question 465: Some integrals , Lambert function

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abstract

This note presents some definite integrals

keywords: Integrals,Lambert function,number pi,Catalan's constant,Euler-Mascheroni constant.

1. Introduction

Lambert function $W(x)$ is defined by

$$W(x)\, e^{W(x)} = x \tag{1}$$

- Equation (1) has an infinite number of solutions "y" for each value of "x" ($x \neq 0$).
- $W(x)$ has an infinite number of branches.
- $W(k, x)$, k integer , is the kth branch of $W(x)$.
- $W(0, x) = W(x)$ is the principal branch .
- $W(x)$ is analytic at 0.

for details see ref.[B] .

In this note we recall some definite integrals.

2. Some Integrals

Notation :

- The number pi : $\pi = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 3.1415 \ldots$
- Catalan's constant : $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.9159 \ldots$
- Euler – Mascheroni constant : $\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n\right) = 0.5772 \ldots$

$$\frac{(\ln 2)^2}{2} = \int_0^1 \frac{\ln(1+x)}{1+x}\, dx \tag{2}$$

Let $u = \frac{\ln 2}{2}$, then

$$\frac{(\ln 2)^2}{2} - \ln 2 = \int_0^u \frac{W(-x)}{x}\, dx \tag{3}$$

$$\frac{\pi^2}{12} = \int_0^1 \frac{\ln(1+x)}{x}\, dx \tag{4}$$

$$2\ln 2 - 1 - \frac{\pi^2}{12} = \int_{\ln 2}^1 \frac{1}{x} W(-1, -x\, e^{-x})\, dx \tag{5}$$

$$\pi = \int_0^\infty \frac{\ln(1+x^2)}{x^2}\, dx \tag{6}$$

$$\pi = \int_0^1 \sqrt{-1 - \frac{1}{x} W(-1, -x\, e^{-x})}\, dx \tag{7}$$

$$\frac{\pi}{2} - \ln 2 = \int_0^1 \frac{\ln(1+x^2)}{x^2}\, dx \tag{8}$$

$$\frac{\pi}{2} + 2\ln 2 = \int_0^{\ln 2} \sqrt{-1 - \frac{1}{x} W(-1, -x\, e^{-x})}\, dx \tag{9}$$

$$\frac{\pi}{2} + \ln 2 = \int_1^\infty \frac{\ln(1+x^2)}{x^2}\, dx \tag{10}$$

$$\frac{\pi}{2} - 2\ln 2 = \int_{\ln 2}^1 \sqrt{-1 - \frac{1}{x} W(-1, -x\, e^{-x})}\, dx \tag{11}$$

$$\frac{\pi^2}{12} = -\int_0^1 \frac{\ln x}{1+x}\, dx \tag{12}$$

$$\frac{\pi^2}{12} = \int_0^\infty \frac{1}{x} W(x\, e^{-x})\, dx = -\int_0^1 \frac{W(-x \ln x)}{x \ln x}\, dx \tag{13}$$

$$\sqrt{\pi} = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}\, dx \tag{14}$$

$$2\sqrt{\pi} = \int_0^\infty W\left(\frac{2}{x^2}\right) dx \tag{15}$$

$$2\sqrt{2\pi} = \int_0^\infty \frac{W(x)}{x^{3/2}}\, dx \tag{16}$$

$$\frac{\pi^2}{6} + \frac{1}{2} = -\int_0^1 W(-1, -x\, e^{-x})\, dx \tag{17}$$

$$\frac{\pi^2}{3} - 1 = -\int_1^\infty \frac{1}{\sqrt{x}} W\left(-\sqrt{x}\, e^{-\sqrt{x}}\right) dx \tag{18}$$

$$\frac{\pi^2}{6} - \frac{1}{2} = -\int_1^\infty W(-x\, e^{-x})\, dx \tag{19}$$

$$\frac{\pi}{\sqrt{3}} = \int_0^\infty \frac{\ln(1+x^3)}{x^3}\, dx \tag{20}$$

$$\frac{\pi}{\sqrt{3}} = \int_0^1 \sqrt[3]{-1 - \frac{1}{x} W(-1, -x\, e^{-x})}\, dx \tag{21}$$

$$\frac{\pi^2}{8} = \int_0^{\pi/2} \frac{\ln(1 + \sin x)}{\sin x} \, dx \tag{22}$$

$$\frac{\pi^2}{8} - \frac{\pi}{2} \ln 2 = \int_{\ln 2}^{1} \sin^{-1}\left(-1 - \frac{1}{x} W(-1, -x e^{-x})\right) dx \tag{23}$$

