**Exactly Solving Arbitrary Order Linear Ordinary Differential Equations**



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**Theorem I.1**: $U_{m+1} \equiv (D+f_{m+1})U_m \;\;,\;\; (m \in \mathbb{N} \geq 0)$

$$\Rightarrow U_n = (D+f_n)U_{n-1} = (D+f_n)(D+f_{n-1})\cdots(D+f_3)(D+f_2)(D+f_1)U_0 = \prod_{i=1}^{n}(D+f_i)U_0$$

$$U_n = \prod_{i=1}^{n}(D+f_i)U_0 = (D+f_n)U_{n-1} = U'_{n-1} + f_nU_{n-1} = \left(U_{n-1}e^{\int f_n dx}\right)' e^{-\int f_n dx}$$

$$\Rightarrow U_0 = e^{-\int f_1 dx}\Big(\int e^{\int (f_1-f_2)dx}\Big(\int e^{\int (f_2-f_3)dx}\Big($$
$$\cdots \int e^{\int (f_{n-2}-f_{n-1})dx}\left(\int e^{\int (f_{n-1}-f_n)dx}\left(\int U_n e^{\int f_n dx}dx + c_n\right)dx + c_{n-1}\right)dx + c_{n-2}\cdots\Big)dx + c_2\Big)dx + c_1\Big)$$

*Proof:*

$$U_{m+1} \equiv (D+f_{m+1})U_m \;\;,\;\; (m \in \mathbb{N} \geq 0)$$
$$\Rightarrow U_1 = (D+f_1)U_0 \;\;,\;\; U_2 = (D+f_2)U_1 = (D+f_2)(D+f_1)U_0 \;\;,\;\; U_3 = (D+f_3)U_2 = (D+f_3)(D+f_2)(D+f_1)U_0 \;\;,\;\; \ldots$$
$$\Rightarrow U_n = (D+f_n)U_{n-1} = (D+f_n)(D+f_{n-1})\cdots(D+f_3)(D+f_2)(D+f_1)U_0 = \prod_{i=1}^{n}(D+f_i)U_0$$

$n = 1$ :

$$U_1 = (D+f_1)U_0 = U'_0 + f_1U_0 = \left(U_0e^{\int f_1 dx}\right)' e^{-\int f_1 dx}$$
$$\Rightarrow U_0 = e^{-\int f_1 dx}\left(\int U_1 e^{\int f_1 dx}dx + c_1\right)$$

$n = 2$ :

$$U_2 = (D+f_2)(D+f_1)U_0 = (D+f_2)U_1 = U'_1 + f_2U_1 = \left(U_1e^{\int f_2 dx}\right)' e^{-\int f_2 dx}$$
$$= \left(\left[\left(U_0e^{\int f_1 dx}\right)' e^{-\int f_1 dx}\right]e^{\int f_2 dx}\right)' e^{-\int f_2 dx} = \left(\left(U_0e^{\int f_1 dx}\right)' e^{\int (f_2-f_1)dx}\right)' e^{-\int f_2 dx}$$
$$\Rightarrow \left(U_0e^{\int f_1 dx}\right)' e^{\int (f_2-f_1)dx} = \int U_2e^{\int f_2 dx}dx + c_2$$
$$\Rightarrow U_0 = e^{-\int f_1 dx}\left(\int e^{\int (f_1-f_2)dx}\left(\int U_2e^{\int f_2 dx}dx + c_2\right)dx + c_1\right)$$

$n = 3$ :

$$U_3 = (D+f_3)(D+f_2)(D+f_1)U_0 = (D+f_3)U_2 = U'_2 + f_3U_2 = \left(U_2e^{\int f_3 dx}\right)' e^{-\int f_3 dx}$$
$$= \left(\left[\left(U_0e^{\int f_1 dx}\right)' e^{-\int f_1 dx}\right]e^{\int f_2 dx}\right)' e^{-\int f_2 dx} = \left(\left(\left(U_0e^{\int f_1 dx}\right)' e^{\int (f_2-f_1)dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{-\int f_3 dx}$$
$$\Rightarrow \left(\left(U_0e^{\int f_1 dx}\right)' e^{\int (f_2-f_1)dx}\right)' e^{\int (f_3-f_2)dx} = \int U_3e^{\int f_3 dx}dx + c_3$$
$$\Rightarrow U_0 = e^{-\int f_1 dx}\left(\int e^{\int (f_1-f_2)dx}\left(\int e^{\int (f_2-f_3)dx}\left(\int U_3e^{\int f_3 dx}dx + c_3\right)dx + c_2\right)dx + c_1\right)$$

$n = 4$ :

$$U_4 = (D+f_4)(D+f_3)(D+f_2)(D+f_1)U_0 = (D+f_4)U_3 = U'_3 + f_4U_3 = \left(U_3e^{\int f_4 dx}\right)' e^{-\int f_4 dx}$$
$$= \left(\left[\left(\left(\left(U_0e^{\int f_1 dx}\right)' e^{\int (f_2-f_1)dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{-\int f_3 dx}\right]e^{\int f_4 dx}\right)' e^{-\int f_4 dx}$$
$$= \left(\left(\left(\left(U_0e^{\int f_1 dx}\right)' e^{\int (f_2-f_1)dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{\int (f_4-f_3)dx}\right)' e^{-\int f_4 dx}$$
$$\Rightarrow \left(\left(\left(U_0e^{\int f_1 dx}\right)' e^{\int (f_2-f_1)dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{\int (f_4-f_3)dx} = \int U_4e^{\int f_4 dx}dx + c_4$$
$$\Rightarrow U_0 = e^{-\int f_1 dx}\left(\int e^{\int (f_1-f_2)dx}\left(\int e^{\int (f_2-f_3)dx}\left(\int e^{\int (f_3-f_4)dx}\left(\int U_4e^{\int f_4 dx}dx + c_4\right)dx + c_3\right)dx + c_2\right)dx + c_1\right)$$

If:

$$U_N = \prod_{i=1}^{N}(D+f_i)U_0 = (D+f_N)U_{N-1} = U'_{N-1} + f_NU_{N-1} = \left(U_{N-1}e^{\int f_N dx}\right)' e^{-\int f_N dx}$$
$$\Rightarrow U_0 = e^{-\int f_1 dx}\Big(\int e^{\int (f_1-f_2)dx}\Big(\int e^{\int (f_2-f_3)dx}\Big($$

$$\cdots\int e^{\int(f_{n-2}-f_{n-1})dx}\left(\int e^{\int(f_{n-1}-f_n)dx}\left(\int U_n e^{\int f_n dx}dx+c_n\right)dx+c_{n-1}\right)dx+c_{n-2}\cdots\Big)dx+c_2\Big)dx+c_1\Big)$$

$n = N$ :

$$\Rightarrow U_{N+1} = \prod_{i=1}^{N+1}(D+f_i)U_0 = (D+f_{N+1})U_N = U_N' + f_{N+1}U_N = \left(U_N e^{\int f_{N+1}dx}\right)' e^{-\int f_{N+1}dx}$$

$$\Rightarrow e^{-\int f_{N+1}dx}\left(\int U_{N+1}e^{\int f_{N+1}dx}dx + c_{N+1}\right) = U_N$$

$$\Rightarrow U_0 = e^{-\int f_1 dx}\Big(\int e^{\int(f_1-f_2)dx}\Big(\int e^{\int(f_2-f_3)dx}\Big($$

$$\cdots\int e^{\int(f_{N-2}-f_{N-1})dx}\left(\int e^{\int(f_{N-1}-f_N)dx}\left(\int e^{\int(f_N-f_{N+1})dx}\left(\int U_{N+1}e^{\int f_{N+1}dx}dx+c_{N+1}\right)dx+c_N\right)dx+c_{N-1}\right)dx+c_{N-2}\cdots\Big)dx+c_2\Big)dx+c_1\Big)$$

□

**Corollary I.1**: $\prod_{i=1}^{n}(D+f_i)y = 0$

$$\Rightarrow y = e^{-\int f_1 dx}\left(\int e^{\int(f_1-f_2)dx}\left(\int e^{\int(f_2-f_3)dx}\left(\cdots\int e^{\int(f_{n-2}-f_{n-1})dx}\left(c_n\int e^{\int(f_{n-1}-f_n)dx}dx+c_{n-1}\right)dx+c_{n-2}\cdots\right)dx+c_2\right)dx+c_1\right)$$

*Proof:*

By theorem I.1:

$$\Rightarrow U_n = (D+f_n)U_{n-1} = (D+f_n)(D+f_{n-1})\cdots(D+f_3)(D+f_2)(D+f_1)U_0 = \prod_{i=1}^{n}(D+f_i)U_0$$

$$U_n = \prod_{i=1}^{n}(D+f_i)U_0 = (D+f_n)U_{n-1} = U_{n-1}' + f_nU_{n-1} = \left(U_{n-1}e^{\int f_n dx}\right)' e^{-\int f_n dx}$$

$$\Rightarrow U_0 = e^{-\int f_1 dx}\Big(\int e^{\int(f_1-f_2)dx}\Big(\int e^{\int(f_2-f_3)dx}\Big($$

$$\cdots\int e^{\int(f_{n-2}-f_{n-1})dx}\left(\int e^{\int(f_{n-1}-f_n)dx}\left(\int U_n e^{\int f_n dx}dx+c_n\right)dx+c_{n-1}\right)dx+c_{n-2}\cdots\Big)dx+c_2\Big)dx+c_1\Big)$$

