Weighted Tribonacci sums*

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Abstract

We derive various weighted summation identities, including binomial and double binomial identities, for Tribonacci numbers. Our results contain some previously known results as special cases.

1 Introduction

For $m \geq 3$, the Tribonacci numbers are defined by

$$T_m = T_{m-1} + T_{m-2} + T_{m-3}\,, \quad T_0 = 0,\, T_1 = T_2 = 1\,. \tag{1.1}$$

By writing $T_{m-1} = T_{m-2} + T_{m-3} + T_{m-4}$ and eliminating T_{m-2} and T_{m-3} between this recurrence relation and the recurrence relation (1.1), a useful alternative recurrence relation is obtained for $m \geq 4$:

$$T_m = 2T_{m-1} - T_{m-4}\,, \quad T_0 = 0\,, \quad T_1 = T_2 = 1\,, \quad T_3 = 2\,. \tag{1.2}$$

Extension of the definition of T_m to negative subscripts is provided by writing the recurrence relation (1.2) as

$$T_{-m} = 2T_{-m+3} - T_{-m+4}\,. \tag{1.3}$$

Anantakitpaisal and Kuhapatanakul [2] proved that

$$T_{-m} = {T_{m-1}}^2 - T_{m-2}T_m. \tag{1.4}$$

The following identity (Feng [3], equation (3.3); Shah [7], (ii)) is readily established by the principle of mathematical induction:

$$T_{m+r} = T_rT_{m-2} + (T_{r-1} + T_r)T_{m-1} + T_{r+1}T_m\,. \tag{1.5}$$

Irmak and Alp [5] derived the following identity for Tribonacci numbers with indices in arithmetic progression:

$$T_{tm+r} = \lambda_1(t)T_{t(m-1)+r} + \lambda_2(t)T_{t(m-2)+r} + \lambda_3(t)T_{t(m-3)+r}\,, \tag{1.6}$$

*AMS Classification: 11B37, 11B39, 65B10

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where,

$$\lambda_1(t) = \alpha^t + \beta^t + \gamma^t, \quad \lambda_2(t) = -(\alpha\beta)^t - (\alpha\gamma)^t - (\beta\gamma)^t, \quad \lambda_3(t) = (\alpha\beta\gamma)^t\,,$$

where α, β and γ are the roots of the characteristic polynomial of the Tribonacci sequence $x^3 - x^2 - x - 1$. Thus,

$$\alpha = \frac{1}{3}\left(1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}\right)\,,$$

$$\beta = \frac{1}{3}\left(1 + \omega\sqrt[3]{19 + 3\sqrt{33}} + \omega^2\sqrt[3]{19 - 3\sqrt{33}}\right)$$

and

$$\gamma = \frac{1}{3}\left(1 + \omega^2\sqrt[3]{19 + 3\sqrt{33}} + \omega\sqrt[3]{19 - 3\sqrt{33}}\right)\,,$$

where $\omega = \exp(2i\pi/3)$ is a primitive cube root of unity. Note that $\lambda_1(t)$, $\lambda_2(t)$ and $\lambda_3(t)$ are integers for any positive integer t [5]; in particular, $\lambda_1(1) = 1 = \lambda_2(1) = \lambda_3(1)$.

2 Weighted sums

Lemma 1 ([1], Lemma 2)**.** *Let $\{X_m\}$ be any arbitrary sequence, where X_m, $m \in \mathbb{Z}$, satisfies a second order recurrence relation $X_m = f_1 X_{m-a} + f_2 X_{m-b}$, where f_1 and f_2 are arbitrary non-vanishing complex functions, not dependent on m, and a and b are integers. Then,*

$$f_2 \sum_{j=0}^{k} \frac{X_{m-ka-b+aj}}{f_1^j} = \frac{X_m}{f_1^k} - f_1 X_{m-(k+1)a}\,, \tag{2.1}$$

$$f_1 \sum_{j=0}^{k} \frac{X_{m-kb-a+bj}}{f_2^j} = \frac{X_m}{f_2^k} - f_2 X_{m-(k+1)b} \tag{2.2}$$

and

$$\sum_{j=0}^{k} \frac{X_{m-(b-a)k+a+(b-a)j}}{(-f_2/f_1)^j} = \frac{f_1 X_m}{(-f_2/f_1)^k} + f_2 X_{m-(k+1)(b-a)}\,. \tag{2.3}$$

for k a non-negative integer.

