The time scale of gravitational collapse

Trevor W. Marshall
Buckingham Centre for Astrobiology, University of Buckingham, Buckingham MK18 1EG, UK.

Abstract In a previous article it was shown that the end state for the dust metric of Oppenheimer and Snyder has most of its mass concentrated just inside the gravitational radius; it is proposed that the resulting object be considered as an idealized *shell collapsar*. Here the treatment is extended to include the family of interior metrics described by Weinberg, and involving the curvature parameter of a Friedmann metric. The end state is again a shell collapsar, with a shell which becomes more concentrated as the curvature parameter increases, which shows that the details of the shell structure are dependent on the initial density profile at the beginning of the collapse. What is lacking in most previous commentaries on the Oppenheimer-Snyder article is the recognition that their matching of the time coordinate at the surface implies a finite upper limit for the comoving time coordinate. A collapse process having all the matter going inside the gravitational radius would require comoving times which go outside that limit.

1 Introduction

Since the inception of General Relativity (GR), solutions have been sought for the evolution of a mass distribution under its own gravity. The first attempt at an *equilibrium* GR solution was the uniform density of Schwarzschild[1] in 1916, but progress on "the problem of motion"[2] came very slowly. The first time-dependent solutions, with the idealized equation of state $p = 0$, were those of Tolman[3]; for simplicity, especially because of the absence of gravitational waves, the Tolman solutions were all spherosymmetric. Models based on this Tolman solution are known as *dust models.*

The 1939 article of Oppenheimer and Snyder[4] (OS) was a particular Tolman solution and played a central role in the birth of the black hole, especially because it was used as a basis for Penrose's[5] Theorem, which states that if certain conditions hold, the end state of a gravitational collapse must be a point singularity having infinite density. Penrose claimed that the OS dust metric, which at that time was the only known solution of the time-dependent field equations, satisfied those conditions. However, it has now been demonstrated[6] that the OS metric, describing collapse from an initially uniform density, has an end state quite different from that described by Penrose; the end state of OS is a shell with most of the dust material concentrated just inside the gravitational radius.

A set of metrics described by Weinberg[7] has made it possible to extend the OS family to nonuniform initial states, and another such family was described more recently by Choquet-Bruhat[8]. The latter author came to the same conclusion as Penrose regarding the end state of OS, and went on to generalize that conclusion to the extended family[8], but Weinberg[7] preserved the analysis of the OS article, and I shall show below that the family described in his article has a similar end state to OS. The part of OS which was not properly implemented, by either Penrose or Choquet-Bruhat, is the mapping by OS, from the comoving coordinates used to describe the interior of the collapsar, onto the exterior Schwarzschild coordinates. OS glued together these two metrics by imposing continuity conditions at the surface. It would seem that the only treatment, since the black-hole era, which maintains the continuity conditions of OS is that of Weinberg[7]. On the whole the OS analysis has been forgotten.

The greater part of the mass of the OS collapsar is concentrated just inside the gravitational radius; in the limit $t \to +\infty$ the density at the surface, like that at the centre of a black hole, becomes infinite, but with this more extended shell version of collapse it is possible that a real collapsar with a nontrivial equation of state will be a smeared out version of an OS shell, a possibility which we discussed in two previous articles[6][9]. Such collapsars may be considered as a revival of the *frozen stars*[10][11] discussed as early alternatives to black holes, a more recent version of which is the *gravastar*[12].

