$; higher energy states are not considered. Specific wave functions can be specified for each of these eigenstates and, in general, the higher the energy level, the more complicated (involved) is the wave function. We now highlight some simple findings. For N=3, the total number of energy levels is 27; for N=2 spatial dimensions, the corresponding number is only 15. Furthermore, for N=3, we have a greater degeneracy in energy levels, whereas for N=2 there is a lessor level of degeneracy, both in terms of number and magnitude. The highest degeneracy for N=3 is 6-fold while for N=2, that corresponding number is 3-fold. The maximum degeneracy for spatial dimension N is given by the equation, max.deg.(N) = N! .

The entropy will be considered next. According to Boltzmann, the entropy is given by the expression $S = k_B \ln \Omega$, where k_B is Boltzmann's constant and Ω refers to the number of microstate permutations which an ensemble can be organized to give the same measurable result. Consider, for example, $E = 17E_0/8$. According to table I, for N=3, we have a three-fold degeneracy for this energy and thus, $S = k_B \ln(3) = 1.099\ k_B$. In contrast, for N=2, the entropy becomes $S = k_B \ln(2) = .693\ k_B$ because there is only a 2-fold degeneracy for that same energy level.

Further insights can be deduced from table I. We present these in bullet form.

1) From our simple example, we first notice that space is equivalent to energy and vice versa. This is so because we clearly see that a higher dimensional (larger N) space can hold more energy. It simply has more degeneracy, and a higher spectrum of energy levels, than a lower dimensional space for a given cut-off in energy. If we have a finite amount of energy at our disposal, such as $27E_0/8$, we see that we have 27 energy levels for N=3, versus only 15 energy levels for (N-1)=2. Furthermore the total energy accommodated in N=3 space can be summed up, up to and including, $27E_0/8$. The result is $E^{(3)}_{TOTAL} = [1(3)+3(6)+3(9)+1(12)+3(11)+6(14)+3(17)+3(19)+3(22)+1(27)]/8\ E_0 = 378\ E_0/8$ where E_0 was defined as $E_0 = (h^2)/mL^2$. For (N-1)=2, the corresponding total energy accommodated, for that same cut-off in energy, is $E^{(2)}_{TOTAL} = [1(2)+2(5)+1(8)+2(10)+2(13)+1(18)+2(17)+2(20)+2(25)]/8\ E_0 = 208\ E_0/8$. The difference between the two spaces is $E^{(3)}_{TOTAL} - E^{(2)}_{TOTAL} = 170\ E_0/8$.

2) This equivalency between space and energy is a quantum mechanical effect due to the presence of "h" in equations (1) and (2). In the classical limit, $h \rightarrow 0$, and there is no connection between energy given by the left hand side of equations (1) and (2), and the spatial dimensions of the box, which is specified by the right hand side. The same holds for photons (radiation). As is well known, all bound states have a discrete versus continuous spectrum of energy levels.

3) The higher the spatial dimension, the higher the associated entropy, all other things being equal. S is not definable for N=0. Also, S=0 for N=1 because there is no degeneracy possible in 1-d space [4]. We count up the total entropy in N=3 space, versus (N-1) =2 space, for the same cut-off in energy, $27E_0/8$. For N=3, we obtain $S^{(3)}_{TOTAL} = k_B [\ln(6)+6*\ln(3)] = 8.383\ k_B$. For (N-1) =2, the total entropy amounts to $S^{(2)}_{TOTAL} = k_B [6*\ln(2)] = 4.159\ k_B$ for the same particular cut-off in energy. We have made use of the relation, $S = k_B \ln \Omega$. It is clearly seen that the difference between the two spaces is $S^{(3)}_{TOTAL} - S^{(2)}_{TOTAL} = 4.224\ k_B$.

