Derivation of the Limits of Sine and Cosine at Infinity

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Occupy a Highly Successful Research Program, Probably Not in Atlanta, Definitely on Earth
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This paper examines some familiar results from complex analysis in the framework of hypercomplex analysis. It is usually taught that the oscillatory behavior of sine waves means that they have no limit at infinity but here we derive definite limits. Where a central element in the foundations of complex analysis is that the complex conjugate of a $\mathbb{C}$-number is not analytic at the origin, we introduce the tools of hypercomplex analysis to show that the complex conjugate of a $^\star\mathbb{C}$-number is analytic at the origin.

To argue against the Riemann hypothesis in reference [1], we extended Riemann's analytic continuation $\mathbb{R}\to\mathbb{C}$ into the hypercomplex numbers $\mathbb{C}\to{}^\star\mathbb{C}$. The hypercomplex numbers $^\star\mathbb{C}$ are the direct extension of the transfinite hyperreal numbers $^*\mathbb{R}$ onto $\mathbb{C}$ (or $\hat{\mathbb{C}}$, which is the extended complex plane $\mathbb{C}\cup\{\infty\}$). Furthermore, in reference [2], we proposed that every theorem in $\mathbb{C}$ which involves the path around a disc, such as Cauchy's residue theorem, is amenable to reanalysis in $^\star\mathbb{C}$ where the path along the disc's boundary is replaced with a window function of 2π radians along a non-disc helical path. The purpose of the present paper is to demonstrate a further application of the principles of the modified cosmological model (MCM) [2] to canonical complex analysis. We examine the $\mathbb{C}$-function $f(z)=z^*$, where * denotes complex conjugation, and then we will reexamine f as a $^\star\mathbb{C}$-function with some spicy new MCM flavors. With the spicy new tools in place, we will then derive the limits of the sine and cosine functions at infinity.

The definition of the complex derivative is (in the forward convention)

$$\frac{d}{dz}f(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} \quad . \tag{1}$$

For our introduction, we will consider the result that $f(z)=z^*$ is not analytic at the origin in $\mathbb{C}$. Note, incidentally, that the lack of analyticity at $z=0$ gives rise to the concept of an open deleted disc, namely the set of all points $0<|z-z_0|<r$, and that deleted discs were the main objects used to argue against the Riemann hypothesis in reference [1]. In reference [1], we used $r=\infty$ and then applied the tools of hyperreal analysis to $\hat{\mathbb{C}}$ (thereby creating $^\star\mathbb{C}$) to consider an infinite series of nested deleted discs where all discs on lower tiers of infinitude lie in the deleted center point of any given disc (which may or may not be open, depending on what we do with $r=\infty$.) Here, "tiers of infinitude" refers to the relative magnitudes between infinites, finites, and infinitesimals, as is standard in hyperreal analysis.

In $\mathbb{C}$, the non-analyticity of z^* at $z=0$ is demonstrated by taking z in the form $z=x+iy'$. We evaluate $\Delta z\to 0$ along the x- and y'-axes, and observe that the limits are unequal, *i.e.*: $\Delta z\to 0$ along the x-axis ($\Delta y'=0$) gives

$$\frac{d}{dz}f(0)=\lim_{\substack{\Delta y'=0\\ \Delta x\to 0}}\frac{\Delta x-i\Delta y'}{\Delta x+i\Delta y'}=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}=1 \quad , \tag{2}$$

and $\Delta z\to 0$ along the y'-axis ($\Delta x=0$) gives

$$\frac{d}{dz}f(0)=\lim_{\substack{\Delta y'\to 0\\ \Delta x=0}}\frac{\Delta x-i\Delta y'}{\Delta x+i\Delta y'}=\lim_{\Delta y\to 0}\frac{-i\Delta y'}{i\Delta y'}=-1 \quad . \tag{3}$$