So, for: $y \equiv U_0$ & $U_n = 0$ :

$$\Rightarrow 0 = U_n = (D+f_n)U_{n-1} = (D+f_n)(D+f_{n-1})\cdots(D+f_3)(D+f_2)(D+f_1)U_0 = \prod_{i=1}^{n}(D+f_i)U_0$$

$$\Rightarrow 0 = U_n = \prod_{i=1}^{n}(D+f_i)y = (D+f_n)U_{n-1} = U_{n-1}' + f_nU_{n-1} = \left(U_{n-1}e^{\int f_n dx}\right)' e^{-\int f_n dx}$$

$$\Rightarrow y = U_0 = e^{-\int f_1 dx}\Big(\int e^{\int(f_1-f_2)dx}\Big(\int e^{\int(f_2-f_3)dx}\Big($$

$$\cdots\int e^{\int(f_{n-2}-f_{n-1})dx}\left(\int e^{\int(f_{n-1}-f_n)dx}\left(\int(0)e^{\int f_n dx}dx+c_n\right)dx+c_{n-1}\right)dx+c_{n-2}\cdots\Big)dx+c_2\Big)dx+c_1\Big)$$

$$= e^{-\int f_1 dx}\left(\int e^{\int(f_1-f_2)dx}\left(\int e^{\int(f_2-f_3)dx}\left(\cdots\int e^{\int(f_{n-2}-f_{n-1})dx}\left(c_n\int e^{\int(f_{n-1}-f_n)dx}dx+c_{n-1}\right)dx+c_{n-2}\cdots\right)dx+c_2\right)dx+c_1\right)$$

□

**Corollary I.2**: $\prod_{i=1}^{n}(D+f_i)y = W$

$$\Rightarrow y = e^{-\int f_1 dx}\left(\int e^{\int(f_1-f_2)dx}\left(\int e^{\int(f_2-f_3)dx}\left(\cdots\int e^{\int(f_{n-2}-f_{n-1})dx}\left(\int e^{\int(f_{n-1}-f_n)dx}\left(\int We^{\int f_n dx}dx+c_n\right)dx+c_{n-1}\right)dx+c_{n-2}\cdots\right)dx+c_2\right)$$

$$\Rightarrow y = e^{-\int f_1 dx}\left(\int e^{\int(f_1-f_2)dx}\left(\int e^{\int(f_2-f_3)dx}\left(\cdots\int e^{\int(f_{n-2}-f_{n-1})dx}\left(c_n\int e^{\int(f_{n-1}-f_n)dx}dx+c_{n-1}\right)dx+c_{n-2}\cdots\right)dx+c_2\right)dx+c_1\right)+$$

$$+e^{-\int f_1 dx}\left(\int e^{\int(f_1-f_2)dx}\left(\int e^{\int(f_2-f_3)dx}\left(\cdots\int e^{\int(f_{n-2}-f_{n-1})dx}\left(\int e^{\int(f_{n-1}-f_n)dx}\left(\int We^{\int f_n dx}dx\right)dx+c_{n-1}\right)dx+c_{n-2}\cdots\right)dx+c_2\right)dx+c$$

$$= y_h + y_p$$

*Proof:*

By theorem I.1:

$$\Rightarrow U_n = (D+f_n)U_{n-1} = (D+f_n)(D+f_{n-1})\cdots(D+f_3)(D+f_2)(D+f_1)U_0 = \prod_{i=1}^{n}(D+f_i)U_0$$

$$U_n = \prod_{i=1}^{n}(D+f_i)U_0 = (D+f_n)U_{n-1} = U_{n-1}' + f_nU_{n-1} = \left(U_{n-1}e^{\int f_n dx}\right)' e^{-\int f_n dx}$$

$$\Rightarrow U_0 = e^{-\int f_1 dx}\Big(\int e^{\int(f_1-f_2)dx}\Big(\int e^{\int(f_2-f_3)dx}\Big($$

$$\cdots\int e^{\int(f_{n-1}-f_{n-1})dx}\left(\int e^{\int(f_{n-1}-f_n)dx}\left(\int U_n e^{\int f_n dx}dx+c_n\right)dx+c_{n-1}\right)dx+c_{n-2}\cdots\Big)dx+c_2\Big)dx+c_1\Big)$$

So, for: $y \equiv U_0$ & $U_n = W$ :

$$\Rightarrow W = U_n = (D+f_n)U_{n-1} = (D+f_n)(D+f_{n-1})\cdots(D+f_3)(D+f_2)(D+f_1)U_0 = \prod_{i=1}^{n}(D+f_i)U_0$$

$$\Rightarrow W = U_n = \prod_{i=1}^{n}(D+f_i)y = (D+f_n)U_{n-1} = U_{n-1}' + f_nU_{n-1} = \left(U_{n-1}e^{\int f_n dx}\right)' e^{-\int f_n dx}$$

$$\Rightarrow y = U_0 = e^{-\int f_1 dx}\Big(\int e^{\int(f_1-f_2)dx}\Big(\int e^{\int(f_2-f_3)dx}\Big($$

$$\cdots\int e^{\int(f_{n-2}-f_{n-1})dx}\left(\int e^{\int(f_{n-1}-f_n)dx}\left(\int(W)e^{\int f_n dx}dx+c_n\right)dx+c_{n-1}\right)dx+c_{n-2}\cdots\Big)dx+c_2\Big)dx+c_1\Big)$$

$$= e^{-\int f_1 dx}\left(\int e^{\int(f_1-f_2)dx}\left(\int e^{\int(f_2-f_3)dx}\left(\cdots\int e^{\int(f_{n-2}-f_{n-1})dx}\left(\int e^{\int(f_{n-1}-f_n)dx}\left(\int We^{\int f_n dx}dx+c_n\right)dx+c_{n-1}\right)dx+c_{n-2}\cdots\right)dx+c_2\right)dx+c_1$$

$$\Rightarrow y = e^{-\int f_1 dx}\left(\int e^{\int(f_1-f_2)dx}\left(\int e^{\int(f_2-f_3)dx}\left(\cdots\int e^{\int(f_{n-2}-f_{n-1})dx}\left(c_n\int e^{\int(f_{n-1}-f_n)dx}dx+c_{n-1}\right)dx+c_{n-2}\cdots\right)dx+c_2\right)dx+c_1\right)+$$

$$
\begin{aligned}
&+ e^{-\int f_1 dx}\left(\int e^{\int (f_1-f_2)dx}\left(\int e^{\int (f_2-f_3)dx}\left(\cdots \int e^{\int (f_{n-2}-f_{n-1})dx}\left(\int e^{\int (f_{n-1}-f_n)dx}\left(\int We^{\int f_n dx} dx\right)dx + c_{n-1}\right)dx + c_{n-2}\cdots\right)dx + c_2\right)dx + c \\
&= y_h + y_p
\end{aligned}
$$

□

**Lemma II.1**: $\sum_{i=1}^{n} u(i,n+1) + \sum_{i_1=1}^{n-1}\sum_{i_2=i_1+1}^{n} u(i_1,i_2) = \sum_{i_1=1}^{n}\sum_{i_2=i_1+1}^{n+1} u(i_1,i_2)$

*Proof:*

$n = 2$ :

$$\sum_{i=1}^{n} u(i,n+1) + \sum_{i_1=1}^{n-1}\sum_{i_2=i_1+1}^{n} u(i_1,i_2) = u(1,2+1) + u(2,2+1) + u(1,2)$$

$$\sum_{i_1=1}^{n}\sum_{i_2=i_1+1}^{n+1} u(i_1,i_2) = u(1,2) + u(1,3) + u(2,3) = \sum_{i=1}^{n} u(i,n+1) + \sum_{i_1=1}^{n-1}\sum_{i_2=i_1+1}^{n} u(i_1,i_2)$$

So, for: $n = N$ :

$$\sum_{i=1}^{N} u(i,N+1) + \sum_{i_1=1}^{N-1}\sum_{i_2=i_1+1}^{N} u(i_1,i_2) = \sum_{i_1=1}^{N}\sum_{i_2=i_1+1}^{N+1} u(i_1,i_2)$$

$$
\begin{aligned}
\Rightarrow \sum_{i_1=1}^{N}\sum_{i_2=i_1+1}^{N+1} u(i_1,i_2) - \sum_{i_1=1}^{N-1}\sum_{i_2=i_1+1}^{N} u(i_1,i_2) &= \sum_{i_1=1}^{N} u(i_1,N+1) + \sum_{i_1=1}^{N}\sum_{i_2=i_1+1}^{N} u(i_1,i_2) - \sum_{i_1=1}^{N-1}\sum_{i_2=i_1+1}^{N} u(i_1,i_2) \\
&= \sum_{i_1=1}^{N} u(i_1,N+1) + \sum_{i_1=1}^{N-1}\sum_{i_2=i_1+1}^{N} u(i_1,i_2) - \sum_{i_1=1}^{N-1}\sum_{i_2=i_1+1}^{N} u(i_1,i_2) = \sum_{i_1=1}^{N} u(i_1,N+1)
\end{aligned}
$$