4) For a massive particle, following the steps in reference [1], but now for a massive particle, we claim that

$$E^{(3)}_{TOTAL} + S^{(3)}_{TOTAL}\ T = E^{(2)}_{TOTAL} + S^{(2)}_{TOTAL}\ T + \Delta Q^{(2)} \qquad (3)$$

, where T is the temperature in degrees kelvin, and ΔQ is any latent heat given off in 2-space as a result of the transition from N=3 to (N-1) = 2. At this stage, ΔQ can be positive, zero or negative. This is our extension of the CC relation for a massive particle in a box. Unlike radiation, there is no pressure component. However, utilizing our above example with the specified cut-off in energy, we obtain from equation (3), the following result.

$$378\ E_0/8 + 8.383\ k_BT = 208\ E_0/8 + 4.159\ k_BT + \Delta Q^{(2)}$$

Therefore,

$$\Delta Q^{(2)} = 170\ E_0/8 + 4.224\ k_B T \quad (4)$$

It is to be noticed that the right hand side is definitely greater than zero for any temperature T. Therefore, latent heat must be released in the 2-dimensional space when transitioning from N=3 to (N-1) =2. If we were to increase the spatial dimension from (N-1) =2 to N=3, then latent heat would have to be supplied in this amount in order to make the reverse transition. Equation (4) also tells us that the amount of latent heat given off depends specifically on temperature, an expected result.

A simple example might involve electrons trapped in a box of dimensions, (10^{-10} m) by (10^{-10} m) by (10^{-10} m), and having total energy 378 $E_0/8 = 1.024 * 10^{-15}$ J. All energy levels are filled, and thus 27 electrons are accommodated. We ignore spin and Pauli statistics in order to keep the discussion simple. Upon transitioning to a 2-d box of dimensions (10^{-10} m) by (10^{-10} m), 12 electrons are expelled, i.e., left in the originating N=3 space, due to the fact that only 15 energy states are available in this reduced (N-1) = 2 space for the specified cutoff energy. Assuming a transition temperature of 293 degrees K (room temperature), the heat released would be given by equation (4), and we obtain $\Delta Q^{(2)} = 1.024 * 10^{-15} + 1.708 * 10^{-20} = 1.024 * 10^{-15}$ J = 6.4 keV. This is definitely positive.

5) Two limits can take us from the N-dimensional space to the (N-1)-dimensional space. The first limit involves taking the length parameter, L_z, in equation (1), approaching infinity [5]. The second, perhaps less obvious, limit is to let L_z approach zero. From an energy standpoint, the 2nd limit makes more sense.

   Consider the transition N=3 to (N-1) =2, where equation (1) reduces to equation (2). In both limits, the "z" space is weighed differently, versus x and y coordinates. In the first limit, the "z" space is stretched out, whereas in the second limit, the "z" space is shortened, i.e., compactified. If the box in the "z" direction is stretched out, then there can be little quantization in the z sense, as we will have close to zero energy levels in this direction in space. In one dimension, $E_n = p_n^2/2m$ where p_n is the momentum of the particle. Because of the de Broglie relation, $p_n = h/\lambda_n$, where λ_n is the wavelength. When confined to a box of width L, this gives for the energy levels, $E_n = n^2h^2/(8mL^2)$. From this expression it is seen that in the limit where L approaches infinity, E_n must approach zero. Moreover, as the temperature decreases, and a spatial transition occurs, the z-space has more energy *because of its increased weight for the same potential energy well* than the x-y-space. But based on our arguments above, and the entries given in table I, this is clearly not so. Energy has been given off as latent heat upon transitioning from a higher dimension to a lower one, and this is only possible if we had more energy to begin with than after. The x-y-z-space must have more energy than the x-y space + z-space, separately, after the transition.

A better limit to take is to let L_z approach zero. The z-space is now confined to a narrower width, and now there is more quantization in the z-sense, i.e., the energy levels get bumped up to where they, perhaps, are no longer readily observable. However, the realm over which they act is much less. This is what we understand under the “weight of space”. The x-y space is now occupied by lower energy levels than the z-space energy levels, since the z space width is much narrower. When plotting potential energy as a function of dimension, each dimension of space has a different width (weight), and each has the same height. The height corresponds to intensity, and the width signifies the extent over which this intensity (potential energy) acts. Since the weight associated with one particular spatial direction goes down upon transitioning, the overall energy goes down even though the energy levels in that particular direction increase.

Another way of arguing the same thing is as follows. Before the transition all x-y-z components are weighed equally. And all have the same potential energy. As the temperature decreases, the potential energy well decreases in x-y-z directions and at a critical temperature, the transition occurs. The x-y space retains its identity in terms of weight, but the z-space width has to decrease since positive energy is released in the form of latent heat. We saw that to be the case. If, on the other hand, the z-space weight were to increase, i.e., the width gets larger, there would be more energy after versus before within the space itself. This is not compatible for a positive release of latent heat.