Let $u = \dfrac{\ln 2}{\sqrt{2}}$, then

$$\frac{\pi \ln 2}{2\sqrt{2}} - \frac{\ln 2}{16\sqrt{\pi}} \left(\Gamma\left(\frac{1}{4}\right)\right)^2 = \int_0^u \sin^{-1}\left(\sqrt{2 - 2\left(\frac{W(x)}{x}\right)^2}\right) dx \tag{24}$$

where $\Gamma(x)$ is the Gamma function.

$$\frac{3}{2}\left(\sqrt[3]{4} - 1\right) \Gamma\left(\frac{1}{3}\right) = \int_0^{\infty} \frac{e^{-x} - e^{-2x}}{x^{5/3}} \, dx \tag{25}$$

$$\frac{9}{10}\left(\sqrt[3]{4} - 1\right) \Gamma\left(\frac{1}{3}\right) = \int_0^{\infty} \left(W\left(\frac{3}{5} x^{-3/5}\right) - \frac{1}{2} W\left(\frac{6}{5} x^{-3/5}\right)\right) dx \tag{26}$$

Let $u = \dfrac{1}{\sqrt{2e}}$, then

$$\sqrt{\pi} = \int_0^u \left(W\left(0, -2x^2\right) - W\left(-1, -2x^2\right)\right) dx \tag{27}$$

$$\sqrt{\pi} = \frac{1}{2\sqrt{2}} \int_e^{\infty} \frac{1}{x^{3/2}} \ln(x \ln(x \ln(x \ldots))) \, dx - 2 \int_0^u x^2 e^{2x^2 e^{2x^2 e^{2x^2 \cdots}}} \, dx \tag{28}$$

$$\frac{\pi}{\sqrt{3}} \ln 3 - \frac{\pi^2}{9} = \int_0^{\infty} \frac{\ln(1 + x^3)}{1 + x^3} \, dx \tag{29}$$

$$\frac{\pi}{\sqrt{3}} \ln 3 - \frac{\pi^2}{9} = \int_0^{e^{-1}} \left(\sqrt[3]{-1 - \frac{1}{x} W(-1, -x)} - \sqrt[3]{-1 - \frac{1}{x} W(0, -x)}\right) dx \tag{30}$$

$$G = -\int_0^1 \frac{\ln x}{1 + x^2} \, dx \tag{31}$$

$$2G = \int_0^{\infty} \sqrt{\frac{1}{x} W(x e^{-x})} \, dx \tag{32}$$

$$G = \int_1^{\infty} \frac{\ln x}{1 + x^2} \, dx \tag{33}$$

Let $u = W\left(e^{-1}\right) = e^{-1-e^{-1-e^{-1-\cdots}}}$, then

$$2G = \int_0^u \left(\sqrt{-\frac{1}{x} W(-1, -x e^{x})} - \sqrt{-\frac{1}{x} W(0, -x e^{x})}\right) dx \tag{34}$$

$$\frac{\pi}{2} \ln 2 - G = \int_0^1 \frac{\ln(1 + x^2)}{1 + x^2} \, dx \tag{35}$$

Let $u = \frac{\ln 2}{2}$, then

$$G + \frac{\ln 2}{2} - \frac{\pi}{2} \ln 2 = \int_0^u \sqrt{-1 - \frac{1}{x} W(-x)} \, dx \tag{36}$$

$$G = -\int_0^1 \frac{1}{1+x^2} \ln\left(\frac{1-x^2}{2}\right) dx \tag{37}$$

$$G - \ln 2 = \int_{\ln 2}^{\infty} \left(1 - \sqrt{1 + \frac{1}{x} W\left(-2\, x\, e^{-2\, x}\right)}\right) dx \tag{38}$$

$$G = \frac{3}{2} \int_{2+\sqrt{3}}^{\infty} \frac{\ln x}{1+x^2} \, dx \tag{39}$$

Let $u = \frac{\ln\left(2+\sqrt{3}\right)}{2\left(2+\sqrt{3}\right)}$, then

$$\frac{4}{3} G + \frac{\ln\left(2+\sqrt{3}\right)}{2} = \int_0^u \sqrt{-\frac{1}{x} W(-1, -x\, e^{x})} \, dx \tag{40}$$

$$\gamma = e^{-1} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\, n!} - \int_0^{e^{-1}} W\left(\frac{1}{x}\right) dx \tag{41}$$

$$\gamma = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\, n!} - 2 \int_0^{e^{-1}} \left(W\left(\frac{1}{x}\right) - \sqrt{\frac{1}{2} W\left(\frac{2}{x^2}\right)}\right) dx \tag{42}$$

References

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