2 The time according to Oppenheimer-Snyder

The Schwarzschild metric describing the space outside a spherosymmetric object is

$$ds^2 = \frac{r - r_0}{r} dt^2 - \frac{r}{r - r_0} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \quad , \tag{1}$$

where, in units $G = c = 1$, the Newtonian far field is $-r_0/2r^2 = -m/r^2$. A test mass starting at rest at $r = \infty$ falls towards the gravitational radius r_0 with speed

$$v = -\frac{dr}{dt} = \frac{r - r_0}{r} \sqrt{\frac{r_0}{r}} \quad , \tag{2}$$

which grows monotonically to $v \approx 0.38c$ at $r = 3r_0$ and then decreases to zero at r_0. Integrating to find $r(t)$, we find

$$t = r_0 \left(-\frac{2}{3} y^{3/2} - 2\sqrt{y} + \ln \frac{\sqrt{y} + 1}{\sqrt{y} - 1} \right) + \text{const.} \quad , \tag{3}$$

where the *cotime* y decreases from $+\infty$ at $t = -\infty$ to 1 at $t = +\infty$. Hence a test mass starting at any finite radius requires an infinite time to reach r_0; we have here the description of an apple, such as was observed by Newton, falling to the ground, where in this case "the ground" is the surface of a completely collapsed object. This result was established in the OS article.

For the interior region $R < R_b$, the OS metric is, using the same units,

$$ds^2 = d\tau^2 - \frac{r^2}{R^2} dR^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \quad , \tag{4}$$

where

$$r = R\left(1 - \frac{3\tau}{2}\sqrt{\frac{r_0}{R_b^3}}\right)^{2/3} \quad , \tag{5}$$

and the *synchronous time* τ is given the value $\tau = 0$ when $r = R_b > r_0$. The OS coordinates are comoving, that is the freefall geodesics are simply $R =$ const., $\theta =$ const., $\phi =$ const. The only nonzero component of the stress tensor is

$$T^{\tau\tau} = -\frac{1}{4\pi}\left[\frac{1}{\sqrt{g_{\theta\theta}}}\frac{\partial^2}{\partial\tau^2}\sqrt{g_{\theta\theta}} + \frac{2}{\sqrt{g_{RR}}}\frac{\partial^2}{\partial\tau^2}\sqrt{g_{RR}}\right] = \frac{3R^3 r_0}{8\pi R_b^3 r^3} \quad , \tag{6}$$

and hence

$$T^{\tau\tau}\sqrt{-g} = \frac{3r_0 R^2 \sin\theta}{8\pi R_b^3} \quad (R < R_b) \quad . \tag{7}$$

OS imposed continuity of the metric at the surface $R = R_b$ by relating the cotime y in the interior (see [4] equation (36), which I shall designate (OS36)) to R

$$y = \frac{rR_b}{r_0 R} + \frac{1}{2}\left(\frac{R^2}{R_b^2} - 1\right) = \frac{R_b}{r_0}\left(1 - \frac{3\tau}{2}\sqrt{\frac{r_0}{R_b^3}}\right)^{2/3} + \frac{1}{2}\left(\frac{R^2}{R_b^2} - 1\right) \quad . \tag{8}$$

The interior OS metric, in the coordinates (t, r, θ, ϕ), is then

$$\begin{aligned} ds^2 = & \frac{R_b r^2 (y-1)^2}{r_0 R(r - r_0 R^3/R_b^3) y^3} dt^2 - \frac{r}{r - r_0 R^3/R_b^3} dr^2 \\ & - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad , \end{aligned} \tag{9}$$

where $R(r, y)$ is the inverse function of (8) and is parametrically defined, for $0 \leq 2\sin\theta \leq \sqrt{3/(2y+1)}$, as

$$R = 2R_b\left(\frac{2y+1}{3}\right)^{1/2}\sin\theta, \quad r = r_0\left(\frac{2y+1}{3}\right)^{3/2}\sin 3\theta \quad . \tag{10}$$

The metrics (1) and (9) are now manifestly continuous at $R = R_b$ where $y = r/r_0$, and this completes the glueing of the interior to the exterior metric.