□

**Theorem II.1**: $\prod_{i=1}^{N}(D+f_i)y = y^{(N)} + \left[\sum_{i=1}^{N} f_i\right] y^{(N-1)} +$

$$
\begin{aligned}
&+\sum_{m=1}^{N-2}\left[\sum_{i_1=1}^{m+1}\sum_{i_2=i_1+1}^{m+2}\cdots\sum_{i_{N-m-2}=i_{N-m-3}+1}^{N-2}\;\sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\;\sum_{i_{N-m}=i_{N-m-1}+1}^{N}\left(\left(\cdots\left(\left(\left(f_{i_1}e^{\int f_{i_2}dx}\right)' e^{-\int f_{i_2}dx}\right)e^{\int f_{i_3}dx}\right)' e^{-\int f_{i_3}dx}\right)\cdots\right)e^{-\int f_N dx}\right] y^{(m)} + \\
&+\left[\left(\left(\cdots\left(\left(\left(f_1 e^{\int f_2 dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{\int (f_4-f_3)dx}\right)'\cdots\right)' e^{\int (f_N-f_{N-1})dx}\right)e^{-\int f_N dx}\right] y
\end{aligned}
$$

*Proof:*

1st Order HLODE:

$$
\begin{aligned}
\prod_{i=1}^{1}(D+f_i)U_0 &= (D+f_1)y \\
&= y' + f_i y = y^{(1)} + \left[\sum_{i=1}^{1} f_i\right] y
\end{aligned}
$$

2nd Order HLODE:

$$
\begin{aligned}
\prod_{i=1}^{2}(D+f_i)U_0 &= (D+f_2)(D+f_1)y = (D+f_2)(y' + f_1 y) \\
&= y'' + (f_1+f_2)y' + [f_1' + f_2 f_1]y \\
&= y^{(2)} + \left[\sum_{i=1}^{2} f_i\right] y^{(1)} + \left[\left(f_1 e^{\int f_2 dx}\right)' e^{-\int f_2 dx}\right] y
\end{aligned}
$$

3rd Order HLODE:

$$
\begin{aligned}
\prod_{i=1}^{3}(D+f_i)U_0 &= (D+f_3)\left(y^{(2)} + \left[\sum_{i=1}^{2} f_i\right] y^{(1)} + \left[\left(f_1 e^{\int f_2 dx}\right)' e^{-\int f_2 dx}\right] y\right) \\
&= y^{(3)} + \left[\sum_{i=1}^{3} f_i\right] y^{(2)} + \left[\left(\sum_{i=1}^{2} f_i\right)' + \left(\left(f_1 e^{\int f_2 dx}\right)' e^{-\int f_2 dx}\right) + f_3\left(\sum_{i=1}^{2} f_i\right)\right] y^{(1)} + \\
&\quad + \left[\left(\left(f_1 e^{\int f_2 dx}\right)' e^{-\int f_2 dx}\right)' + f_3\left(\left(f_1 e^{\int f_2 dx}\right)' e^{-\int f_2 dx}\right)\right] y \\
&= y^{(3)} + \left[\sum_{i=1}^{3} f_i\right] y^{(2)} + \left[\left(f_1 e^{-\int f_3 dx}\right)' e^{\int f_3 dx} + \left(f_1 e^{\int f_2 dx}\right)' e^{-\int f_2 dx} + \left(f_2 e^{-\int f_3 dx}\right)' e^{\int f_3 dx}\right] y^{(1)} + \\
&\quad + \left[\left(\left(\left(f_1 e^{\int f_2 dx}\right)' e^{-\int f_2 dx}\right)e^{\int f_3 dx}\right)' e^{-\int f_3 dx}\right] y \\
&= y^{(3)} + \left[\sum_{i=1}^{3} f_i\right] y^{(2)} + \left[\sum_{i_1=1}^{1+1}\sum_{i_2=i_1+1}^{1+2}\left(f_{i_1} e^{\int f_{i_2} dx}\right)' e^{-\int f_{i_2} dx}\right] y^{(1)} + \\
&\quad + \left[\left(\left(\left(f_1 e^{\int f_2 dx}\right)' e^{-\int f_2 dx}\right)e^{\int f_3 dx}\right)' e^{-\int f_3 dx}\right] y \\
&= y^{(3)} + \left[\sum_{i=1}^{3} f_i\right] y^{(2)} + \left[\left(f_1 e^{\int f_2 dx}\right)' e^{-\int f_2 dx} + \left(f_1 e^{-\int f_3 dx}\right)' e^{\int f_3 dx} + \left(f_2 e^{-\int f_3 dx}\right)' e^{\int f_3 dx}\right] y^{(1)} + \\
&\quad + \left[\left(\left(\left(f_1 e^{\int f_2 dx}\right)' e^{-\int f_2 dx}\right)e^{\int f_3 dx}\right)' e^{-\int f_3 dx}\right] y \\
&= y^{(3)} + \left[\sum_{i=1}^{3} f_i\right] y^{(2)} + \left[\sum_{i_1=1}^{3-1}\left(\sum_{i_2=i+1}^{3}\left(f_{i_1} e^{\int f_{i_2} dx}\right)' e^{-\int f_{i_2} dx}\right)\right] y^{(1)} + \\
&\quad + \left[\left(\left(f_1 e^{\int f_2 dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{-\int f_3 dx}\right] y
\end{aligned}
$$

$$= y^{(3)} + \left[\sum_{i=1}^{3} f_i\right] y^{(2)} + \left[\sum_{i_1=1}^{3-1}\left(\sum_{i_2=i+1}^{3}\left(f_{i_1} e^{\int f_{i_2}dx}\right)' e^{-\int f_{i_2}dx}\right)\right] y^{(1)} + \left[\left(\left(f_1 e^{\int f_2 dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{-\int f_3 dx}\right] y$$

4th Order HLODE:

$$\prod_{i=1}^{4}(D+f_i)U_0 = (D+f_4)\Big\{ y^{(3)} + \left[\sum_{i=1}^{3} f_i\right] y^{(2)} + \left[\sum_{i_1=1}^{3-1}\left(\sum_{i_2=i+1}^{3}\left(f_{i_1} e^{\int f_{i_2}dx}\right)' e^{-\int f_{i_2}dx}\right)\right] y^{(1)} +$$
$$+ \left[\left(\left(f_1 e^{\int f_2 dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{-\int f_3 dx}\right] y\Big\}$$

$$= y^{(4)} + \left[\sum_{i=1}^{4} f_i\right] y^{(3)} +$$
$$+ \left(\left[\sum_{i=1}^{3} f_i\right]' + f_4\left[\sum_{i=1}^{3} f_i\right] + \left[\sum_{i_1=1}^{3-1}\left(\sum_{i_2=i+1}^{3}\left(f_{i_1} e^{\int f_{i_2}dx}\right)' e^{-\int f_{i_2}dx}\right)\right]\right) y^{(2)} +$$
$$+ \left(\left[\sum_{i_1=1}^{3-1}\left(\sum_{i_2=i+1}^{3}\left(f_{i_1} e^{\int f_{i_2}dx}\right)' e^{-\int f_{i_2}dx}\right)\right]' + f_4\left[\sum_{i_1=1}^{3-1}\left(\sum_{i_2=i+1}^{3}\left(f_{i_1} e^{\int f_{i_2}dx}\right)' e^{-\int f_{i_2}dx}\right)\right] + \left[\left(\left(f_1 e^{\int f_2 dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{-\int f_3 dx}\right]\right)$$
$$+ \left(\left[\left(\left(f_1 e^{\int f_2 dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{-\int f_3 dx}\right]' + f_4\left[\left(\left(f_1 e^{\int f_2 dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{-\int f_3 dx}\right]\right) y$$

$$= y^{(4)} + \left[\sum_{i=1}^{4} f_i\right] y^{(3)} +$$
$$+ \left(\left(\left[\sum_{i=1}^{3} f_i\right] e^{\int f_4 dx}\right)' e^{-\int f_4 dx} + \left[\sum_{i_1=1}^{3-1}\left(\sum_{i_2=i+1}^{3}\left(f_{i_1} e^{\int f_{i_2}dx}\right)' e^{-\int f_{i_2}dx}\right)\right]\right) y^{(2)} +$$
$$+ \left(\left(\left[\sum_{i_1=1}^{3-1}\left(\sum_{i_2=i+1}^{3}\left(f_{i_1} e^{\int f_{i_2}dx}\right)' e^{-\int f_{i_2}dx}\right)\right] e^{\int f_4 dx}\right)' e^{-\int f_4 dx} + \left[\left(\left(f_1 e^{\int f_2 dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{-\int f_3 dx}\right]\right) y^{(1)} +$$
$$+ \left(\left(\left[\left(\left(f_1 e^{\int f_2 dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{-\int f_3 dx}\right] e^{\int f_4 dx}\right)' e^{-\int f_4 dx}\right) y$$

$$= y^{(4)} + \left[\sum_{i=1}^{4} f_i\right] y^{(3)} +$$
$$+ \left(\left[\sum_{i_1=1}^{3-1}\left(\sum_{i_2=i+1}^{3}\left(f_{i_1} e^{\int f_{i_2}dx}\right)' e^{-\int f_{i_2}dx}\right)\right] + \left(\left[\sum_{i=1}^{3} f_i\right] e^{\int f_4 dx}\right)' e^{-\int f_4 dx}\right) y^{(2)} +$$
$$+ \left(\left(\left[\sum_{i_1=1}^{3-1}\left(\sum_{i_2=i+1}^{3}\left(f_{i_1} e^{\int f_{i_2}dx}\right)' e^{-\int f_{i_2}dx}\right)\right] e^{\int f_4 dx}\right)' e^{-\int f_4 dx} + \left[\left(\left(f_1 e^{\int f_2 dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{-\int f_3 dx}\right]\right) y^{(1)} +$$
$$+ \left(\left(\left[\left(\left(f_1 e^{\int f_2 dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{-\int f_3 dx}\right] e^{\int f_4 dx}\right)' e^{-\int f_4 dx}\right) y$$

