On Finsler Geometry, MOND and Diffeomorphic Metrics to the Schwarzschild Solution

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Abstract

We revisit the construction of diffeomorphic but *not* isometric solutions to the Schwarzschild metric. The solutions relevant to Black Holes are those which require the introduction of non-trivial areal-radial functions that are characterized by the key property that the radial horizon's location is *displaced* continuously towards the singularity ($r = 0$). In the limiting case scenario the location of the singularity and horizon *merges* and any infalling observer hits a null singularity at the very moment he/she crosses the horizon. This fact may have important consequences for the resolution of the firewall problem and the complementarity controversy in black holes. Next we show how modified Newtonian dynamics (MOND) can be obtained from solutions to Finsler gravity, and which in turn, can also be modeled by metrics which are diffeomorphic but not isometric to the Schwarzschild metric. The key point now is that one will have to dispense with the asymptotic flatness condition, by choosing an areal radial function which is $finite$ at $r = \infty$. Consequently, changing the boundary condition at $r = \infty$ leads to MONDian dynamics. We conclude with some discussions on the role of scale invariance and Born's Reciprocal Relativity Theory based on the existence of a maximal proper force.

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1 Introduction : Diffeomorphic Metrics to the Schwarzschild Solution, Firepoints and Firewalls

The static spherically symmetric (SSS) *vacuum* solution of Einstein's field equations [1] that we learned from the text books is actually the Hilbert form of the original Schwarzschild [2] solution

$$(ds)^2 \ = \ (1 - \frac{2GM}{r}) \ (dt)^2 \ - \ (1 - \frac{2GM}{r})^{-1} \ (dr)^2 \ - \ r^2 \ (d\Omega)^2. \tag{1.1}$$

Birkoff's theorem states that all static spherically symmetric vacuum solutions to Einstein's equations are $diffeomorphic$ to the Hilbert-Schwarzchild solution. There are an infinite number of metrics which are diffeomorphic but **not** isometric to the Hilbert form of the Schwarzschild [2] solution. In particular, given an *areal* radial function $\rho(r) \neq r$ (in $c = 1$ units), the metric

$$(ds)^2 \ = \ (1 - \frac{2GM}{\rho(r)}) \ (dt)^2 \ - \ (1 - \frac{2GM}{\rho(r)})^{-1} \ (d\rho)^2 \ - \ \rho^2(r) \ (d\Omega)^2. \tag{1.2}$$

is diffeomorphic but **not** isometric to the Hilbert form of the Schwarzschild [2] solution. $(d\rho)^2 = (d\rho(r)/dr)^2(dr)^2$, and the solid angle infinitesimal element is $(d\Omega)^2 = (d\phi)^2 + sin^2(\phi)(d\theta)^2$. The surface area at each point r is now given by $4\pi(\rho(r))^2$ so that $\rho(r)$ plays the role of an effective radius and hence the name of "areal-radial" function for $\rho(r)$.

It is clear that the metric (1.2) is diffeomorphic but *not* isometric to the Hilbert form (1.1) of the Schwarzschild [2] solution because the area elements $r^2 \ (d\Omega)^2 \neq \rho^2(r) \ (d\Omega)^2$ are *not* equal, except in the trivial case when $\rho(r) = r$. The (active) diffeomorphisms are simply established by the mappings $r \to \rho(r)$. We have not *relabeled* the radial variable r by giving it another "name" and calling it "ρ", because $\rho(r)$ is itself a function of r. Furthermore, one has *not* performed a naive change of variables by writing $r = r(r')$ because $\rho(r) \neq r = r(r')$. The metric (1.2) leads to modifications of the Newtonian potential. One recovers the Newtonian potential in the regime when $\rho(r) \simeq r$. In the Appendix it is shown explicitly that the metric (1.2) is a solution to Einstein's vacuum field equations.

It is well known to the experts that the *extended* Schwarzschild metric solution for $r < 0$ with $M > 0$, corresponds to a solution in the region $r > 0$ with $M < 0$. Negative masses are associated with repulsive gravity. For this reason, the domain of values of r will be chosen to span the whole real axis $-\infty \leq r \leq \infty$.

