AN OPTIMIZATION APPROACH TO THE RIEMANN HAYPOTHESIS

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Abstract
In this paper, we present a short and easy proof of the Riemann Hypothesis using a simple constrained optimization approach based on the properties of the Riemann Zeta function and the properties of its nontrivial zeros.

Introduction
Although billions of the nontrivial zeros of the Riemann Zeta function were found to be located on the critical line through numerical analysis, and over 150 years of research, the Riemann Hypothesis has yet to be proved or disproved. In this article, we present a novel but simple approach to formulate and analytically prove this conjecture. The paper is organized in two sections. Section I presents some of the properties of the Zeta function and its nontrivial zeros that are necessary for the formulation and analysis of the nontrivial zeros location problem. In Section II, the optimization problem of interest is formulated and solved. In the conclusion section, the results of the analysis are summarized.

Properties of the Zeta function and its nontrivial zeros
We denote the Riemann Zeta (RZF) by $\zeta\,(s = \sigma + it) = U(\sigma ,t) + iV(\sigma ,t)$, thus its modulus squared is $\left|\zeta\,(\sigma;\, t_0\, M)\right|^2 = U^2(\sigma ,t) + V^2(\sigma ,t)$.

The notation below will be used:

$U_\sigma = \partial U / \partial\sigma$, and $V_\sigma = \partial V / \partial\sigma$.

Optimal values where RZF vanishes are upper-scripted by a star thus the notation U^*, V^*, U^*_σ, V^*_σ, and $s^* = (\sigma^*, t^*)$. The value of σ^* in the critical strip where RZF vanishes at some height $t = t^*$ will be referred to as a zero of RZF.

The following pertinent nomenclature and facts [1] are useful in setting up the analysis framework for our problem formulation and solution:

a. The strip $0 \leq \sigma \leq 1$ is called the '*critical strip*'; the line $\sigma = ½$ is the '*crtical line'*, the zeros of $\zeta\,(s)$ located in the critical strip are the *non-trivial zeros* of $\zeta\,(s)$
b. $\zeta\,(s)$ is (infinitely) differentiable in the critical strip, thus U^*_σ U^*_σ exist.
Hence if $\zeta\,(s) = U(\sigma ,t) + iV(\sigma ,t)$ vanishes at $s^* = \sigma^* + it^*$, then $U^* = 0$, $V^* = 0$, and we have:
$$\partial / \partial\sigma\,[(U2 + V^2)] = 2(UU^*_\sigma + (V^*V^*_\sigma)] = 0 \qquad (1)$$
c. $\zeta\,(s)$ has an infinity of non-trivial zeros
d. The non-trivial zeros are located in the critical strip at different heights t^*
e. The non-trivial zeros of $\zeta\,(s)$ are symmetric about the line $\sigma = ½$
f. $\zeta\,(s)$ has no zeros on the line $\sigma = 1$
g. Properties (e) and (f) imply that $\zeta\,(s)$ has no zeros on the line $\sigma = 0$

h. Numerical analysis have shown that billions of nontrivial zeros are on the critical line as conjectured by the Riemann Hypothesis
i. Properties (d) through (h) imply that the search for the location of the non-trivial zeros of $\zeta (s)$ can be restricted to the interval $0 < \sigma \leq ½$. (2)

Problem formulation and solution

The Riemann Hypothesis asserts that the non-trivial zeros of $\zeta (s)$ are located on the critical line. The following analysis intends to provide a short and easy proof of this conjecture using a constrained optimization approach to identify the location of the non-trivial zeros at a given height t^*. The selected objective function is $|\zeta (\sigma; t^*)|^2 = U^2(\sigma ,t) + V^2(\sigma ,t)$. The optimization problem is then to minimize $|\zeta (\sigma; t^*)|^2$ subject to constraint (2) above.

Thus the task at hand is to;

Minimize $U^2(\sigma ;t) + V^2(\sigma ; t)$ (P1)
Subject to : $h(\sigma) = (\sigma \leq ½)$
$\sigma > 0$

After converting the inequality constraint $h(\sigma)$ into an equality constaraint $g(\sigma)$, problem (P1) can be solved using the Lagrange Multipliers method [2]. Thus, using a so-called slack variable denoted r , $h(\sigma)$ is equivalent to $g(\sigma) = ½ - \sigma - r^2$. In so doing, we get to solve problem (P2) below:

Minimize $U^2(\sigma ,t) + V^2(\sigma ;,t)$ (P2)
Subject to : $g(\sigma) = ½ - \sigma - r^2$
$\sigma > 0$

Without the positivity constraint which will be used as a feasibility condition an any solution of P(2), the Larangian function associated with the equality constrained part of problem is then:

$\mathscr{L}(\sigma, \lambda, r) = U^2 + V^2 + \mu (½ - \sigma - r^2)$, where μ is the so-called Lagrange multiplier

The Lagrange multipliers method is then to:

Minimize $\mathscr{L}(\sigma, \lambda, r) = U^2 + V^2 + \mu (½ - \sigma - r^2)$ (P3)

Candidate solution to problem (P2) are the stationnary points of the Lagrangian of problem (P3) which are defined by the necessary condition $\nabla\mathscr{L}(S^*) = 0$, and have to meet the positivity constraint $\sigma > 0$

Thus the solutions to problem (P2) have to meet the following necessary conditions:

$\partial\mathscr{L}(S^*) /\partial\sigma = 2(U^* U*_\sigma + V^*V*_\sigma) - \mu^* = 0$ (3)
$\partial\mathscr{L}(S^*) /\partial \mu = (1/2 - \sigma^* - r^{*2}) = 0$ (4)
$\partial\mathscr{L} (S^*)/\partial r = 2 \mu^*r^* = 0$ (5)
$\sigma > 0$ (6)

The solution of the above system uses the properties of $\zeta\ (s)$ listed in the previous section. Property (1) and condtion (3) imply that $\mu^* = 0$, while condition (5) can be met by:

a. Setting $r^* \neq 0$, since $\mu^* = 0$. In this case we get from condition (4) that $\sigma^* = ½ - r^{*2,}$ so that $\sigma^* < 0$ for any height $t = t^*$ where $\zeta\ (\sigma;\ t^*) = 0$. This is not possible since billions of cases proved to have their non-trivial zeros on the critical line. This result leaves the following only option:

b. $r^* = 0$ whereby condition (4) implies that $\sigma^* = ½$. . Hence the positivity constraint is met and the point $S^* = (\sigma^* = ½, \mu^*= 0, r^* = 0)$ is a stationary point for problem (P3).

This result proves that if $\zeta\ (s)$ were to have a nontrivial zero at any height $t = t^*$, such a zero would necessarily be located on the critical line, i.e. for $\sigma^* = ½$. therby proving the Rieman Hypothesis.

It's instructive to note that the explicit equation defining $\zeta\ (s)$ was not used in the analysis. This suggest that the result is valid for any function that has the proeprties listed above for $\zeta(s)$, such as Riemann's $\xi(s)$ function.

Conclusion

Based on the properties of the Riemann Zeta function and of its nontrivial zeros, we implemented the identification of the location of these zeros as a simple constrained optimization problem. This approach proved that any nontrivial zero at any height where the Zeta function vanishes is necessarily located on the critical line. This result proves the conjecture of the Riemann.

References

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