$$= y^{(4)} + \left[\sum_{i=1}^{4} f_i\right] y^{(3)} +$$
$$+ \Big(\left(f_1 e^{\int f_2 dx}\right)' e^{-\int f_2 dx} + \left(f_1 e^{-\int f_3 dx}\right)' e^{\int f_3 dx} + \left(f_2 e^{-\int f_3 dx}\right)' e^{\int f_3 dx} +$$
$$+ \left(f_1 e^{\int f_4 dx}\right)' e^{-\int f_4 dx} + \left(f_2 e^{\int f_4 dx}\right)' e^{-\int f_4 dx} + \left(f_3 e^{\int f_4 dx}\right)' e^{-\int f_4 dx}\Big) y^{(2)} +$$
$$+ \Big(\left(\left[\left(f_1 e^{\int f_2 dx}\right)' e^{-\int f_2 dx} + \left(f_1 e^{-\int f_3 dx}\right)' e^{\int f_3 dx} + \left(f_2 e^{-\int f_3 dx}\right)' e^{\int f_3 dx}\right] e^{\int f_4 dx}\right)' e^{-\int f_4 dx} +$$
$$+ \left(\left(\left(f_1 e^{\int f_2 dx}\right)' e^{-\int f_2 dx}\right) e^{\int f_3 dx}\right)' e^{-\int f_3 dx}\Big) y^{(1)} +$$
$$+ \left(\left(\left[\left(\left(f_1 e^{\int f_2 dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{-\int f_3 dx}\right] e^{\int f_4 dx}\right)' e^{-\int f_4 dx}\right) y$$

$$= y^{(4)} + \left[\sum_{i=1}^{4} f_i\right] y^{(3)} +$$
$$+ \Big(\left(f_1 e^{\int f_2 dx}\right)' e^{-\int f_2 dx} + \left(f_1 e^{-\int f_3 dx}\right)' e^{\int f_3 dx} + \left(f_1 e^{\int f_4 dx}\right)' e^{-\int f_4 dx} +$$
$$+ \left(f_2 e^{-\int f_3 dx}\right)' e^{\int f_3 dx} + \left(f_2 e^{\int f_4 dx}\right)' e^{-\int f_4 dx} + \left(f_3 e^{\int f_4 dx}\right)' e^{-\int f_4 dx}\Big) y^{(2)} +$$
$$+ \Big(\left(\left(\left(f_1 e^{\int f_2 dx}\right)' e^{-\int f_2 dx}\right) e^{\int f_3 dx}\right)' e^{-\int f_3 dx} + \left(\left(\left(f_1 e^{\int f_2 dx}\right)' e^{-\int f_2 dx}\right) e^{\int f_4 dx}\right)' e^{-\int f_4 dx} +$$
$$+ \left(\left(\left(f_1 e^{-\int f_3 dx}\right)' e^{\int f_3 dx}\right) e^{\int f_4 dx}\right)' e^{-\int f_4 dx} + \left(\left(\left(f_2 e^{-\int f_3 dx}\right)' e^{\int f_3 dx}\right) e^{\int f_4 dx}\right)' e^{-\int f_4 dx}\Big) y^{(1)} +$$
$$+ \left(\left(\left[\left(\left(f_1 e^{\int f_2 dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{-\int f_3 dx}\right] e^{\int f_4 dx}\right)' e^{-\int f_4 dx}\right) y$$

$$= y^{(4)} + \left[\sum_{i=1}^{4} f_i\right] y^{(3)} + \left[\sum_{i_1=1}^{4-1}\sum_{i_2=i_1+1}^{4}\left(f_{i_1} e^{\int f_{i_2}dx}\right)' e^{-\int f_{i_2}dx}\right] y^{(2)} +$$
$$+ \left[\sum_{i_1=1}^{4-2}\sum_{i_2=i_1+1}^{4-1}\sum_{i_3=i_2+1}^{4}\left(\left(\left(f_{i_1} e^{\int f_{i_2}dx}\right)' e^{-\int f_{i_2}dx}\right) e^{\int f_{i_3}dx}\right)' e^{-\int f_{i_3}dx}\right] y^{(1)} +$$
$$+ \left[\left(\left(\left(f_1 e^{\int f_2 dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{\int (f_4-f_3)dx}\right)' e^{-\int f_4 dx}\right] y$$

5th Order HLODE:

$$
\begin{aligned}
\prod_{i=1}^{5}(D+f_i)U_0 =\ & (D+f_5)\Big\{y^{(4)}+\left[\sum_{i=1}^{4}f_i\right]y^{(3)}+\left[\sum_{i_1=1}^{4-1}\sum_{i_2=i_1+1}^{4}\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right]y^{(2)}+\\
&+\left[\sum_{i_1=1}^{4-2}\sum_{i_2=i_1+1}^{4-1}\sum_{i_3=i_2+1}^{4}\left(\left(\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right)e^{\int f_{i_3}dx}\right)'e^{-\int f_{i_3}dx}\right]y^{(1)}+\\
&+\left[\left(\left(\left(f_1e^{\int f_2dx}\right)'e^{\int(f_3-f_2)dx}\right)'e^{\int(f_4-f_3)dx}\right)'e^{-\int f_4dx}\right]y\Big\}\\
=\ & y^{(4+1)}+\left[\sum_{i=1}^{4}f_i\right]y^{(3+1)}+\left[\sum_{i=1}^{4}f_i\right]'y^{(3)}+\\
&+\left[\sum_{i_1=1}^{4-1}\sum_{i_2=i_1+1}^{4}\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right]y^{(2+1)}+\left[\sum_{i_1=1}^{4-1}\sum_{i_2=i_1+1}^{4}\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right]'y^{(2)}\\
&+\left[\sum_{i_1=1}^{4-2}\sum_{i_2=i_1+1}^{4-1}\sum_{i_3=i_2+1}^{4}\left(\left(\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right)e^{\int f_{i_3}dx}\right)'e^{-\int f_{i_3}dx}\right]y^{(1+1)}+\\
&+\left[\sum_{i_1=1}^{4-2}\sum_{i_2=i_1+1}^{4-1}\sum_{i_3=i_2+1}^{4}\left(\left(\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right)e^{\int f_{i_3}dx}\right)'e^{-\int f_{i_3}dx}\right]'y^{(1)}+\\
&+\left[\left(\left(\left(f_1e^{\int f_2dx}\right)'e^{\int(f_3-f_2)dx}\right)'e^{\int(f_4-f_3)dx}\right)'e^{-\int f_4dx}\right]y^{(1)}+\\
&+\left[\left(\left(\left(f_1e^{\int f_2dx}\right)'e^{\int(f_3-f_2)dx}\right)'e^{\int(f_4-f_3)dx}\right)'e^{-\int f_4dx}\right]'y+\\
&+f_5y^{(4)}+f_5\left[\sum_{i=1}^{4}f_i\right]y^{(3)}+f_5\left[\sum_{i_1=1}^{4-1}\sum_{i_2=i_1+1}^{4}\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right]y^{(2)}+\\
&+f_5\left[\sum_{i_1=1}^{4-2}\sum_{i_2=i_1+1}^{4-1}\sum_{i_3=i_2+1}^{4}\left(\left(\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right)e^{\int f_{i_3}dx}\right)'e^{-\int f_{i_3}dx}\right]y^{(1)}+\\
&+f_5\left[\left(\left(\left(f_1e^{\int f_2dx}\right)'e^{\int(f_3-f_2)dx}\right)'e^{\int(f_4-f_3)dx}\right)'e^{-\int f_4dx}\right]y\\
=\ & y^{(5)}+\left[\sum_{i=1}^{5}f_i\right]y^{(4)}+\left[\sum_{i=1}^{4}f_i\right]'y^{(3)}+f_5\left[\sum_{i=1}^{4}f_i\right]y^{(3)}+\\
&+\left[\sum_{i_1=1}^{4-1}\sum_{i_2=i_1+1}^{4}\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right]'y^{(2)}+f_5\left[\sum_{i_1=1}^{4-1}\sum_{i_2=i_1+1}^{4}\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right]y^{(2)}+\\
&+\left[\sum_{i_1=1}^{4-2}\sum_{i_2=i_1+1}^{4-1}\sum_{i_3=i_2+1}^{4}\left(\left(\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right)e^{\int f_{i_3}dx}\right)'e^{-\int f_{i_3}dx}\right]'y^{(1)}+\\
&+f_5\left[\sum_{i_1=1}^{4-2}\sum_{i_2=i_1+1}^{4-1}\sum_{i_3=i_2+1}^{4}\left(\left(\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right)e^{\int f_{i_3}dx}\right)'e^{-\int f_{i_3}dx}\right]y^{(1)}+\\
&+\left[\left(\left(\left(f_1e^{\int f_2dx}\right)'e^{\int(f_3-f_2)dx}\right)'e^{\int(f_4-f_3)dx}\right)'e^{-\int f_4dx}\right]'y+\\
&+f_5\left[\left(\left(\left(f_1e^{\int f_2dx}\right)'e^{\int(f_3-f_2)dx}\right)'e^{\int(f_4-f_3)dx}\right)'e^{-\int f_4dx}\right]y+\\
&+\left[\sum_{i_1=1}^{4-1}\sum_{i_2=i_1+1}^{4}\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right]y^{(2+1)}+\\
&+\left[\sum_{i_1=1}^{4-2}\sum_{i_2=i_1+1}^{4-1}\sum_{i_3=i_2+1}^{4}\left(\left(\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right)e^{\int f_{i_3}dx}\right)'e^{-\int f_{i_3}dx}\right]y^{(1+1)}+\\
&+\left[\left(\left(\left(f_1e^{\int f_2dx}\right)'e^{\int(f_3-f_2)dx}\right)'e^{\int(f_4-f_3)dx}\right)'e^{-\int f_4dx}\right]y^{(1)}\\
=\ & y^{(5)}+\left[\sum_{i=1}^{5}f_i\right]y^{(4)}+\\
&+\left[\left(\sum_{i=1}^{4}f_ie^{\int f_5dx}\right)'e^{-\int f_5dx}\right]y^{(3)}+\left[\sum_{i_1=1}^{4-1}\sum_{i_2=i_1+1}^{4}\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right]y^{(2+1)}+\\
&+\left[\left(\left[\sum_{i_1=1}^{4-1}\sum_{i_2=i_1+1}^{4}\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right]e^{\int f_5dx}\right)'e^{-\int f_5dx}\right]y^{(2)}+\\
&+\left[\sum_{i_1=1}^{4-2}\sum_{i_2=i_1+1}^{4-1}\sum_{i_3=i_2+1}^{4}\left(\left(\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right)e^{\int f_{i_3}dx}\right)'e^{-\int f_{i_3}dx}\right]y^{(1+1)}+\\
&+\left[\left(\left[\sum_{i_1=1}^{4-2}\sum_{i_2=i_1+1}^{4-1}\sum_{i_3=i_2+1}^{4}\left(\left(\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right)e^{\int f_{i_3}dx}\right)'e^{-\int f_{i_3}dx}\right]e^{\int f_5dx}\right)'e^{-\int f_5dx}\right]y^{(1)}+\\
&+\left[\left(\left(\left(f_1e^{\int f_2dx}\right)'e^{\int(f_3-f_2)dx}\right)'e^{\int(f_4-f_3)dx}\right)'e^{-\int f_4dx}\right]y^{(1)}+\\
&+\left[\left(\left[\left(\left(\left(f_1e^{\int f_2dx}\right)'e^{\int(f_3-f_2)dx}\right)'e^{\int(f_4-f_3)dx}\right)'e^{-\int f_4dx}\right]e^{\int f_5dx}\right)'e^{-\int f_5dx}\right]y
\end{aligned}
$$