Theorem 1. *The following identities hold for any integers m and k:*

$$\sum_{j=0}^{k} 2^{-j} T_{m-k-4+j} = 2T_{m-k-1} - 2^{-k} T_m\,, \tag{2.4}$$

$$2\sum_{j=0}^{k} (-1)^j T_{m-4k-1+4j} = (-1)^k T_m + T_{m-4k-4} \tag{2.5}$$

and

$$\sum_{j=0}^{k} 2^j T_{m-3k+1+3j} = 2^{k+1} T_m - T_{m-3k-3}\,. \tag{2.6}$$

Proof. From the recurrence relation (1.2), make the identifications $f_1 = 2$, $f_2 = -1$, $a = 1$ and $b = 4$ and use these in Lemma 1 with $X = T$. $\square$

Particular instances of identities (2.4)–(2.6) are the following identities:

$$\sum_{j=0}^{k} 2^{-j} T_j = 4 - 2^{-k} T_{k+4}\,, \tag{2.7}$$

giving,

$$\sum_{j=0}^{\infty} 2^{-j} T_j = 4\,, \tag{2.8}$$

and

$$2 \sum_{j=0}^{k} (-1)^j T_{4j} = (-1)^k T_{4k+1} - 1 \tag{2.9}$$

and

$$\sum_{j=0}^{k} 2^j T_{3j} = 2^{k+1} T_{3k-1}\,. \tag{2.10}$$

Lemma 2 (Partial sum of an n^{th} order sequence)**.** *Let $\{X_j\}$ be any arbitrary sequence, where X_j, $j \in \mathbb{Z}$, satisfies a n^{th} order recurrence relation $X_j = f_1 X_{j-c_1} + f_2 X_{j-c_2} + \cdots + f_n X_{j-c_n} = \sum_{m=1}^{n} f_m X_{j-c_m}$, where f_1, f_2, $\ldots$, f_n are arbitrary non-vanishing complex functions, not dependent on j, and c_1, c_2, $\ldots$, c_n are fixed integers. Then, the following summation identity holds for arbitrary x and non-negative integer k :*

$$\sum_{j=0}^{k} x^j X_j = \frac{\sum_{m=1}^{n} \left\{ x^{c_m} f_m \left(\sum_{j=1}^{c_m} x^{-j} X_{-j} - \sum_{j=k-c_m+1}^{k} x^j X_j \right) \right\}}{1 - \sum_{m=1}^{n} x^{c_m} f_m}\,.$$

Proof. Recurrence relation:

$$X_j = \sum_{m=1}^{n} f_m X_{j-c_m}\,.$$

We multiply both sides by x^j and sum over j to obtain

$$\sum_{j=0}^{k} x^j X_j = \sum_{m=1}^{n} \left(f_m \sum_{j=0}^{k} x^j X_{j-c_m} \right) = \sum_{m=1}^{n} \left(x^{c_m} f_m \sum_{j=-c_m}^{k-c_m} x^j X_j \right),$$

after shifting the summation index j. Splitting the inner sum, we can write

$$\sum_{j=0}^{k} x^j X_j = \sum_{m=1}^{n} x^{c_m} f_m \left(\sum_{j=-c_m}^{-1} x^j X_j + \sum_{j=0}^{k} x^j X_j + \sum_{j=k+1}^{k-c_m} x^j X_j \right).$$

Since

$$\sum_{j=-c_m}^{-1} x^j X_j \equiv \sum_{j=1}^{c_m} x^{-j} X_{-j} \quad \text{and} \quad \sum_{j=k+1}^{k-c_m} x^j X_j \equiv - \sum_{j=k-c_m+1}^{k} x^j X_j\,,$$

the preceding identity can be written

$$\sum_{j=0}^{k} x^j X_j = \sum_{m=1}^{n} x^{c_m} f_m \left(\sum_{j=1}^{c_m} x^{-j} X_{-j} + \sum_{j=0}^{k} x^j X_j - \sum_{j=k-c_m+1}^{k} x^j X_j \right) .$$