The boundary condition obeyed by the areal radial function $\rho(r)$ at the origin is $\rho(r = 0) = 0$. At infinity, and asymptotically flat metric would require $\rho(r \to \infty) \sim r \to \infty$. The Hilbert textbook (black hole) solution [5] when $\rho(r) = r$ obeys the boundary conditions but the Abrams-Brillouin [4] choice $\rho(r) = r + 2GM$ does *not*. The *original* solution of 1916 found by Schwarzschild for $\rho(r)$ did not obey the boundary condition $\rho(r = 0) = 0$ as well. The condition $\rho(r = 0) = 2GM$ has a serious $flaw$ and is : how is it *possible* for a point-mass at $r = 0$ to have a non-zero area $4\pi(2GM)^2$ and a *zero* volume *simultaneously* ? Therefore one cannot have Abrams-Brillouin's [4] choice. It is known that fractals have unusual properties related to their lengths, areas, volumes, dimensions but we are not focusing on fractal spacetimes at the moment. For instance, one could have a *fractal* horizon surface of infinite area but zero volume (space-filling fractal surface). The finite area of $4\pi(2GM)^2$ could then be seen as a regularized value of the infinite area of a "fractal horizon".

The Hilbert choice for the areal radial function $\rho(r) = r$ is ultimately linked to the actual form of the Newtonian potential $V_N = -(Gm_1m_2/r)$. In the last few decades corrections to Newton's law of gravitation and constraints on them have become the subject of considerable study, see the monograph [6]. Yukawa-type corrections to Newton's gravitational law from two recent measurements of the Casimir interaction between metallic surfaces was studied by [7]. A Yukawa-like correction to the Newtonian potential could be chosen to be

$$V(r) \;=\; -\,\frac{Gm_1m_2}{r}\,(1 - \lambda\; e^{-r/2GM}), \quad \lambda > 0 \qquad (1.3a)$$

where λ and $r_o = 2GM$ are the strength and interaction range of the Yukawa-type correction. One may notice that the potential (1.3a) can be rewritten in terms of an areal-radial function $\rho(r)$ as

$$V(r) \;=\; -\,\frac{Gm_1m_2}{\rho(r)}, \quad \rho(r) \;=\; \frac{r}{1 - \lambda\; e^{-r/2GM}}, \quad \lambda \neq 1 \qquad (1.3b)$$

One has the correct boundary conditions for the areal radial function when $\lambda \neq 1$

$$\rho(r = 0) \;=\; 0; \quad \rho_\lambda(r \to \infty) \to r, \quad \rho(r = r_h) \;=\; 2GM; \quad 0 \leq r_h \leq 2GM \qquad (1.3c)$$

so that the location of the horizon radius r_h has been *shifted* towards the singularity. In the asymptotic regime one has as expected $\rho(r \to \infty) \;\to\; r$, so that the areal-radial function tends to r (as in the Hilbert choice) and the expression for the potential is asymptotic to the Newtonian one. At the end of this section we shall discuss the case when $\lambda = 1$.

Instead of the Yukawa-type areal radial function (1.3b), one could have had many other areal-radial functions $\rho(r)$ [1] obeying the boundary conditions. In particular, the metric solutions (1.2) are *invariant* under the transformations $r \to -r; M \to -M$ for our particular choice of the areal radial functions given in eq-(1.3b) due to the condition $\rho(-r, -M) = -\rho(r, M)$. This allows us to extended the solutions to the $r < 0$ region. For a recent analysis of the properties of the maximal extensions (in regions $r < 0$) of the Kerr and Kerr-Newman spacetimes with *negative* mass, see [8].

Given the particular choice of the areal radial function in eq-(1.3b), it is important to emphasize that the Newtonian potential is recovered in the regime when $r >> 2GM$, so that $V(r) = -\frac{Gm_1m_2}{\rho(r)} \simeq -\frac{Gm_1m_2}{r}$. For example, in the case of the sun its Schwarzschild radius $2GM$ is of the order of 3 Kms which is much smaller than the solar radius and the scale of the planetary orbits. Consequently, in the regime when $r >> 2GM$, all the

[1] We thank Matej Pavsic for a discussion on other choices for the radial functions

metric solutions in eq-(1.2) reduce to the standard textbook Hilbert solution in eq-(1.1), and the Newtonian potential is always recovered from an infinity of modified potentials.

In the next section we will show why modified Newtonian dynamics (MOND) [14] associated with galaxies can be obtained from Finsler gravity, and which in turn, can be modeled by metrics (1.2) which are diffeomorphic but not isometric to the Hilbert-Schwarzschild metric. The key point is that one will have to dispense with the asymptotic flatness condition, by choosing a $different$ areal radial function than the ones discussed above, and instead introduce an infrared-cutoff for the metric at $r = \infty$ in the form of $\rho(r = \infty) = \rho_o = finite$.

The solutions to Einstein's equations are defined modulo diffeomorphisms. Therefore, all these mathematically distinct solutions obtained via the *active* diffeomorphisms $r \to \rho(r)$, and obeying the *same* boundary conditions at $r = 0, \infty$, are *not* physically distinguishable, but they represent *one* and the *same* physical solution of the field equations. It will be shown in the next section that changing the boundary condition at $r = \infty$ leads to MONDian dynamics. For a historical account of the role of active and passive diffeomorphisms within the context of the "hole argument" that much troubled Einstein we refer to [10] [2].