$$= y^{(5)} + \left[\sum_{i=1}^{5} f_i\right] y^{(4)} + \left[\sum_{i_1=1}^{5-1}\sum_{i_2=i_1+1}^{5}\Big(f_{i_1}e^{\int f_{i_2}dx}\Big)' e^{-\int f_{i_2}dx}\right] y^{(3)} +$$

$$+\left[\sum_{i_1=1}^{5-2}\sum_{i_2=i_1+1}^{5-1}\sum_{i_3=i_2+1}^{5}\Big(\Big(\Big(f_{i_1}e^{\int f_{i_2}dx}\Big)' e^{-\int f_{i_2}dx}\Big)e^{\int f_{i_3}dx}\Big)' e^{-\int f_{i_3}dx}\right] y^{(2)} +$$

$$+\left[\Big(\sum_{i_1=1}^{5-3}\sum_{i_2=i_1+1}^{5-2}\sum_{i_3=i_2+1}^{5-1}\sum_{i_4=i_3+1}^{5}\Big(\Big(\Big(\Big(f_{i_1}e^{\int f_{i_2}dx}\Big)' e^{-\int f_{i_2}dx}\Big)e^{\int f_{i_3}dx}\Big)' e^{-\int f_{i_3}dx}\Big)e^{\int f_{i_4}dx}\Big)' e^{-\int f_{i_4}dx}\right] y^{(1)} +$$

$$+\left[\Big(\Big(\Big(\Big(f_1e^{\int f_2dx}\Big)' e^{\int(f_3-f_2)dx}\Big)' e^{\int(f_4-f_3)dx}\Big)' e^{\int(f_5-f_4)dx}\Big)' e^{-\int f_5dx}\right] y$$

So:
If for: $n = N$ :

$$\prod_{i=1}^{N}(D+f_i)y = y^{(N)} + \left[\sum_{i=1}^{N} f_i\right]y^{(N-1)} +$$

$$+\sum_{m=1}^{N-2}\left[\sum_{i_1=1}^{m+1}\sum_{i_2=i_1+1}^{m+2}\cdots\sum_{i_{N-m-2}=i_{N-m-3}+1}^{N-2}\sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\sum_{i_{N-m}=i_{N-m-1}+1}^{N}\Big(\Big(\cdots\Big(\Big(\Big(f_{i_1}e^{\int f_{i_2}dx}\Big)' e^{-\int f_{i_2}dx}\Big)e^{\int f_{i_3}dx}\Big)' e^{-\int f_{i_3}dx}\Big)\cdots\Big)e^{-\int f_N dx}\right]y^{(m)} +$$

$$+\left[\Big(\Big(\cdots\Big(\Big(\Big(f_1e^{\int f_2dx}\Big)' e^{\int(f_3-f_2)dx}\Big)' e^{\int(f_4-f_3)dx}\Big)'\cdots\Big)' e^{\int(f_N-f_{N-1})dx}\Big)e^{-\int f_N dx}\right]y$$

$$= y^{(N)} + \left[\sum_{i=1}^{N} f_i\right]y^{(N-1)} +$$

$$+\sum_{m=1}^{N-2}\left[\sum_{i_1=1}^{N-m}\sum_{i_2=i_1+1}^{N-m+1}\cdots\sum_{i_{m-1}=i_{m-2}+1}^{N-2}\sum_{i_m=i_{m-1}+1}^{N-1}\sum_{i_{m+1}=i_m+1}^{N}\Big(\Big(\cdots\Big(\Big(\Big(f_{i_1}e^{\int f_{i_2}dx}\Big)' e^{-\int f_{i_2}dx}\Big)e^{\int f_{i_3}dx}\Big)' e^{-\int f_{i_3}dx}\Big)\cdots\Big)e^{-\int f_N dx}\right]y^{(N-1-m)} +$$

$$+\left[\Big(\Big(\cdots\Big(\Big(\Big(f_1e^{\int f_2dx}\Big)' e^{\int(f_3-f_2)dx}\Big)' e^{\int(f_4-f_3)dx}\Big)'\cdots\Big)' e^{\int(f_N-f_{N-1})dx}\Big)e^{-\int f_N dx}\right]y$$

then:

$$\prod_{i=1}^{N+1}(D+f_i)y = (D+f_{N+1})\prod_{i=1}^{N}(D+f_i)y = D\prod_{i=1}^{N}(D+f_i)y + f_{N+1}\prod_{i=1}^{N}(D+f_i)y$$

$$= y^{(N+1)} + f_{N+1}y^{(N)} + \left[\sum_{i=1}^{N} f_i\right]y^{(N)} + \left[\sum_{i=1}^{N} f_i\right]' y^{(N-1)} + f_{N+1}\left[\sum_{i=1}^{N} f_i\right]y^{(N-1)} +$$

$$+\sum_{m=1}^{N-2}\left[\sum_{i_1=1}^{m+1}\sum_{i_2=i_1+1}^{m+2}\cdots\sum_{i_{N-m-2}=i_{N-m-3}+1}^{N-2}\sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\sum_{i_{N-m}=i_{N-m-1}+1}^{N}\Big(\Big(\cdots\Big(\Big(\Big(f_{i_1}e^{\int f_{i_2}dx}\Big)' e^{-\int f_{i_2}dx}\Big)e^{\int f_{i_3}dx}\Big)' e^{-\int f_{i_3}dx}\Big)\cdots\Big)e^{-\int f_N dx}\right]' y^{(m)} +$$

$$+f_{N+1}\sum_{m=1}^{N-2}\left[\sum_{i_1=1}^{m+1}\sum_{i_2=i_1+1}^{m+2}\cdots\sum_{i_{N-m-2}=i_{N-m-3}+1}^{N-2}\sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\sum_{i_{N-m}=i_{N-m-1}+1}^{N}\Big(\Big(\cdots\Big(\Big(\Big(f_{i_1}e^{\int f_{i_2}dx}\Big)' e^{-\int f_{i_2}dx}\Big)e^{\int f_{i_3}dx}\Big)' e^{-\int f_{i_3}dx}\Big)\cdots\Big)e^{-\int f_N dx}\right]y^{(m)} +$$

$$+\left[\Big(\Big(\cdots\Big(\Big(\Big(f_1e^{\int f_2dx}\Big)' e^{\int(f_3-f_2)dx}\Big)' e^{\int(f_4-f_3)dx}\Big)'\cdots\Big)' e^{\int(f_N-f_{N-1})dx}\Big)e^{-\int f_N dx}\right]' y +$$

$$+f_{N+1}\left[\Big(\Big(\cdots\Big(\Big(\Big(f_1e^{\int f_2dx}\Big)' e^{\int(f_3-f_2)dx}\Big)' e^{\int(f_4-f_3)dx}\Big)'\cdots\Big)' e^{\int(f_N-f_{N-1})dx}\Big)e^{-\int f_N dx}\right]y +$$