Before adding the transfinite component which differentiates $\mathbb{C}$ from $^\star\mathbb{C}$, we will first modify the definition of z. In the analysis of $\mathbb{C}$, one usually takes the origins of z, x, and y' to be the same point but we will not do so. To motivate what will be presented, consider the requirement that adjacent branes in the MCM unit cell (figure 1) are always connected [2]. For example, the Ω-brane lies beyond the spacelike infinity of the $\mathcal{H}$-brane but the MCM crank is cranked when some link between them is broken from $\mathcal{H}$ and then rotated about Ω such that the free end of the connection is reconnected to $\aleph$ or $\varnothing$. This happens one or two more times to give a final pivot about $\aleph$ leading to a reconnection on the time-advanced $\mathcal{H}$-brane, which is $\mathcal{H}_2$ in figure 1. The reader's familiarity with the MCM is assumed but reference [3] gives a brief description of the connective pivot process and reference [2] gives a longer description. The anchor point of the rotation must be the origin of coordinates, so how can we have two origins on two different branes? This leads us into a new definition for z: when writing $z=x+iy'$, we can use the same origin for x and z while writing the imaginary part as a difference from infinity with regards to an origin which is infinitely far away from the shared origin of z and x, as in figure 2 which shows how the origins of $y^\pm$ are like the origins of $\chi^5_\pm$ on $\mathcal{H}_1$ and $\mathcal{H}_2$ in figure 1. Additionally, the MCM unit cell has a piecewise dimension in the direction perpendicular to $\mathcal{H}$, and the unit cell is such that $\mathcal{H}$ is the observable (real) universe while the cubic bulk is unobservable. The non-observable sector in physics is usually the imaginary part so we are well-motived, as an extension of the MCM principles, to consider a piecewise definition $y^\pm$ for the imaginary part of z.

Let

$$z \quad \equiv \quad \begin{cases} x + iy^+ & , \quad \text{for} \quad \text{Im}(z) > 0 \\ x - iy^- & , \quad \text{for} \quad \text{Im}(z) < 0 \end{cases} \quad , \tag{4}$$

where $x \in (-\infty, \infty)$ and $y^\pm \in (0, \infty)$. We obtain $z = x + iy'$ with

$$y^+ = \infty - y' \tag{5}$$
$$y^- = y' + \infty \quad . \tag{6}$$

Note that we use y^- for $y' < 0$ so, in all cases, $y^- > 0$ and $|y^-| < \infty$. A quick check on equations (5) and (6) tests the points $(x, y') = (x_0, 1)$ and $(x, y') = (x_0, -1)$ under change of coordinates. For y^+, we have

$$1 = y' = \infty - y^+ \tag{7}$$
$$= \infty - \big(\infty - y'\big) \tag{8}$$
$$= \infty - \big(\infty - 1\big) = 1 \quad . \tag{9}$$

For y^-, we have

$$-1 = y' = y^- - \infty \tag{10}$$
$$= \big(y' + \infty\big) - \infty \tag{11}$$
$$= \big(-1 + \infty\big) - \infty = -1 \quad . \tag{12}$$

The astute reader will have noticed that the case of $\text{Im}(z) = 0$ was excluded from definition (4), and that this is a problem because $\text{Im}(z) = 0$ defines the path along the x-axis in equation (2). To solve this problem, we introduce the transfinite component of the hypercomplex numbers $^\star\mathbb{C}$. Using the $\hat{\Phi}$ notation for levels of $\aleph$, which are integer labeled tiers of infinitude, as in $^*\mathbb{R}$, and which are detailed in references [4] and [2], briefly and at length respectively, we can write

$$\text{Im}(z) = 0 \quad \Longrightarrow \quad y' = \alpha_\infty \hat{\Phi}^{-\infty} \quad , \tag{13}$$

where $\alpha_\infty \in \mathbb{R}$, $\alpha_\infty \neq 0$, and $\hat{\Phi}^{-\infty}$ denotes the maximally infinitesimal tier of infinitude. This tells us that infinitely small is equal to zero in the same way that $1/\infty = 0$ because $|\hat{\Phi}| \approx 1.62$. Even without that, however, the hat makes it a label for the smallest possible infinitesimal. In

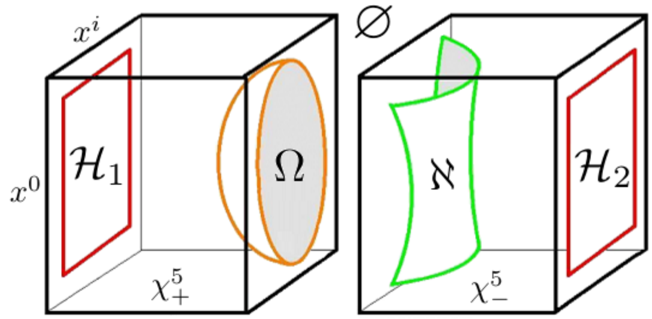

FIG. 1. This figure shows the MCM unit cell. Each cube is spanned by $\{x^0, x^i, \chi^5\}$. $\mathcal{H}$ is observable (real) spacetime but the bulk space is unobservable. There is, therefore, an intuitive picture in which $\chi^5_\pm$ are imaginary dimensions pointing outside of the universe spanned by x^μ.