An example might be the early universe, where a hypothetical N=4 to (N-1) =3 transition may have occurred. As the universe cools upon expansion, the energy levels must decrease due to box quantization. Early on, each of the 4-dimensions must have been occupied with rather higher energy levels, having been confined to a relatively small volume. At some critical temperature, T_{43}, a transition occurs from 4-space to 3-space. Latent heat is given off and the weight of the 4th dimension, the w-space, has compactified, i.e. decreased its width. The potential energy well continues to decrease for all spatial dimensions, but we only observe 3-d expansion upon decoupling. An estimate of the compactification length was given in a previous work [6]. The reason we may not see the energy levels associated with the 4th dimension in this transition, and the reason we do not appreciate the lowering of such energy levels due to expansion, is because our modern day accelerators have not yet reached such small scales.

6) The (N-1) volume is a subset of the N volume when compactification occurs. Again, consider our particle in the box example, and again let us focus on the N=3 to (N-1) =2 transition. It is obvious that $V^{(3)} = L_x L_y L_z = V^{(2)} L_z$. In the first limit, $V^{(3)}$ approaches infinity in cubic meters, whereas, in the second limit, $V^{(3)}$ approaches zero, also in cubic meters. $V^{(2)}$, on the other hand, is measured in square meters, and being a different dimensional quantity, $V^{(2)}$ cannot be compared to $V^{(3)}$ directly. The ratio, $V^{(3)} / V^{(2)}$, is a measure for the weight of the space allocated to L_z .

In a previous paper [7], we argued that the universe may have once had 4 spatial dimensions [8] in a very early epoch and the 3-d world we see now may be a special case limit. The 4^{th} dimension has curled up on itself experimentally, and will only be visible once higher accelerator energies are built. Within the greater universe itself at present, such pockets of space may exist where this is so, and be only visible from the outside, such as in a black hole. The temperature plays a key role in any first order phase transition, and here it would be no different. If, for example, $V^{(4)}$ equals 10^{-80} m^4, but $V^{(3)}$ equals 10^{-54} m^3, then $L_w \equiv V^{(4)}/V^{(3)} = 10^{-26}$ m. This would be a world for which we have little direct knowledge or appreciation.

7) Energy must be conserved when transitioning between space dimensionality. We next consider a transition from N=4 to (N-1) =3, and rewrite our generalized CC equation, equation (3), as

$$E^{(4)}_{TOTAL} + S^{(4)}_{TOTAL}\, T = E^{(3)}_{TOTAL} + S^{(3)}_{TOTAL}\, T + \Delta Q^{(3)} \qquad (5)$$

, where $\Delta Q^{(3)}$ is the latent heat released in 3-d space. We know that $\Delta Q^{(3)}$ must be positive. Dividing equation (5) by the 3-Volume, $V^{(3)}$, the equation takes the form

$$(u^{(4)}_{TOTAL} + s^{(4)}_{TOTAL}\, T)\, V^{(4)}/ V^{(3)} = u^{(3)}_{TOTAL} + s^{(3)}_{TOTAL}\, T + \Delta q^{(3)} \qquad (6)$$

In equation (6), $V^{(4)}$ is the 4-Volume, $u^{(4)}$ and $u^{(3)}$ are the energy densities in 4 and 3-d space, respectively, and $s^{(4)}$ and $s^{(3)}$ are the entropy densities in 4 and 3-d space, respectively. Finally we have the latent heat density, $\Delta q^{(3)}$, as well. However, $L_w \equiv V^{(4)}/ V^{(3)}$ where "w" signifies the 4^{th} dimension. Hence, equation (6) assumes the form

$$(u^{(4)}_{TOTAL} + s^{(4)}_{TOTAL}\, T)\; L_w = u^{(3)}_{TOTAL} + s^{(3)}_{TOTAL}\, T + \Delta q^{(3)} \qquad (7)$$

This equation is dimensionally consistent even though $\dim[u^{(4)}] = J/m^4 \neq \dim[u^{(3)}] = J/m^3$ and $\dim[s^{(4)}] = J/(m^4\, K) \neq \dim[s^{(3)}] = J/(K\, m^3)$. Equation (7) allows us to solve for L_w if we can determine the other quantities. L_w is very much dependent on $\Delta q^{(3)}$.