$$+\sum_{m=1}^{N-2}\left[\sum_{i_1=1}^{m+1}\sum_{i_2=i_1+1}^{m+2}\cdots\sum_{i_{N-m-2}=i_{N-m-3}+1}^{N-2}\sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\sum_{i_{N-m}=i_{N-m-1}+1}^{N}\Big(\Big(\cdots\Big(\Big(\Big(f_{i_1}e^{\int f_{i_2}dx}\Big)' e^{-\int f_{i_2}dx}\Big)e^{\int f_{i_3}dx}\Big)' e^{-\int f_{i_3}dx}\Big)\cdots\Big)e^{-\int f_N dx}\right]y^{(m+1)} +$$

$$+\left[\Big(\Big(\cdots\Big(\Big(\Big(f_1e^{\int f_2dx}\Big)' e^{\int(f_3-f_2)dx}\Big)' e^{\int(f_4-f_3)dx}\Big)'\cdots\Big)' e^{\int(f_N-f_{N-1})dx}\Big)e^{-\int f_N dx}\right]y'$$

$$= y^{(N+1)} + \left[\sum_{i=1}^{N+1} f_i\right]y^{(N)} + \left[\sum_{i=1}^{N}\Big(f_ie^{\int f_{N+1}dx}\Big)' e^{-\int f_{N+1}dx}\right]y^{(N-1)} +$$

$$+\sum_{m=1}^{N-2}\Bigg[\sum_{i_1=1}^{m+1}\sum_{i_2=i_1+1}^{m+2}\cdots\sum_{i_{N-m-2}=i_{N-m-3}+1}^{N-2}\sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\sum_{i_{N-m}=i_{N-m-1}+1}^{N}\Big(\Big(\Big(\Big(\cdots\Big(\Big(\Big(f_{i_1}e^{\int f_{i_2}dx}\Big)' e^{-\int f_{i_2}dx}\Big)e^{\int f_{i_3}dx}\Big)' e^{-\int f_{i_3}dx}\Big)\cdots\Big)e^{-\int f_N dx}\Big)e^{\int f_{N+1}dx}\Big)$$

$$+\left[\Big(\Big(\Big(\Big(\Big(\cdots\Big(\Big(\Big(f_1e^{\int f_2dx}\Big)' e^{\int(f_3-f_2)dx}\Big)' e^{\int(f_4-f_3)dx}\Big)'\cdots\Big)' e^{\int(f_N-f_{N-1})dx}\Big)e^{-\int f_N dx}\Big)e^{\int f_{N+1}dx}\Big)' e^{-\int f_{N+1}dx}\right]y +$$

$$+\sum_{m=1}^{N-2}\left[\sum_{i_1=1}^{m+1}\sum_{i_2=i_1+1}^{m+2}\cdots\sum_{i_{N-m-2}=i_{N-m-3}+1}^{N-2}\sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\sum_{i_{N-m}=i_{N-m-1}+1}^{N}\Big(\Big(\cdots\Big(\Big(\Big(f_{i_1}e^{\int f_{i_2}dx}\Big)' e^{-\int f_{i_2}dx}\Big)e^{\int f_{i_3}dx}\Big)' e^{-\int f_{i_3}dx}\Big)\cdots\Big)e^{-\int f_N dx}\right]y^{(m+1)} +$$

$$+\left[\Big(\Big(\cdots\Big(\Big(\Big(f_1e^{\int f_2dx}\Big)' e^{\int(f_3-f_2)dx}\Big)' e^{\int(f_4-f_3)dx}\Big)'\cdots\Big)' e^{\int(f_N-f_{N-1})dx}\Big)e^{-\int f_N dx}\right]y'$$

$$= y^{(N+1)} + \left[\sum_{i=1}^{N+1} f_i\right]y^{(N)} + \left[\sum_{i=1}^{N}\Big(f_ie^{\int f_{N+1}dx}\Big)' e^{-\int f_{N+1}dx}\right]y^{(N-1)} +$$

$$+\sum_{m=2}^{N-2}\Bigg[\sum_{i_1=1}^{m+1}\sum_{i_2=i_1+1}^{m+2}\cdots\sum_{i_{N-m-2}=i_{N-m-3}+1}^{N-2}\sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\sum_{i_{N-m}=i_{N-m-1}+1}^{N}\Big(\Big(\Big(\Big(\cdots\Big(\Big(\Big(f_{i_1}e^{\int f_{i_2}dx}\Big)' e^{-\int f_{i_2}dx}\Big)e^{\int f_{i_3}dx}\Big)' e^{-\int f_{i_3}dx}\Big)\cdots\Big)e^{-\int f_N dx}\Big)e^{\int f_{N+1}dx}\Big)$$

$$+\Bigg[\sum_{i_1=1}^{1+1}\sum_{i_2=i_1+1}^{1+2}\cdots\sum_{i_{N-m-2}=i_{N-m-3}+1}^{N-2}\sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\sum_{i_{N-m}=i_{N-m-1}+1}^{N}\Big(\Big(\Big(\Big(\cdots\Big(\Big(\Big(f_{i_1}e^{\int f_{i_2}dx}\Big)' e^{-\int f_{i_2}dx}\Big)e^{\int f_{i_3}dx}\Big)' e^{-\int f_{i_3}dx}\Big)\cdots\Big)e^{-\int f_N dx}\Big)e^{\int f_{N+1}dx}\Big)' e$$

$$+\left[\left(\left(\left(\left(\cdots\left(\left(\left(f_1 e^{\int f_2 dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{\int (f_4-f_3)dx}\right)' \cdots\right)' e^{\int (f_N-f_{N-1})dx}\right) e^{-\int f_N dx}\right) e^{\int f_{N+1} dx}\right)' e^{-\int f_{N+1} dx}\right] y +$$

$$+\left[\sum_{i_1=1}^{N-2+1}\sum_{i_2=i_1+1}^{N-2+2}\left(\left(f_{i_1} e^{\int f_{i_2} dx}\right)' e^{-\int f_{i_2} dx}\right)\right] y^{(N-2+1)} +$$

$$+\sum_{m=1}^{N-3}\left[\sum_{i_1=1}^{m+1}\sum_{i_2=i_1+1}^{m+2}\cdots\sum_{i_{N-m-2}=_{N-m-3}+1}^{N-2}\ \sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\ \sum_{i_{N-m}=i_{N-m-1}+1}^{N}\left(\left(\cdots\left(\left(\left(f_{i_1} e^{\int f_{i_2} dx}\right)' e^{-\int f_{i_2} dx}\right) e^{\int f_{i_3} dx}\right)' e^{-\int f_{i_3} dx}\right)\cdots\right) e^{-\int f_N dx}\right] y^{(m+1)} +$$

$$+\left[\left(\left(\cdots\left(\left(\left(f_1 e^{\int f_2 dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{\int (f_4-f_3)dx}\right)' \cdots\right)' e^{\int (f_N-f_{N-1})dx}\right) e^{-\int f_N dx}\right] y'$$

$$= y^{(N+1)} + \left[\sum_{i=1}^{N+1} f_i\right] y^{(N)} +$$

$$+\left[\sum_{i=1}^{N}\left(f_i e^{\int f_{N+1} dx}\right)' e^{-\int f_{N+1} dx}\right] y^{(N-1)} + \left[\sum_{i_1=1}^{N-2+1}\sum_{i_2=i_1+1}^{N-2+2}\left(\left(f_{i_1} e^{\int f_{i_2} dx}\right)' e^{-\int f_{i_2} dx}\right)\right] y^{(N-2+1)} +$$

$$+\sum_{m=2}^{N-2}\left[\sum_{i_1=1}^{m+1}\sum_{i_2=i_1+1}^{m+2}\cdots\sum_{i_{N-m-2}=_{N-m-3}+1}^{N-2}\ \sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\ \sum_{i_{N-m}=i_{N-m-1}+1}^{N}\left(\left(\left(\left(\cdots\left(\left(\left(f_{i_1} e^{\int f_{i_2} dx}\right)' e^{-\int f_{i_2} dx}\right) e^{\int f_{i_3} dx}\right)' e^{-\int f_{i_3} dx}\right)\cdots\right) e^{-\int f_N dx}\right) e^{\int f_{N+1} dx}\right)\right.$$

$$+\left[\sum_{i_1=1}^{1+1}\sum_{i_2=i_1+1}^{1+2}\cdots\sum_{i_{N-m-2}=_{N-m-3}+1}^{N-2}\ \sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\ \sum_{i_{N-m}=i_{N-m-1}+1}^{N}\left(\left(\left(\left(\cdots\left(\left(\left(f_{i_1} e^{\int f_{i_2} dx}\right)' e^{-\int f_{i_2} dx}\right) e^{\int f_{i_3} dx}\right)' e^{-\int f_{i_3} dx}\right)\cdots\right) e^{-\int f_N dx}\right) e^{\int f_{N+1} dx}\right)' e\right.$$

$$+\left[\left(\left(\left(\left(\cdots\left(\left(\left(f_1 e^{\int f_2 dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{\int (f_4-f_3)dx}\right)' \cdots\right)' e^{\int (f_N-f_{N-1})dx}\right) e^{-\int f_N dx}\right) e^{\int f_{N+1} dx}\right)' e^{-\int f_{N+1} dx}\right] y +$$