Thus, we have

$$S = \sum_{m=1}^{n} x^{c_m} f_m \left(\sum_{j=1}^{c_m} x^{-j} X_{-j} + S - \sum_{j=k-c_m+1}^{k} x^j X_j \right) ,$$

where

$$S = S_k(x) = \sum_{j=0}^{k} x^j X_j \, .$$

Removing brackets, we have

$$S = \sum_{m=1}^{n} x^{c_m} f_m \left(\sum_{j=1}^{c_m} x^{-j} X_{-j} - \sum_{j=k-c_m+1}^{k} x^j X_j \right) + S \sum_{m=1}^{n} x^{c_m} f_m \, ,$$

from which the result follows by grouping the S terms. □

Lemma 3 (Generating function)**.** *Under the conditions of Lemma 2, if additionally $x^k X_k$ vanishes in the limit as k approaches infinity, then*

$$S_\infty(x) = \sum_{j=0}^{\infty} x^j X_j = \frac{\sum_{m=1}^{n} \left(x^{c_m} f_m \sum_{j=1}^{c_m} x^{-j} X_{-j} \right)}{1 - \sum_{m=1}^{n} x^{c_m} f_m} \, ,$$

so that $S_\infty(x)$ is a generating function for the sequence $\{X_j\}$.

Theorem 2 (Sum of Tribonacci numbers with indices in arithmetic progression)**.** *For arbitrary x, any integers t and r and any non-negative integer k, the following identity holds:*

$$\begin{aligned}\left(1 - \lambda_1(t)x - \lambda_2(t)x^2 - \lambda_3(t)x^3\right) \sum_{j=0}^{k} x^j T_{tj+r} = T_r &+ (x\lambda_2(t) + x^2\lambda_3(t))T_{r-t} \\ &+ x\lambda_3(t)T_{r-2t} - x^{k+1}T_{(k+1)t+r} \\ &- x^{k+2}(\lambda_2(t) + x\lambda_3(t))T_{kt+r} \\ &- x^{k+2}\lambda_3(t)T_{(k-1)t+r} \, ,\end{aligned}$$

where,

$$\lambda_1(t) = \alpha^t + \beta^t + \gamma^t, \quad \lambda_2(t) = -(\alpha\beta)^t - (\alpha\gamma)^t - (\beta\gamma)^t, \quad \lambda_3(t) = (\alpha\beta\gamma)^t \, ,$$

where α, β and γ are the roots of the characteristic polynomial of the Tribonacci sequence $x^3 - x^2 - x - 1$.

Proof. Write identity (1.6) as $X_j = f_1 X_{j-1} + f_2 X_{j-2} + f_3 X_{j-3}$ and identify the sequence $\{X_j\} = \{T_{tj+r}\}$ and the constants $c_1 = 1$, $c_2 = 2$, $c_3 = 3$ and the functions $f_1 = \lambda_1(t)$, $f_2 = \lambda_2(t)$, $f_3 = \lambda_3(t)$, and use these in Lemma 2.

□

Corollary 3 (Generating function of the Tribonacci numbers with indices in arithmetic progression)**.** *For any integers t and r, any non-negative integer k and arbitrary x for which x^kT_k vanishes as k approaches infinity , the following identity holds:*

$$\sum_{j=0}^{\infty} x^j T_{tj+r} = \frac{T_r + (x\lambda_2 + x^2\lambda_3)T_{r-t} + x\lambda_3 T_{r-2t}}{1 - \lambda_1 x - \lambda_2 x^2 - \lambda_3 x^3}\,,$$

where,

$$\lambda_1 = \alpha^t + \beta^t + \gamma^t, \quad \lambda_2 = -(\alpha\beta)^t - (\alpha\gamma)^t - (\beta\gamma)^t, \quad \lambda_3 = (\alpha\beta\gamma)^t\,,$$

where α, β and γ are the roots of the characteristic polynomial of the Tribonacci sequence $x^3 - x^2 - x - 1$.