By (8) the limiting value of $y = 1$, corresponding to $t \to +\infty$, is

$$\tau = \tau_0(R) = \frac{2}{3}\sqrt{\frac{R_b^3}{r_0}} - \frac{r_0}{3\sqrt{2}}\left(3 - \frac{R^2}{R_b^2}\right)^{3/2} \quad . \tag{11}$$

This result is equivalent to (OS37), which led to a main conclusion of their OS article[4], namely

> From this relation we see that for a fixed value of R as t tends to infinity, τ tends to a finite value τ_0 which increases with R.

For given R, there is no time t corresponding to τ later than $\tau_0(R)$, which means that synchronous times later than $\tau_0(R)$ fall outside physical time, or one could say they are "beyond infinity".

3 The Penrose version of OS

From his interpretation of [4], Penrose[5] deduced that

> The general situation with regard to a spherically symmetrical body is well known[4]. For a sufficiently great mass, there is no final equilibrium state. When sufficient thermal energy has been radiated away, the body contracts and continues to contract until a physical singularity is encountered at $r = 0$. As measured by local comoving observers, the body passes within its Schwarzschild radius $r = r_0$.

I emphasize that Penrose described no collapse process apart from that of OS, and did not give any indication that his interpretation differs from that of OS themselves. Choquet-Bruhat repeated the Penrose analysis, concluding similarly[8] that

> If the density[1] is uniform (Oppenheimer-Snyder case), the dust shells all arrive at the same time at the centre.

Both of these statements are in contradiction with the OS analysis of the previous section; they ignore the continuity requirement of OS, and the consequent restriction in the range of τ. The source of the contradiction is shown in the second of these quotations; Choquet-Bruhat is clearly referring to the relation (5), carrying the time τ through to

$$\tau_0 = \frac{2}{3}\sqrt{\frac{R_b^3}{r_0}} \quad , \tag{12}$$

which, according to (OS37), that is (11), lies outside the physical range of τ.

Yet, paradoxically, Penrose, as recently as November 2016, continues to say[14] that the OS article was the "first description of a black hole". I would say, on the contrary, that OS, together with some generalizations described in the next but one section, are *the only* full descriptions we have of a collapse process (as distinct from an alleged collapsed final state). So the fact that OS and its generalizations have no singularity should cast doubt on the validity of all black hole physics.

4 The shell collapsar

Substituting (11), that is (OS37), in (5), that is (OS25), gives immediately

$$r(\tau_0) = \frac{r_0 R}{2R_b}\left(3 - \frac{R^2}{R_b^2}\right) \quad . \tag{13}$$

This is the end state of OS collapse. It is appropriate to call it a *shell collapsar*, because its density is strongly concentrated near $R = R_b$. This may be seen

[1] meaning initial density

from the fact that half of the total dust for large y lies within $R_b > R > 2^{-1/3}R_b = 0.794R_b$, and in the end state this maps into $r_0 > r > 0.941r_0$. Furthermore, because in this limit $dr/dR \to 0$, the density at the surface r_0 actually becomes infinite. When we take account of the relativistic mass effect of the dust particles[6], by analysing the stress tensor (7) in the (r, t) coordinates, the concentration is even more pronounced, and the *half-mass shell* lies between r_0 and $0.947r_0$.

Note that a given dust particle follows the geodesic

$$r = \frac{r_0 R}{2R_b}\left(2y + 1 - \frac{R^2}{R_b^2}\right) \quad , \tag{14}$$

so that, for $y \to +\infty$, that is $t \to -\infty$, r is proportional to R, and therefore in this limit the dust density is uniform in both the comoving and external coordinate frames.

A recent treatment of the OS dust model by Zakir[13] supports the conclusion that no collapse occurs beyond the gravitational radius; this author concludes that OS is an example of a frozen star [10][11].