$$+\sum_{m=1}^{N-3}\left[\sum_{i_1=1}^{m+1}\sum_{i_2=i_1+1}^{m+2}\cdots\sum_{i_{N-m-2}=_{N-m-3}+1}^{N-2}\ \sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\ \sum_{i_{N-m}=i_{N-m-1}+1}^{N}\left(\left(\cdots\left(\left(\left(f_{i_1} e^{\int f_{i_2} dx}\right)' e^{-\int f_{i_2} dx}\right) e^{\int f_{i_3} dx}\right)' e^{-\int f_{i_3} dx}\right)\cdots\right) e^{-\int f_N dx}\right] y^{(m+1)} +$$

$$+\left[\left(\left(\cdots\left(\left(\left(f_1 e^{\int f_2 dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{\int (f_4-f_3)dx}\right)' \cdots\right)' e^{\int (f_N-f_{N-1})dx}\right) e^{-\int f_N dx}\right] y'$$

$$= y^{(N+1)} + \left[\sum_{i=1}^{N+1} f_i\right] y^{(N)} +$$

$$+\left[\sum_{i=1}^{N}\left(f_i e^{\int f_{N+1} dx}\right)' e^{-\int f_{N+1} dx} + \sum_{i_1=1}^{N-1}\sum_{i_2=i_1+1}^{N}\left(\left(f_{i_1} e^{\int f_{i_2} dx}\right)' e^{-\int f_{i_2} dx}\right)\right] y^{(N-1)} +$$

$$+\sum_{m=2}^{N-2}\left[\sum_{i_1=1}^{m+1}\sum_{i_2=i_1+1}^{m+2}\cdots\sum_{i_{N-m-2}=_{N-m-3}+1}^{N-2}\ \sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\ \sum_{i_{N-m}=i_{N-m-1}+1}^{N}\left(\left(\left(\left(\cdots\left(\left(\left(f_{i_1} e^{\int f_{i_2} dx}\right)' e^{-\int f_{i_2} dx}\right) e^{\int f_{i_3} dx}\right)' e^{-\int f_{i_3} dx}\right)\cdots\right) e^{-\int f_N dx}\right) e^{\int f_{N+1} dx}\right)\right.$$

$$+\sum_{n=2}^{N-2}\left[\sum_{i_1=1}^{n}\sum_{i_2=i_1+1}^{n+1}\cdots\sum_{i_{N-m-2}=_{N-m-3}+1}^{N-2}\ \sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\ \sum_{i_{N-m}=i_{N-m-1}+1}^{N}\left(\left(\cdots\left(\left(\left(f_{i_1} e^{\int f_{i_2} dx}\right)' e^{-\int f_{i_2} dx}\right) e^{\int f_{i_3} dx}\right)' e^{-\int f_{i_3} dx}\right)\cdots\right) e^{-\int f_N dx}\right] y^{(n)} +$$

$$+\left[\sum_{i_1=1}^{1+1}\sum_{i_2=i_1+1}^{1+2}\cdots\sum_{i_{N-m-2}=_{N-m-3}+1}^{N-2}\ \sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\ \sum_{i_{N-m}=i_{N-m-1}+1}^{N}\left(\left(\left(\left(\cdots\left(\left(\left(f_{i_1} e^{\int f_{i_2} dx}\right)' e^{-\int f_{i_2} dx}\right) e^{\int f_{i_3} dx}\right)' e^{-\int f_{i_3} dx}\right)\cdots\right) e^{-\int f_N dx}\right) e^{\int f_{N+1} dx}\right)' e\right.$$

$$+\left[\left(\left(\cdots\left(\left(\left(f_1 e^{\int f_2 dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{\int (f_4-f_3)dx}\right)' \cdots\right)' e^{\int (f_N-f_{N-1})dx}\right) e^{-\int f_N dx}\right] y'$$

$$+\left[\left(\left(\left(\cdots\left(\left(\left(f_1 e^{\int f_2 dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{\int (f_4-f_3)dx}\right)' \cdots\right)' e^{\int (f_N-f_{N-1})dx}\right)' e^{\int (f_{N+1}-f_N)dx}\right)' e^{-\int f_{N+1} dx}\right] y$$

$$= y^{(N+1)} + \left[\sum_{i=1}^{N+1} f_i\right] y^{(N)} + \left[\sum_{i_1=1}^{N}\sum_{i_2=i_1+1}^{N+1}\left(f_{i_1} e^{\int f_{i_2} dx}\right)' e^{-\int f_{i_2} dx}\right] y^{(N-1)} +$$

$$+\sum_{m=2}^{N-2}\left\{\sum_{i_1=1}^{m+1}\sum_{i_2=i_1+1}^{m+2}\cdots\sum_{i_{N-m-2}=_{N-m-3}+1}^{N-2}\ \sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\ \sum_{i_{N-m}=i_{N-m-1}+1}^{N}\left(\left(\left(\left(\cdots\left(\left(\left(f_{i_1} e^{\int f_{i_2} dx}\right)' e^{-\int f_{i_2} dx}\right) e^{\int f_{i_3} dx}\right)' e^{-\int f_{i_3} dx}\right)\cdots\right) e^{-\int f_N dx}\right) e^{\int f_{N+1} dx}\right.\right.$$

$$+\sum_{i_1=1}^{m}\sum_{i_2=i_1+1}^{m+1}\cdots\sum_{i_{N-m-2}=_{N-m-3}+1}^{N-2}\ \sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\ \sum_{i_{N-m}=i_{N-m-1}+1}^{N}\left(\left(\cdots\left(\left(\left(f_{i_1} e^{\int f_{i_2} dx}\right)' e^{-\int f_{i_2} dx}\right) e^{\int f_{i_3} dx}\right)' e^{-\int f_{i_3} dx}\right)\cdots\right) e^{-\int f_N dx}\right\} y^{(m)} +$$

$$+\left\{\sum_{i_1=1}^{1+1}\sum_{i_2=i_1+1}^{1+2}\cdots\sum_{i_{N-m-2}=_{N-m-3}+1}^{N-2}\ \sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\ \sum_{i_{N-m}=i_{N-m-1}+1}^{N}\left(\left(\left(\left(\cdots\left(\left(\left(f_{i_1} e^{\int f_{i_2} dx}\right)' e^{-\int f_{i_2} dx}\right) e^{\int f_{i_3} dx}\right)' e^{-\int f_{i_3} dx}\right)\cdots\right) e^{-\int f_N dx}\right) e^{\int f_{N+1} dx}\right)' e\right.$$

$$\left.+\left(\left(\cdots\left(\left(\left(f_1 e^{\int f_2 dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{\int (f_4-f_3)dx}\right)' \cdots\right)' e^{\int (f_N-f_{N-1})dx}\right) e^{-\int f_N dx}\right\} y'$$

$$+\left[\left(\left(\left(\cdots\left(\left(\left(f_1 e^{\int f_2 dx}\right)' e^{\int (f_3-f_2)dx}\right)' e^{\int (f_4-f_3)dx}\right)' \cdots\right)' e^{\int (f_N-f_{N-1})dx}\right)' e^{\int (f_{N+1}-f_N)dx}\right)' e^{-\int f_{N+1} dx}\right] y$$

$$= y^{(N+1)} + \left[\sum_{i=1}^{N+1} f_i\right] y^{(N)} + \left[\sum_{i_1=1}^{N}\sum_{i_2=i_1+1}^{N+1}\left(f_{i_1} e^{\int f_{i_2} dx}\right)' e^{-\int f_{i_2} dx}\right] y^{(N-1)} +$$

$$+\sum_{m=2}^{N-2}\left\{\sum_{i_1=1}^{m+1}\sum_{i_2=i_1+1}^{m+2}\cdots\sum_{i_{N-m-2}=_{N-m-3}+1}^{N-2}\ \sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\ \sum_{i_{N-m}=i_{N-m-1}+1}^{N}\left(\left(\left(\left(\cdots\left(\left(\left(f_{i_1} e^{\int f_{i_2} dx}\right)' e^{-\int f_{i_2} dx}\right) e^{\int f_{i_3} dx}\right)' e^{-\int f_{i_3} dx}\right)\cdots\right) e^{-\int f_N dx}\right) e^{\int f_{N+1} dx}\right.\right.$$

$$+\sum_{i_1=1}^{m}\sum_{i_2=i_1+1}^{m+1}\cdots\sum_{i_{N-m-2}=_{N-m-3}+1}^{N-2}\ \sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\ \sum_{i_{N-m}=i_{N-m-1}+1}^{N}\left(\left(\cdots\left(\left(\left(f_{i_1} e^{\int f_{i_2} dx}\right)' e^{-\int f_{i_2} dx}\right) e^{\int f_{i_3} dx}\right)' e^{-\int f_{i_3} dx}\right)\cdots\right) e^{-\int f_N dx}\right\} y^{(m)} +$$

$$+\Big\{\sum_{i_1=1}^{1+1}\sum_{i_2=i_1+1}^{1+2}\cdots\sum_{i_{N-m-2}=i_{N-m-3}+1}^{N-2}\sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\sum_{i_{N-m}=i_{N-m-1}+1}^{N}\left(\left(\left(\left(\cdots\left(\left(\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right)e^{\int f_{i_3}dx}\right)'e^{-\int f_{i_3}dx}\right)\cdots\right)e^{-\int f_N dx}\right)e^{\int f_{N+1}dx}\right)'e$$

$$+\left(\left(\cdots\left(\left(\left(f_1e^{\int f_2dx}\right)'e^{\int(f_3-f_2)dx}\right)'e^{\int(f_4-f_3)dx}\right)'\cdots\right)'e^{\int(f_N-f_{N-1})dx}\right)e^{-\int f_N dx}\Big\}y'$$