Many instances of Theorem 2 may be explored. In particular, we have

$$\begin{aligned}(\lambda_1(t) + \lambda_2(t) + \lambda_3(t) - 1)\sum_{j=0}^{k} T_{tj+r} &= -T_r - (\lambda_2(t) + \lambda_3(t))T_{r-t}\\ &\quad - \lambda_3(t)T_{r-2t} + T_{(k+1)t+r}\\ &\qquad + (\lambda_2(t) + \lambda_3(t))T_{kt+r} + \lambda_3(t)T_{(k-1)t+r}\,,\end{aligned} \tag{2.11}$$

which at $r = 0$ gives

$$\begin{aligned}(\lambda_1(t) + \lambda_2(t) + \lambda_3(t) - 1)\sum_{j=0}^{k} T_{tj} &= -(\lambda_2(t) + \lambda_3(t))(T_{t-1}^2 - T_{t-2}T_t)\\ &\quad - \lambda_3(t)(T_{2t-1}^2 - T_{2t-2}T_{2t}) + T_{(k+1)t}\\ &\qquad + (\lambda_2(t) + \lambda_3(t))T_{kt} + \lambda_3(t)T_{(k-1)t}\,;\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}(1 + \lambda_1(t) - \lambda_2(t) + \lambda_3(t))\sum_{j=0}^{k} (-1)^j T_{tj+r} &= T_r + (\lambda_3(t) - \lambda_2(t))T_{r-t}\\ &\quad - \lambda_3(t)T_{r-2t} + (-1)^k T_{(k+1)t+r}\\ &\qquad + (-1)^k(\lambda_3(t) - \lambda_2(t))T_{kt+r}\\ &\qquad - (-1)^k\lambda_3(t)T_{(k-1)t+r}\,,\end{aligned} \tag{2.13}$$

which at $r = 0$ gives

$$\begin{aligned}(1 + \lambda_1(t) - \lambda_2(t) + \lambda_3(t))\sum_{j=0}^{k} (-1)^j T_{tj} &= (\lambda_3(t) - \lambda_2(t))(T_{t-1}^2 - T_{t-2}T_t)\\ &\quad - \lambda_3(t)(T_{2t-1}^2 - T_{2t-2}T_{2t}) + (-1)^k T_{(k+1)t}\\ &\quad + (-1)^k(\lambda_3(t) - \lambda_2(t))T_{kt}\\ &\quad - (-1)^k\lambda_3(t)T_{(k-1)t}\,.\end{aligned} \tag{2.14}$$

Many previously known results are particular instances of the identities (2.11) and (2.13). For example, Theorem 5 of [6] is obtained from identity (2.12) by setting $t = 4$. Sums of

Tribonacci numbers with indices in arithmetic progression are also discussed in references [4, 5, 6] and references therein, using various techniques.

Weighted sums of the form $\sum_{j=0}^{k} j^p T_{tj+r}$, where p is a non-negative integer, may be evaluated by setting $x = e^y$ in the identity of Theorem 2, differentiating both sides p times with respect to y and then setting $y = 0$. The simplest examples in this category are the following:

$$\begin{aligned} 2\sum_{j=0}^{k} jT_{j+r} &= -T_{r-2} + 3T_{r+1} + (k-1)T_{k+r-1} \\ &\qquad + (2k-1)T_{k+r} + (k-2)T_{k+r+1} \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} 2\sum_{j=0}^{k} j^2 T_{j+r} &= -3T_{r-1} - 5T_r - 6T_{r+1} \\ &\qquad + (k^2 - 2k + 3)T_{k+r-1} + (2k^2 - 2k + 5)T_{k+r} \\ &\qquad + (k^2 - 4k + 6)T_{k+r+1}\,, \end{aligned} \tag{2.16}$$

with the particular cases

$$2\sum_{j=0}^{k} jT_j = 2 + (k-1)T_{k-1} + (2k-1)T_k + (k-2)T_{k+1} \tag{2.17}$$

and

$$\begin{aligned} 2\sum_{j=0}^{k} j^2 T_j &= -6 + (k^2 - 2k + 3)T_{k+r-1} \\ &\qquad + (2k^2 - 2k + 5)T_k \\ &\qquad + (k^2 - 4k + 6)T_{k+1}\,. \end{aligned} \tag{2.18}$$