5 Generalizations of Oppenheimer-Snyder

We consider now the Weinberg interior metric[2][7] (see equation (11.9.16))

$$ds^2 = d\tau^2 - \frac{z^2}{1 + kR^2}dR^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad , \tag{15}$$

where $z(\tau) = r/R$ is given parametrically ([7] (11.9.25)) by

$$z = k^{-1}\sinh^2\sqrt{k}\eta, \quad \tau = \frac{2}{3} + k^{-1}\left(\eta - k^{-1/2}\sinh\sqrt{k}\eta\cosh\sqrt{k}\eta\right) \quad ; \tag{16}$$

OS is the case $k \to 0$. We may rewrite the metric as

$$\begin{aligned} ds^2 \quad = \quad & \frac{z^2(1 + kR^2)}{(1 + kz)(z - R^2)}\left[dz + \frac{(1 + kz)RdR}{1 + kR^2}\right]^2 \\ & - \frac{z}{z - R^2}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad . \end{aligned} \tag{17}$$

At this point Weinberg constructed an *integrating factor*

$$dy = f(R, z)\left[dz + \frac{(1 + kz)RdR}{1 + kR^2}\right] \quad (f(1, z) = 1) \quad , \tag{18}$$

with the property that dy is a perfect differential. In this case the solution is simple, namely

$$f(R, z) = \sqrt{\frac{1 + kR^2}{1 + k}} \quad , \tag{19}$$

[2] I have changed the sign of k; the negative k case describes not a collapsing but a spherically pulsating object. I have also simplified the algebra, by suitable choice of units, so that $r_0 = 1$ and $R_b = 1$.

leading to

$$y = \frac{1}{k}\left[(1+kz)\sqrt{\frac{1+kR^2}{1+k}} - 1\right] = \frac{1}{k}\left[\frac{R+kr}{R}\sqrt{\frac{1+kR^2}{1+k}} - 1\right] \quad . \tag{20}$$

Note that, for $k \to 0$, this gives the OS value (8). Then the metric becomes

$$ds^2 = \frac{r^2(1+k)}{(R+kr)(r-R^3)}dy^2 - \frac{r}{r-R^3}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad , \tag{21}$$

with $R(r, y)$ given as the inversion of (20). As with OS, this metric becomes continuous with the exterior metric at $R = 1$ once a suitable transformation $t(y)$ is made, namely

$$\left(\frac{dy}{dt}\right)^2 = \frac{(y-1)^2(1+ky)}{y^3(1+k)} \quad ; \tag{22}$$

the resulting metric is[3]

$$ds^2 = \frac{r^2(y-1)^2(1+ky)}{(r-R^3)y^3(R+kr)}dt^2 - \frac{r}{r-R^3}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad , \tag{23}$$

and, as with OS, the continuity property is manifested by putting $R = 1$.

We may now plot $r(R, y)$ at $y = 1$, and, following analysis of the stress tensor as in[6], we find that the final half-mass shell has radius 0.961 at $k = 0.5$ and 0.967 at $k = 1$. Comparison with the OS value 0.947 at $k = 0$ shows that the density of the shell is greater than in the corresponding OS shell.

A somewhat wider generalization of OS was given by Choquet-Bruhat[8], namely

$$ds^2 = d\tau^2 - \left(\frac{\partial r}{\partial R}\right)^2 dR^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad , \tag{24}$$

where

$$r(\tau, R) = R\left(1 - \frac{3\tau\sqrt{F(R)}}{2}\right)^{2/3} \quad , \tag{25}$$

with the surface at $F(1) = 1$; F(R) is simply related to the comoving mass density. An analysis similar to that of Weinberg is possible, but the integrating factor is less simple and must be obtained by a numerical integration. I have been able to do this with the function $F(R)$ given by

$$F(R) = 1 + F_0(1-R), \quad (F_0 > -1) \quad , \tag{26}$$

and again we find a limiting density which is a shell near to $r = 1$, with a shell thickness depending on F_0. Of course, in the case $F_0 = 0$ the density corresponds to OS. As stated earlier, Choquet-Bruhat, unlike Weinberg, did not do the matching of the t-coordinate.