$$+\left[\left(\left(\left(\cdots\left(\left(\left(f_1e^{\int f_2dx}\right)'e^{\int(f_3-f_2)dx}\right)'e^{\int(f_4-f_3)dx}\right)'\cdots\right)'e^{\int(f_N-f_{N-1})dx}\right)'e^{\int(f_{N+1}-f_N)dx}\right)'e^{-\int f_{N+1}dx}\right]y$$

$$= y^{(N+1)}+\left[\sum_{i=1}^{N+1}f_i\right]y^{(N)}+\left[\sum_{i_1=1}^{N}\sum_{i_2=i_1+1}^{N+1}\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right]y^{(N-1)}+$$

$$+\sum_{m=2}^{N-2}\left[\sum_{i_1=1}^{m+1}\sum_{i_2=i_1+1}^{m+2}\cdots\sum_{i_{N-m-2}=i_{N-m-3}+1}^{N-2}\sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\sum_{i_{N-m}=i_{N-m-1}+1}^{N}\left(\left(\cdots\left(\left(\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right)e^{\int f_{i_3}dx}\right)'e^{-\int f_{i_3}dx}\right)\cdots\right)e^{-\int f_N dx}\right]y^{(m)}+$$

$$+\left[\sum_{i_1=1}^{1+1}\sum_{i_2=i_1+1}^{1+2}\cdots\sum_{i_{N-m-2}=i_{N-m-3}+1}^{N-2}\sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\sum_{i_{N-m}=i_{N-m-1}+1}^{N}\left(\left(\cdots\left(\left(\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right)e^{\int f_{i_3}dx}\right)'e^{-\int f_{i_3}dx}\right)\cdots\right)e^{-\int f_N dx}\right]y^{(1)}+$$

$$+\left[\left(\left(\left(\cdots\left(\left(\left(f_1e^{\int f_2dx}\right)'e^{\int(f_3-f_2)dx}\right)'e^{\int(f_4-f_3)dx}\right)'\cdots\right)'e^{\int(f_N-f_{N-1})dx}\right)'e^{\int(f_{N+1}-f_N)dx}\right)'e^{-\int f_{N+1}dx}\right]y$$

$$= y^{(N+1)}+\left[\sum_{i=1}^{N+1}f_i\right]y^{(N)}+\left[\sum_{i_1=1}^{N}\sum_{i_2=i_1+1}^{N+1}\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right]y^{(N-1)}+$$

$$+\sum_{m=1}^{N-2}\left[\sum_{i_1=1}^{m+1}\sum_{i_2=i_1+1}^{m+2}\cdots\sum_{i_{N-m-2}=i_{N-m-3}+1}^{N-2}\sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\sum_{i_{N-m}=i_{N-m-1}+1}^{N}\left(\left(\cdots\left(\left(\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right)e^{\int f_{i_3}dx}\right)'e^{-\int f_{i_3}dx}\right)\cdots\right)e^{-\int f_N dx}\right]y^{(m)}+$$

$$+\left[\left(\left(\left(\cdots\left(\left(\left(f_1e^{\int f_2dx}\right)'e^{\int(f_3-f_2)dx}\right)'e^{\int(f_4-f_3)dx}\right)'\cdots\right)'e^{\int(f_N-f_{N-1})dx}\right)'e^{\int(f_{N+1}-f_N)dx}\right)'e^{-\int f_{N+1}dx}\right]y$$

$$= y^{(N+1)}+\left[\sum_{i=1}^{N+1}f_i\right]y^{((N+1)-1)}+\left[\sum_{i_1=1}^{N}\sum_{i_2=i_1+1}^{N+1}\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right]y^{((N+1)-2)}+$$

$$+\sum_{m=1}^{(N+1)-3}\left[\sum_{i_1=1}^{m+1}\sum_{i_2=i_1+1}^{m+2}\cdots\sum_{i_{N-m-2}=i_{N-m-3}+1}^{N-2}\sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\sum_{i_{N-m}=i_{N-m-1}+1}^{N}\left(\left(\cdots\left(\left(\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right)e^{\int f_{i_3}dx}\right)'e^{-\int f_{i_3}dx}\right)\cdots\right)e^{-\int f_N dx}\right]y^{(m)}+$$

$$+\left[\left(\left(\left(\cdots\left(\left(\left(f_1e^{\int f_2dx}\right)'e^{\int(f_3-f_2)dx}\right)'e^{\int(f_4-f_3)dx}\right)'\cdots\right)'e^{\int(f_N-f_{N-1})dx}\right)'e^{\int(f_{N+1}-f_N)dx}\right)'e^{-\int f_{N+1}dx}\right]y$$

$$= y^{(N+1)}+\left[\sum_{i=1}^{N+1}f_i\right]y^{((N+1)-1)}+$$

$$+\sum_{m=1}^{(N+1)-2}\left[\sum_{i_1=1}^{m+1}\sum_{i_2=i_1+1}^{m+2}\cdots\sum_{i_{N-m-2}=i_{N-m-3}+1}^{N-2}\sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\sum_{i_{N-m}=i_{N-m-1}+1}^{N}\left(\left(\cdots\left(\left(\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right)e^{\int f_{i_3}dx}\right)'e^{-\int f_{i_3}dx}\right)\cdots\right)e^{-\int f_N dx}\right]y^{(m)}+$$

$$+\left[\left(\left(\left(\cdots\left(\left(\left(f_1e^{\int f_2dx}\right)'e^{\int(f_3-f_2)dx}\right)'e^{\int(f_4-f_3)dx}\right)'\cdots\right)'e^{\int(f_N-f_{N-1})dx}\right)'e^{\int(f_{N+1}-f_N)dx}\right)'e^{-\int f_{N+1}dx}\right]y$$

□

**Corollary II.1**: $\prod_{i=1}^{n}(D+f_i)y = 0$

$$= y^{(n)}+\left[\sum_{i=1}^{N+1}f_i\right]y^{(n-1)}+$$

$$+\sum_{m=1}^{n-2}\left[\sum_{i_1=1}^{m+1}\sum_{i_2=i_1+1}^{m+2}\cdots\sum_{i_{N-m-2}=i_{N-m-3}+1}^{N-2}\sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\sum_{i_{N-m}=i_{N-m-1}+1}^{N}\left(\left(\cdots\left(\left(\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right)e^{\int f_{i_3}dx}\right)'e^{-\int f_{i_3}dx}\right)\cdots\right)e^{-\int f_N dx}\right]y^{(m)}+$$

$$+\left[\left(\left(\left(\cdots\left(\left(\left(f_1e^{\int f_2dx}\right)'e^{\int(f_3-f_2)dx}\right)'e^{\int(f_4-f_3)dx}\right)'\cdots\right)'e^{\int(f_N-f_{N-1})dx}\right)'e^{\int(f_{N+1}-f_N)dx}\right)'e^{-\int f_{N+1}dx}\right]y$$

$$\Rightarrow y = e^{-\int f_1dx}\left(\int e^{\int(f_1-f_2)dx}\left(\int e^{\int(f_2-f_3)dx}\left(\cdots\int e^{\int(f_{n-1}-f_{n-1})dx}\left(c_n\int e^{\int(f_{n-1}-f_n)dx}dx+c_{n-1}\right)dx+c_{n-2}\cdots\right)dx+c_2\right)dx+c_1\right)$$

*Proof:*

By corollary I.1 & theorem II.1.

□

**Corollary II.2**: If $\exists f_{i_1}$ :

$$P_m = \begin{cases}\left[\sum_{i_1=1}^{m+1}\sum_{i_2=i_1+1}^{m+2}\cdots\sum_{i_{N-m-2}=i_{N-m-3}+1}^{N-2}\sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\sum_{i_{N-m}=i_{N-m-1}+1}^{N}\left(\left(\cdots\left(\left(\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right)e^{\int f_{i_3}dx}\right)'e^{-\int f_{i_3}dx}\right)\cdots\right)e^{-\int f_N dx}\right], & (m\in\mathbb{N})\\ \left[\sum_{i_1=1}^{m+1}\sum_{i_2=i_1+1}^{m+2}\cdots\sum_{i_{N-m-2}=i_{N-m-3}+1}^{N-2}\sum_{i_{N-m-1}=i_{N-m-2}+1}^{N-1}\sum_{i_{N-m}=i_{N-m-1}+1}^{N}\left(\left(\cdots\left(\left(\left(f_{i_1}e^{\int f_{i_2}dx}\right)'e^{-\int f_{i_2}dx}\right)e^{\int f_{i_3}dx}\right)'e^{-\int f_{i_3}dx}\right)\cdots\right)e^{-\int f_N dx}\right], & (m=0)\end{cases}$$

and: $y^{(n)}+\sum_{m=1}^{n-1}P_my^{(m)}+P_0y = 0$

then:

$$\Rightarrow y = e^{-\int f_1 dx}\left(\int e^{\int (f_1 - f_2)dx}\left(\int e^{\int (f_2 - f_3)dx}\left(\cdots \int e^{\int (f_{n-1} - f_{n-1})dx}\left(c_n \int e^{\int (f_{n-1} - f_n)dx} dx + c_{n-1}\right)dx + c_{n-2} \cdots\right)dx + c_2\right)dx + c_1\right)$$

*Proof:*

By corollary II.1.

□

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