3 Weighted binomial sums

Lemma 4 ([1], Lemma 3)**.** *Let $\{X_m\}$ be any arbitrary sequence. Let X_m, $m \in \mathbb{Z}$, satisfy a second order recurrence relation $X_m = f_1 X_{m-a} + f_2 X_{m-b}$, where f_1 and f_2 are non-vanishing complex functions, not dependent on m, and a and b are integers. Then,*

$$\sum_{j=0}^{k} \binom{k}{j} \left(\frac{f_1}{f_2}\right)^j X_{m-bk+(b-a)j} = \frac{X_m}{f_2^k}\,, \tag{3.1}$$

$$\sum_{j=0}^{k} \binom{k}{j} \frac{X_{m+(a-b)k+bj}}{(-f_2)^j} = \left(-\frac{f_1}{f_2}\right)^k X_m \tag{3.2}$$

and

$$\sum_{j=0}^{k} \binom{k}{j} \frac{X_{m+(b-a)k+aj}}{(-f_1)^j} = \left(-\frac{f_2}{f_1}\right)^k X_m\,, \tag{3.3}$$

for k a non-negative integer.

Theorem 4. *The following identities hold for any integer m and any non-negative integer k:*

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}2^jT_{m-4k+3j}=(-1)^kT_m\,, \tag{3.4}$$

$$\sum_{j=0}^{k}\binom{k}{j}T_{m-3k+4j}=2^kT_m \tag{3.5}$$

and

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}2^{-j}T_{m+3k+j}=2^{-k}T_m\,. \tag{3.6}$$

Proof. Identify $X=T$ in Lemma 4 and use the f_1, f_2, a and b values found in the proof of Theorem 1. □

Particular cases of (3.4), (3.5) and (3.6) are the following identities:

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}2^jT_{3j}=(-1)^kT_{4k}\,, \tag{3.7}$$

$$\sum_{j=0}^{k}\binom{k}{j}T_{4j}=2^kT_{3k} \tag{3.8}$$

and

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}2^{-j}T_j=2^{-k}(T_{3k-1}^2-T_{3k-2}T_{3k})\,. \tag{3.9}$$

4 Weighted double binomial sums

Lemma 5. *Let $\{X_m\}$ be any arbitrary sequence, X_m satisfying a third order recurrence relation $X_m=f_1X_{m-a}+f_2X_{m-b}+f_3X_{m-c}$, where f_1, f_2 and f_3 are arbitrary nonvanishing functions and a, b and c are integers. Then, the following identities hold:*

$$\sum_{j=0}^{k}\sum_{s=0}^{j}\binom{k}{j}\binom{j}{s}\left(\frac{f_2}{f_3}\right)^j\left(\frac{f_1}{f_2}\right)^sX_{m-ck+(c-b)j+(b-a)s}=\frac{X_m}{{f_3}^k}\,, \tag{4.1}$$

$$\sum_{j=0}^{k}\sum_{s=0}^{j}\binom{k}{j}\binom{j}{s}\left(\frac{f_3}{f_2}\right)^j\left(\frac{f_1}{f_3}\right)^sX_{m-bk+(b-c)j+(c-a)s}=\frac{X_m}{{f_2}^k}\,, \tag{4.2}$$

$$\sum_{j=0}^{k}\sum_{s=0}^{j}\binom{k}{j}\binom{j}{s}\left(\frac{f_3}{f_1}\right)^j\left(\frac{f_2}{f_3}\right)^sX_{m-ak+(a-c)j+(c-b)s}=\frac{X_m}{{f_1}^k}\,, \tag{4.3}$$