[3]The metric is identical to Weinberg's[7] equations (11.9.31-2); to see this we have to change k to $-k$, and then make the changes in notation $(a, R, S) = (1, kz, ky)$.

6 Discussion

We have shown how complete matching of the interior with the exterior metric, leading to a proper understanding of the limits of the time variables, is intimately linked with the shell structure of the end state. This step, requiring the construction (18) of the integrating factor $f(R, z)$, was an essential part of the analysis of the original OS article, and also of Weinberg[7], but was not considered by Penrose[5] or Choquet-Bruhat[8].

Of course, we should recognize that only a limited set of inferences may be drawn from these highly idealized dust models, but, since they are the only complete time-dependent solutions for collapse, they should properly be taken account of in trying to describe collapsars with more realistic equations of state. A contemporary statement from Christodoulou[15] along these lines is

> An important remark at this point is that it is not a priori obvious that closed trapped surfaces are evolutionary. That is, it is not obvious whether closed trapped surfaces can form in evolution starting from initial conditions in which no such surfaces are present. What is more important, the physically interesting problem is the problem where the initial conditions are of arbitrarily low compactness, that is, arbitrarily far from already containing closed trapped surfaces, and we are asked to follow the long time evolution and show that, under suitable circumstances, closed trapped surfaces eventually form. Only an analysis of the dynamics of gravitational collapse can achieve this aim.

The alleged evolution of a collapsar towards a trapped surface inside the horizon is, of course a necessary step in the black-hole conjecture. The argument of the previous sections was almost entirely geometrical, but the latter quotation emphasizes the need for a field-theoretic content. A first step in that direction was already taken in [6], by considering the stress tensor, but a more ambitious and revealing programme will necessitate studying, for example, the Landau energy pseudotensor[16]; this is an area which we investigated in a previous publication[9].

I draw attention to two aspects of my stress-tensor analysis which indicate the role played by gravitational energy. First there is the obvious question of what physical force intervenes to prevent gravitational attraction causing collapse to a black hole. With due respect to fifty years of tradition, I submit that this is the wrong question. My analysis shows that Einstein's insistence[17] on a proper treatment of time dilation, involving infinite red shift as an external object approaches the gravitational radius[18], has led us to particle paths in the interior of the collapsar which are continuous with those of an exterior object, and which go asymptotically towards the surface and not to the centre. The conclusion then seems inescapable; the force taking such particles towards the surface is one of *gravitational repulsion*[19]. A second question is posed by the form of the mass density revealed in [6]. The total mass increases steadily as y decreases and tends to infinity as y tends to 1. This is an extreme example of a long recognized behaviour in collapsing bodies first noted by Cameron[20], who discovered that a neutron star of gravitational mass a little under $2M_\odot$ has a "proper mass" of about $3M_\odot$ and a radius about 1.35 times its gravitational radius. The factor linking these two masses, discussed also by Weinberg[7] (see his section 11.1), is the time dilation factor given as $y/(y-1)$ in [6]. So, in the highly idealized dust models, which are the only ones giving exact solutions, repulsive gravity is associated with both an infinite surface density and an infinite energy.

Both of the latter features of the dust collapsar indicate the need to take account of the nonzero pressures inside real collapsars, a theme we have discussed previously[6][9][19]. In dense bodies like neutron stars and galactic centres there is a core of negative gravitational energy with negative mass, which not only produces repulsion but also cancels out a proportion of the "proper mass" contained in the stellar material. A nonzero pressure is what prevents repulsive gravity effects going to the extremes of the dust models. The pioneering work in this area was the article of Oppenheimer and Volkoff[21] (OV), published a few months prior to the OS article. To the extent that such an investigation has been carried through to black hole related models, it has been completely dominated by what may be considered a newtonian insistence that gravity can only be attractive, with the consequence that nuclear material is squeezed to very high central densities, giving birth to exotic material such as hyperons and quarks. Note, however, that Oppenheimer and Volkoff, in their footnote 10, conceded the possibility of varying the central boundary condition, but did not investigate it further in the light of the OS article. In our articles cited above we indicated the profound way in which the incorporation of repulsive gravity changes such theories, through changed boundary conditions at the centre. It should be borne in mind that the pressures required to prevent infinite density at the surface are less by many orders of magnitude than would be required to prevent an infinite central density. It should also be noted that collapsars with shell-like density profiles, and with a realistic equation of state, have been proposed in the context of metrics having an empty de Sitter metric[12] at the centre.