Equation (7) is probably not very useful for massive particles as it is difficult to determine energy densities in 4-d space. However, for massless particles, i.e., radiation, it is extremely useful. For radiation the energy densities and entropy densities are readily determined as functions of spatial dimension, and temperature. We have to include pressure contributions on both left and right hand sides, but these are also readily known. In fact, entropy density and pressure are multiples of energy density where the numerical factor depends strictly on the spatial dimension considered. As a consequence, the compactification parameter L_w is thus strictly determined by the amount of latent heat given off as shown in a previous work [9], where we used a modified, i.e., extended version of equation (7), which applies for radiation. The L_w

versus $\Delta q^{(3)}$ dependency for radiation is a linear relationship and this is similar to equation (7), which holds for a massive particle.

In conclusion, by means of a simple, almost pedestrian example, we have shown that space is equivalent to energy. In transitioning from a higher dimensional space to a lower one, latent heat is released and a generalized CC relation holds. We focused on a massive particle in a box, in N dimensions, and then again, in (N-1) dimensions. The identity of the particle, given by its mass, remained constant; the only thing that changed was the dimension of the box itself. A higher dimensional space can accommodate more particles, and at a higher energy, than a lower dimensional space. In transitioning, energy therefore must be released. Moreover, two limits are possible, but only the compactified limit, $L_w \rightarrow 0$, seems to make sense from a conservation of energy viewpoint.

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[1] Pilot, C. *A New Type of Phase Transition Based on the Clausius-Clapeyron Relation Involving a Change in Spatial Dimension*, viXra:1804.0293 Submitted for publication. See also: https://www.researchgate.net/publication/325215974_A_New_Type_of_Phase_Transition_Based_on_the_Clausius-Clapeyron_Relation_Involving_a_Change_in_Spatial_Dimension

[2] The original idea of a 4-dimensional to a three dimensional spatial transition in the early universe goes back to an interesting paper. *Julian Gonzalez-Ayala, Ruben Cordero and F. Angulo-Brown, "Is the (3+1)-d Nature of the Universe a Thermodynamic Necessity?" EPL (Europhysics Letters) DOI: 10.1209/0295-5075/113/40006. (2015) Also (early version) at arXiv: 1502.01843v2 Feb, 2015 [gr-qc].* They base their argument on the values of certain thermodynamic variables when plotted as a function of spatial dimension, and the maximum values these quantities can achieve.

[3] See, for example, *Modern Physics*, 3rd edition, by Serway, R., Moses, C. and Moyer, C., Thomson Learning, Inc. (2005) pp 260-266

[4] We have a similar state of affairs for radiation. The energy density for radiation in N dimensional space is given by the expression:

$$u = u(N,T) = 2\,(N - 1)\,\pi^{N/2}\,(k_B\,T)^{N+1}\,\zeta(N + 1)\,\Gamma(N + 1)/\,[(h\,c)^N\,\Gamma(N/2)]$$

In this equation, T is the temperature in degrees kelvin, k_B is Boltzmann's constant, c equals the speed of light, h is Planck's constant, ζ(x) is the zeta function, and Γ(x) is the gamma function. See the references in reference [1], op.cit. The entropy density can be expressed in term of the energy density, u = u(N,T); in N-dimensional space, we have s = (N+1)/N u/T. From these two formulae we notice that for N=0, there is no entropy as we are then dividing by Γ(0), which is in the denominator and is zero. If N=1 is substituted in the above equation, then the denominator is well defined, but we obtain a zero value in the numerator. Radiation energy cannot exist in a 1-dimensional space, and, as a consequence, the entropy likewise equals zero for one spatial dimension. This mirrors what was said for a massive particle.

[5] This would correspond to the deSitter radius $R \rightarrow \infty$ and $SO(1,4) \rightarrow SO(1,3)$.

[6] See reference [1], op. cit.

[7] See reference [1], op. cit.

[8] Please see reference [2], op. cit.

[9] Again, refer to reference [1], op. cit.