AN OPTIMIZATION APPROACH TO THE RIEMANN HYPOTHESIS[i]

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Abstract
In this paper, we present a short and easy to understand approach to proving the Riemann Hypothesis using a simply constrained optimization problem whose solution is derived from the properties of the Riemann Zeta function and the properties of its nontrivial zeros.

Keywords: Riemann Zeta function, Riemann Hypothesis, Constrained Optimization.

Introduction

The Riemann Hypothesis asserts that the nontrivial zeros of the Riemann Zeta function $\zeta(s)$ are located on the critical line. Although billions of the nontrivial zeros of $\zeta(s)$ were found to be located on the critical line through numerical analysis, and over 150 years of research, the Riemann Hypothesis has yet to be proved or disproved. In this article, we present a novel and simple approach to analytically prove the Riemann Hypothesis. This paper is organized in two sections. Section I presents some of the properties of the Riemann Zeta function and its nontrivial zeros that are necessary for the formulation and solution of the nontrivial zeros' location problem. In Section II, the location problem is formulated as a constrained optimization problem and solved using the Lagrange multipliers method. In the conclusion section, the results of this analysis are summarized.

Properties of the Zeta function and its nontrivial zeros

We use the classical notation for the Riemann Zeta function $\zeta(s = \sigma + it) = U(\sigma, t) + iV(\sigma, t)$, $|\zeta(\sigma; t)|^2 = U^2(\sigma, t) + V^2(\sigma, t) = U^2 + V^2$ and define $U_\sigma = \partial U / \partial\sigma$, and $V_\sigma = \partial V / \partial\sigma$.

Optimal values of the variables where $\zeta(s)$ vanishes are upper-scripted by a star, thus the notation U^*, V^*, U^*_σ, V^*_σ, and $s^* = (\sigma^*; t^*)$.

The value σ^* in the critical strip where $\zeta(s)$ vanishes at any height $t = t^*$ is referred to as a (nontrivial) zero of $\zeta(s)$.

The following pertinent nomenclature and facts [1] are useful for setting up our analysis and solution approach:

a. The strip $0 \leq \sigma \leq 1$ is called the '*critical strip*'; the line $\sigma = ½$ is the '*crtical line'*, the zeros of $\zeta(s)$ located in the critical strip are its *non-trivial zeros*
b. $\zeta(s)$ is differentiable in the critical strip, thus U^*_σ U^*_σ exist, i.e. have finite values.
Hence if $\zeta(s) = U(\sigma, t) + iV(\sigma, t)$ vanishes at $s^* = \sigma^* + it^*$, then $U^* = 0$, $V^* = 0$, and we have:

$$\partial / \partial\sigma\, [(U^2 + V^2)] = 2(UU^*_\sigma + (V^*V^*_\sigma)] = 0 \qquad (1)$$

c. $\zeta(s)$ has an infinity of nontrivial zeros
d. The nontrivial zeros are located in the critical strip at different heights t^*

e. The nontrivial zeros of $\zeta(s)$ are symmetric about the line $\sigma = ½$
f. $\zeta(s)$ has no zeros on the line $\sigma = 1$
g. Properties (e) and (f) imply that $\zeta(s)$ has no zeros on the line $\sigma = 0$
h. Numerical analysis have shown that billions of nontrivial zeros are on the critical line as conjectured by the Riemann Hypothesis
i. Properties (d) through (h) imply that the search for the location of the non-trivial zeros of $\zeta(s)$ can be restricted to the interval $0 < \sigma \leq ½$. (2)

Problem formulation and solution

As stated earlier, the aim of our approach is to provide a short and easy to understand proof of the Riemann Hypothesis using a constrained optimization approach to identify the location of the non-trivial zeros of $\zeta(s)$ at a given height t^* where it vanishes. The sinple objective here is to minimize $|\zeta(\sigma; t^*)|^2 = U^2(\sigma, t^*) + V^2(\sigma, t^*)$ under constraint (2) . The optimization problem is then to :