References

[1] K. Schwarzschild, On the gravitational field of a ball of incompressible fluid according to Einstein's theory, Conference reports of the Royal Prussian Academy of Sciences, Berlin, Phys.-Math. Class, 424-434 (1916)

[2] A. Einstein, L. Infeld and B. Hoffmann The gravitational equations and the problem of motion Ann. Math. 39, 65 (1938)

[3] R. C. Tolman, Effect of inhomogeneity on cosmological models, Proc. Natl. Acad. Sci. U. S., 20, 19-176 (1934)

[4] J. R. Oppenheimer and H. Snyder, On continued gravitational contraction *Phys. Rev.*, **1939**, *56,* 455-459.

[5] R. Penrose, Gravitational collapse and space-time singularities, *Phys. Rev. Lett.* , **1965**, *14*, 57-59.

[6] T. W. Marshall, The shell collapsar – a possible alternative to black holes *Entropy*, **2016**, *18*, 363-373

[7] S. Weinberg, *Gravitation and Cosmology*; John Wiley, New York, **1972**

[8] Y. Choquet-Bruhat, *General Relativity and the Einstein Equations*, **2009** Oxford University Press, New York, pp 96-103.

[9] M. K. Wallis and T. W. Marshall, Energy in General Relativity the case of the neutron star. In *Physical Interpretations of Relativity Theory; Proceedings of International Meeting, Bauman State Technical University, Moscow*, **2015**; pp 544-556. Also at http://tinyurl.com/PIRT-Moscow-15-MarshallWallis.

[10] Ruffini, R. and Wheeler, J.A. (1971). Introducing the black hole. Physics Today , January 1971, 3041.

[11] S. N. Zhang arXiv:1003.1359v1 (2010) Physics Letters B 679 (2009) p88 with Y. Liu On the Solution to the Frozen Star Paradox, Nature of Astrophysical Black Holes, non-Existence of Gravitational Singularity in the Physical Universe and Applicability of the Birkhoffs Theorem

[12] M. Visser and D. I. Wiltshire, Stable gravastar – an alternative to black-holes?, *Class. Quantum Gravity*, **2004**, *21* 1-17.

[13] Z. Zakir, *Astrophys Space Sci*, **363**:30 (2018) On the consistency of the Oppenheimer-Snyder solution for a dust star. Reply to Marshall's criticism.

[14] "A Life Scientific" BBC interview with R. Penrose, 22 November 2016 www.bbc.co.uk/programmes/b082ymnx

[15] D. Christodoulou, arxiv.org/pdf/0805.3880.pdf The formation of black holes in General Relativity

[16] L. D. Landau and E. M. Lifshitz, *Classical Theory of Fields*; Pergamon, Oxford, UK, **1962**, section 110

[17] A. Einstein, On a stationary system with spherical symmetry consisting of many gravitating masses, *Ann. Math.*, **1939**, *40*, 922-36

[18] R. J. Spivey, Dispelling black-hole pathologies through theory and observation *Progress in Physics* **2015**

[19] T. W. Marshall and M. K. Wallis, *Supermassive galactic centre with repulsive gravity* **2013** arxiv.org/pdf/1303.5604.pdf

[20] A. G. W. Cameron, Neutron star models *Ap.J.*, **1959**,*130*, 884-894.

[21] J. R. Oppenheimer and G. M. Volkoff, On massive neutron cores *Phys. Rev.* **1939**, *55* 374-381.