$$\sum_{j=0}^{k}\sum_{s=0}^{j}\binom{k}{j}\binom{j}{s}\left(\frac{f_2}{f_3}\right)^j\left(-\frac{1}{f_2}\right)^sX_{m-(c-a)k+(c-b)j+bs}=\left(-\frac{f_1}{f_3}\right)^kX_m\,, \tag{4.4}$$

$$\sum_{j=0}^{k}\sum_{s=0}^{j}\binom{k}{j}\binom{j}{s}\left(\frac{f_1}{f_3}\right)^j\left(-\frac{1}{f_1}\right)^sX_{m-(c-b)k+(c-a)j+as}=\left(-\frac{f_2}{f_3}\right)^kX_m\,, \tag{4.5}$$

and

$$\sum_{j=0}^{k}\sum_{s=0}^{j}\binom{k}{j}\binom{j}{s}\left(\frac{f_1}{f_2}\right)^j\left(-\frac{1}{f_1}\right)^s X_{m-(b-c)k+(b-a)j+as}=\left(-\frac{f_3}{f_2}\right)^k X_m\,. \tag{4.6}$$

Proof. Only identity (4.1) needs to be proved as identities (4.2)–(4.6) are obtained from (4.1) by re-arranging the recurrence relation. The proof of (4.1) is by induction on k, similar to the proof of Lemma 3 of [1]. □

Theorem 5. *The following identities hold for non-negative integer k, integer m and integer $r\notin\{-17,-4,-1,0\}$:*

$$\sum_{j=0}^{k}\sum_{s=0}^{j}\binom{k}{j}\binom{j}{s}(T_{r-1}+T_r)^{j-s}\frac{T_{r+1}^s}{T_r^j}T_{m-(r+2)k+j+s}=\frac{T_m}{T_r^k}\,, \tag{4.7}$$

$$\sum_{j=0}^{k}\sum_{s=0}^{j}\binom{k}{j}\binom{j}{s}\frac{T_r^{j-s}T_{r+1}^s}{(T_{r-1}+T_r)^j}T_{m-(r+1)k-j+2s}=\frac{T_m}{(T_{r-1}+T_r)^k}\,, \tag{4.8}$$

$$\sum_{j=0}^{k}\sum_{s=0}^{j}\binom{k}{j}\binom{j}{s}\frac{T_{r-1}^{j-s}(T_{r-2}+T_{r-1})^s}{T_r^j}T_{m-(r-1)k-2j+s}=\frac{T_m}{T_r^k}\,, \tag{4.9}$$

$$\sum_{j=0}^{k}\sum_{s=0}^{j}(-1)^s\binom{k}{j}\binom{j}{s}\frac{(T_{r-1}+T_r)^{j-s}}{T_r^j}T_{m-2k+j+(r+1)s}=(-1)^k\left(\frac{T_{r+1}}{T_r}\right)^k T_m\,, \tag{4.10}$$

$$\sum_{j=0}^{k}\sum_{s=0}^{j}(-1)^s\binom{k}{j}\binom{j}{s}\frac{T_{r+1}^{j-s}}{T_r^j}T_{m-k+2j+rs}=(-1)^k\left(\frac{T_{r-1}+T_r}{T_r}\right)^k T_m \tag{4.11}$$

and

$$\sum_{j=0}^{k}\sum_{s=0}^{j}(-1)^s\binom{k}{j}\binom{j}{s}\frac{T_{r+1}^{j-s}}{(T_{r-1}+T_r)^j}T_{m+k+j+rs}=(-1)^k\left(\frac{T_r}{T_{r-1}+T_r}\right)^k T_m\,. \tag{4.12}$$

Proof. Write the identity (1.5) as $T_m=T_rT_{m-r-2}+(T_{r-1}+T_r)T_{m-r-1}+T_{r+1}T_{m-r}$, identify $f_1=T_r$, $f_2=T_{r-1}+T_r$, $f_3=T_{r+1}$, $a=r+2$, $b=r+1$, $c=r$ and use these in Lemma 5 with $X=T$. □

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