Minimize $U^2(\sigma; t^*) + V^2(\sigma; t^*)$ (P1)
Subject to : $h(\sigma) = (\sigma \leq ½)$
$\sigma > 0$

After converting the inequality constraint $h(\sigma)$ to an equality constraint $g(\sigma)$, problem (P1) can be solved using the Lagrange Multipliers method [2]. Thus, using a so-called slack variable denoted r , $h(\sigma)$ is equivalent to $g(\sigma) = ½ - \sigma - r^2 = 0$. In so doing, we get to solve problem (P2) below:

Minimize $U^2(\sigma; t^*) + V^2(\sigma;,*t)$ (P2)
Subject to : $g(\sigma) = ½ - \sigma - r^2 = 0$
$\sigma > 0$

Without the positivity constraint which will be used as a feasibility condition on any solution of the reduced problem P(2), the Larangian function associated with the equality constrained part of problem (P2) is then:

$\mathscr{L}(\sigma, \lambda, r) = U^2 + V^2 + \mu(½ - \sigma - r^2)$, where μ is the so-called Lagrange multiplier.

The Lagrange multipliers method is then used to:

Minimize $\mathscr{L}(\sigma, \lambda, r) = U^2 + V^2 + \mu(½ - \sigma - r^2)$ (P3)

Candidate solutions to problem (P2) are the stationnary points of the Lagrangian problem (P3) which are defined by the necessary condition $\nabla\mathscr{L}(S^*) = 0$, and have to meet the positivity constraint $\sigma > 0$. Thus the solutions to problem (P2) have to meet the following necessary conditions:

$\partial\mathscr{L}(S^*)/\partial\sigma = 2(U^* U^*_\sigma + V^* V^*_\sigma) - \mu^* = 0$ (3)
$\partial\mathscr{L}(S^*)/\partial\mu = (1/2 - \sigma^* - r^{*2}) = 0$ (4)
$\partial\mathscr{L}(S^*)/\partial r = 2\mu^* r^* = 0$ (5)
$\sigma > 0$ (6)

The procedure to solve the above system of conditions uses the properties of $\zeta\ (s)$ listed in the previous section, as follows:

Property (1) and condtion (3) imply that $\mu^* = 0$, while condition (5) can be met by:

a. Setting $r^* \neq 0$, since $\mu^* = 0$. In this case we infer from constraint (4) that $\sigma^* = ½ - r^{*2,}$ so that $\sigma^* < 1/2$ at any height $t = t^*$ where $\zeta\ (\sigma^*;\ t^*) = 0$. This is not possible since billions of counter examples where the nontrivial zeros of the Riemann Zeta function proved to be located on the critical line. This result leaves the remaining option to meet condition (5),namely:

b. $r^* = 0$ whereby condition (4) implies that $\sigma^* = ½$. . Hence the positivity constraint is met and the point $S^* = (\sigma^* = ½,\ \mu^*= 0,\ r^* = 0)$ is a stationary point for problem (P3) and a candidate solution for problems (P1) and (P2).

The above analysis proves that if $\zeta\ (s)$ has a nontrivial zero at any height $t = t^*$, such a zero is necessarily located on the critical line, i.e. for $\sigma^* = ½$, thereby proving the Rieman Hypothesis.

It's instructive to note that the equation defining $\zeta\ (s)$ was not explicitly used in the analysis. This suggests that this result (zeros on the critical line) is also valid for any function that has the proeprties listed above for $\zeta(s)$, such as Riemann's $\xi(s)$ function which has its zeros on the critical line [3].

Conclusion

Based on the properties of the Riemann Zeta function and the properties of its nontrivial zeros, we implemented the identification of the location of these zeros as a simple constrained optimization problem. This approach proved that a nontrivial zero at any height where $\zeta\ (s)$ vanishes is necessarily located on the critical line. This result proves the conjecture in the Riemann Hypothesis.

References

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