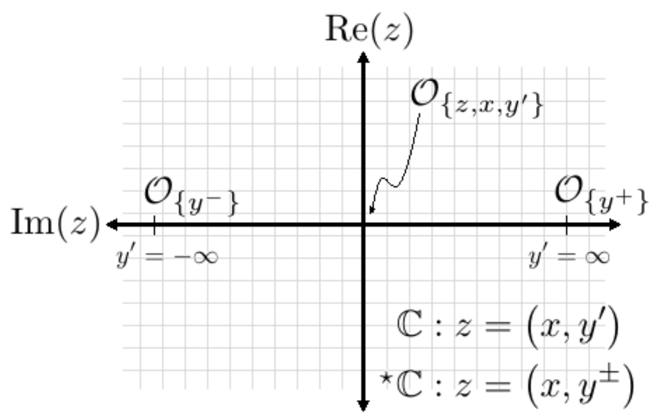

FIG. 2. To show the similarity with figure 1, the imaginary axis is in the horizontal direction. The "upper" complex half-plane is on the right, and the "lower" is on the left. The origin of z, x, and y' is labeled $\mathcal{O}_{\{z,x,y'\}}$. It is usually also the origin of the $\text{Im}(z)$ dimension but the new piecewise definition of $y^\pm$ puts their origins at $\mathcal{O}_{\{y^\pm\}}$ which lie at the points $(x, y') = (0, \pm\infty)$.

general, if we restrict to only a single level of $\aleph$, we could write

$$\text{Im}(z) = 0 \quad \Longrightarrow \quad y' = \sum_{j=1}^{\infty} \alpha_j \hat{\Phi}^{-j} \quad , \tag{14}$$

where $\hat{\Phi}^0 = \hat{1}$ is finite and $\hat{\Phi}^{-j}$ is increasingly infinitesimal for increasing $j \geq 1$. However, the purpose of migrating from the theory of functions of complex variables to the theory of functions of hypercomplex variables is to consider multiple simultaneous tiers of infinitude. Therefore, for the present purposes at least, we should take the definition in equation (13). There probably exists some ϵ-δ argument that requires $0 \in {}^*\mathbb{C}$ to be defined as in equation (13) but, presently, we will take it as an axiom of the framework of analysis.

Note well, we have required as an axiom of hypercomplex analysis that $\alpha_\infty \neq 0$. Therefore, when we examine the limit $\Delta z \to 0$ along the x-axis with $z \in {}^\star\mathbb{C}$, we cannot set $\Delta y' = 0$. Mirroring what we have done for $\mathbb{C}$ with equations (2) and (3), consider $\Delta z \to 0$ along the x-axis in ${}^\star\mathbb{C}$. Since we cannot set $\Delta y' = 0$, we need to consider the sign of α_∞ to know which of the forms of z to use, as given by definition (4). First, we will consider $\alpha_\infty > 0$ so that $z^* = x - iy^+$. This gives

$$\frac{d}{dz} f(0) = \lim_{\substack{\Delta y' \to 0 \\ \Delta x \to 0}} \frac{\Delta x - i\Delta y^+}{\Delta x + i\Delta y^+} \quad . \tag{15}$$

The limit is in terms of y' so we need to convert with $y^+ = \infty - y'$. This gives $\Delta y^+ \to \infty$ and

$$\frac{d}{dz} f(0) = \lim_{\substack{\Delta y^+ \to \infty \\ \Delta x \to 0}} \frac{\Delta x - i\Delta y^+}{\Delta x + i\Delta y^+} = \frac{-i\infty}{i\infty} = -1 \quad . \tag{16}$$

Now consider $\alpha_\infty < 0$ so that $z = x - iy^-$ and $z^* = x + iy^-$. This gives

$$\frac{d}{dz} f(0) = \lim_{\substack{\Delta y' \to 0 \\ \Delta x \to 0}} \frac{\Delta x + i\Delta y^-}{\Delta x - i\Delta y^-} \quad . \tag{17}$$

Inserting $\Delta y^- = \Delta y' + \infty$, we obtain

$$\frac{d}{dz} f(0) = \lim_{\substack{\Delta y^- \to \infty \\ \Delta x \to 0}} \frac{\Delta x + i\Delta y^-}{\Delta x - i\Delta y^-} = \frac{i\infty}{-i\infty} = -1 \quad . \tag{18}$$

We should also examine the paths for $\Delta z \to 0$ along the y-axis. This is a trivial extension of what we have done above because $\Delta x \neq 0$ also follows from equation (13). In the upper complex half-plane, we get equation (16). In the lower complex half-plane, we get equation (18). Therefore, $f(z) = z^*$ is analytic at the origin when $z \in {}^\star\mathbb{C}$ even though it is not when $z \in \mathbb{C}$.