To model the scenario when the horizon *merges* precisely with the singularity one needs an area radial function defined as follows

$$\rho(r = 0) = 0, \quad \rho(r) \ = \ \frac{r}{1 \ - \ e^{-r/2GM}}, \quad r > 0 \tag{1.4}$$

Under $r \to -r;\ M \to -M$ one has that $\rho(r) \to -\rho(r)$ so one can ensure the invariance of the metric (1.2) under these transformations and extend the solutions to the $r < 0$ region.

Hence, we have in eq-(1.4) that $\rho(r = 0^+; M) = 2GM$, and $\rho(r = 0^-; -M) = -2GM$, but $\rho(r = 0) = 0$ since a point mass must have zero area and zero volume. The horizon is located at $r_h = 0^+$ and the singularity at $r = 0$. There is a *discontinuity* of $\rho(r)$ at $r = 0$. The right $r = 0^+$, and left $r = 0^-$ limits of $\rho(r)$ give respectively $\pm 2GM$, while $\rho(r = 0) = 0$ which is the arithmetic mean of $2GM$ and $-2GM$.

In the Appendix it is shown that the areal-radial function $\rho(r)$ given by eq-(1.4) (along with an infinite number of C^∞ differentiable functions) solves the vacuum field equations. The finite discontinuity of $\rho(r)$ occurs at one *single* point $r = 0$ (the origin), whereas $(d\rho/dr) = \infty$ at $r = 0^\pm$. Hence, the derivatives of $\rho(r)$ are continuous while $\rho(r)$ is discontinuous at $r = 0$. A typical example of this behavior is the tangent function $tan(\theta)$. At $\theta = \pi/2$, the tangent exhibits a discontinuity as it goes from ∞ to $-\infty$, whereas the derivative remains the same and equal to ∞. Fractal curves on the other hand are continuous everywhere but nowhere differentiable. Concluding, the derivatives of $\rho(r)$ are continuous, and the metric (1.2) whose areal-radial function $\rho(r)$ is given by eq-(1.4) solves the vacuum field equations as shown in the Appendix.

Because a point mass is an infinitely compact source of infinite density, there is nothing *wrong* with the possibility of having a *discontinuity* of the metric at the location of the singularity $r = 0$. Due to the boundary condition $\rho(r = 0) = 0$, there is a curvature

[2] If it much troubled Einstein imagine what it has done to us during the past decades in trying to decipher what dark matter is, and whether or not it exists

tensor singularity and the Kretschmann invariant $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \sim (2GM)^2/\rho(r)^6$ diverges at $\rho(r = 0) = 0$. In this extreme case, when the the location of the horizon merges with the singularity, there is a null-line singularity at $r = 0$ and a null-surface at $r = 0^+$. This may sound quite paradoxical but it is a consequence of the metric *discontinuity* at $r = 0$, the location of the point mass (singularity). This key fact may have important consequences for the resolution of the firewall problem and the complementarity controversy in black holes. We may label the null-line singularity at $r = 0$ as a $firepoint$, and the null-surface at $r = 0^+$ as a $firewall$. A recent discussion of the notion of a $firepoint$ can be found in [11] where they propose the singularity itself as a "firepoint" capable to break the entanglement between the "in" and the "out" states created through the Hawking process.

When the areal-radial function $\rho(r)$ has the actual form in eq-(1.4) there is *no interior* region beyond the horizon $r = 0^+$, so that the metric (1.2) is truly *static* everywhere. The Fronsdal-Kruskal-Szekeres analytical continuation of the metric (1.1) inside the horizon is *not* static. Klinkhamer [12] provided earlier on a regularization of the standard Schwarzschild solution with a curvature singularity at the center by removing the interior region of a ball and identifying the antipodal points on the boundary. The resulting four-dim manifold has now the topology $R \times \tilde{M}_3$ where $\tilde{M}_3$ is a nonsimply-connected manifold, which up to a point (the center), is homeomorphic to the 3-dim real projective space RP^3. In our case, there is no need to remove the interior region by surgery. The discontinuity $\rho(r = 0) = 0; \rho(r = 0^+) = 2GM$ of the areal-radial function amounts to a sort of "point-splitting" creating a *void* (hole) in spacetime, isolating and expunging the curvature singularity at the center from the remaining region of spacetime. The topology of the region free of the singularity at the center is $\tilde{M}_4 = R \times \tilde{M}_3$, where $\tilde{M}_3 = R^3 - \{0\}$ is the punctured 3-dim space.