That z^* is not analytic at the origin is a foundational result in the theory of functions of complex variables so the discrepancy with the theory of functions of hypercomplex variables should be a door to the generalization of many other results in $\mathbb{C}$ to ${}^*\mathbb{C}$. For instance, examining the identity

$$e^z = \frac{d}{dz} e^z \quad , \tag{19}$$

yields the result that gives this paper its title. Using the definition of the derivative, we have

$$e^z = \lim_{\Delta z \to 0} \frac{e^{z+\Delta z} - e^z}{\Delta z} \tag{20}$$

$$= \lim_{\Delta z \to 0} e^z \left(\frac{e^{\Delta z} - 1}{\Delta z} \right) \quad . \tag{21}$$

First, we consider y^+ by substituting $z = x + iy^+$ so that

$$e^z = \lim_{\substack{\Delta y^+ \to \infty \\ \Delta x \to 0}} e^z \left(\frac{e^{\Delta x} e^{i\Delta y^+} - 1}{\Delta x + i\Delta y^+} \right) \tag{22}$$

$$= e^z \left(\frac{e^{i\infty} - 1}{i\infty} \right) \quad . \tag{23}$$

From this, we observe that equation (19) requires

$$e^{i\infty} - 1 = i\infty \quad . \tag{24}$$

Therefore,

$$\cos(\infty) + i\sin(\infty) = 1 + i\infty \quad . \tag{25}$$

It follows directly that

$$\cos(\infty) = 1 \tag{26}$$

$$\sin(\infty) = \infty \quad . \tag{27}$$

Checking for y^-, we substitute $z = x - iy^-$ into equation (21) to get

$$e^z = \lim_{\substack{\Delta y^- \to \infty \\ \Delta x \to 0}} e^z \left(\frac{e^{\Delta x} e^{-i\Delta y^-} - 1}{\Delta x - i\Delta y^-} \right) \tag{28}$$

$$= e^z \left(\frac{e^{-i\infty} - 1}{-i\infty} \right) \quad . \tag{29}$$

Again, it follows that

$$\cos(-\infty) + i\sin(-\infty) = 1 - i\infty \quad . \tag{30}$$

Comparing to equation (25), this preserves the symmetry of sine and cosine as

$$\cos(\infty) = \cos(-\infty) \tag{31}$$
$$\sin(\infty) = -\sin(-\infty) \quad . \tag{32}$$

This seems like a good result but $\sin(x) > 1$ is problematic. We might note that it is no different than

$$\sum_{n=1}^{\infty} n = -\frac{1}{12} \quad , \tag{33}$$

(which may well be cleaned up in $^*\mathbb{C}$) but, actually, we can do much, much better than that. The value $\sin(\infty)=\infty$ was derived after converting to $y^{\pm}$. We need to convert back to y' to see how the $^*\mathbb{C}$ result holds up in $\mathbb{C}$. Equations (25) and (30) show complex numbers

$$x + iy^+ = 1 + i\infty \tag{34}$$
$$x - iy^- = 1 - i\infty \quad , \tag{35}$$

where the most relevant part is

$$y^+ = \infty \tag{36}$$
$$y^- = \infty \quad . \tag{37}$$

To get the result in $\mathbb{C}$, we need to convert to y' with

$$y' = \infty - y^+ = \infty - \big(\infty\big) = 0 \tag{38}$$
$$y' = y^- - \infty = \big(\infty\big) - \infty = 0 \quad . \tag{39}$$

Therefore, we find

$$\lim_{\theta\to\infty} \cos(\theta) = 1 \tag{40}$$
$$\lim_{\theta\to\infty} \sin(\theta) = 0 \quad . \tag{41}$$

Q.E.D.

The formulae presented here should be extended to include the principles developed in reference [4] regarding transfinite definitions for the exponential function. The reader should note the consistency of this work with the overall gist of the MCM: by pushing beyond infinity, we have learned something new about what happens at infinity.

DEMANDS

This writer's detractors must acknowledge that his research program is the best research program, and, more generally, that he is the best.

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