2 Finsler Geometry, Modified Newtonian Dynamics and Areal Radial Functions

It has long been known that if one recurs to Newton's inverse-square law of gravity, the observed baryonic matter cannot provide enough force to attract the matter (stars) present in the outer edges of the galaxies [17]. Postulating that galaxies are surrounded by massive, non-luminous dark matter is one of the most widely accepted proposals to solve the problem. No dark matter has been detected yet. An extensive overview with a vast number of references of the tests and problems of the standard model in Cosmology can be found in [19].

Some models have been built as an alternative to the dark matter hypothesis. The main ideas are based on assuming that the Newtonian gravity or Newtons dynamics is invalid on galactic scales. In the MOND model (modified Newtonian dynamics) of Milgrom [14] it assumes that the Newtonian dynamics does not hold on galactic scales. In Extended Theories of Gravity, like $f(R)$ gravity [18] it is shown that several gravitating structures like stars, spiral galaxies, elliptical galaxies and clusters of galaxies can be self-consistently described without dark matter. There are other MONDian theories, for example, by introducing several scalar, vector and tensor fields, Bekenstein [15] rewrote the MOND model of Milgrom in a covariant formalism (the TeVeS model). There is the Einstein-aether theory [16] admitting a preferred reference frame and broken local Lorentz invariance.

The accurate measurement of the speed of gravitational waves (GW) compared to the speed of light in 2017 ruled out modified gravity theories, termed "dark matter emulators", which dispense with the need for dark matter by making ordinary matter couple to a different metric from that of GW. These models have the property that, in the extreme weak field regime relevant to cosmology, gravitational waves propagate on different geodesics from those followed by photons and neutrinos. Therefore, the differential Shapiro delay between GWs and photons/neutrinos is due to the gravitational potential of only the dark matter. Some examples of these Dark Matter (DM) emulator theories include Bekenstein's TeVeS theory [15] and Moffat's Scalar-TensorVector gravity theory [21]. It is important to understand that dark matter emulators constitute a special class of modified gravity theories which attempt to dispense with dark matter. Many modifications of gravity do not fall within this class [20], including Milgrom's bi-metric formulation of MOND [22], nonlocal MOND [23], such as superfluid dark matter [25], or dipolar dark matter [24]. Nor does it apply to certain types of Einstein-Aether theories [26] whose vector kinetic terms are properly chosen. Therefore, other kinds of modified gravity theories which dispense with the need for dark matter and are still viable [20].

In this section we shall review the main ingredients of Finsler geometry [27], [28], [29]; present a solution of the vacuum field equation in Finsler gravity, in the weak field approximation [30], and show how it reproduces the main results of MOND. The solution depends on the rotational velocity of the galaxy consistent with the relationship between the Tully-Fisher relation [36] and MOND. We finalize by showing how this Finsler gravity solution leads to a metric that is diffeomorphic (but not isometric) to the Hilbert-Schwarzschild metric. The most salient feature is that the metric is *not* asymptotically flat due to the infrared cutoff of the areal radial function, and resulting from imposing different boundary conditions for the metric at $r = \infty$ than in the Hilbert-Schwarzschild metric case.

We shall begin with a very brief discussion of Finsler geometry [27], [28], [29] before discussing the gravitational vacuum field equations. Finsler geometry is based on a non-negative real function $F(\mathbf{x}, \mathbf{y} = \frac{d\mathbf{x}}{d\tau})$, obeying $F(\mathbf{x}, \lambda\mathbf{y}) = \lambda F(\mathbf{x}, \mathbf{y})$, and defined on the tangent bundle TM represented by the coordinates $\mathbf{x} \equiv x^0, x^1, x^2, \cdots x^n; \mathbf{y} \equiv \frac{dx^0}{d\tau}, \frac{dx^1}{d\tau}, \cdots \frac{dx^n}{d\tau}$. The fundamental metric tensor is given as

$$g_{\mu\nu}(\mathbf{x}, \mathbf{y}) \equiv \frac{1}{2} \frac{\partial^2 F^2}{\partial y^\mu \partial y^\nu} \tag{2.1}$$

The arc-length is

$$L \;=\; \int\; F(x^0,x^1,\cdots x^n; y^0,y^1,\cdots y^n)\; d\tau \;=\; \int\; F(x^0,x^1,\cdots x^n; \frac{dx^0}{d\tau},\frac{dx^1}{d\tau},\cdots \frac{dx^n}{d\tau})\; d\tau \tag{2.2}$$

The Cartan tensor (which measures the deviation from a Riemannian manifold) is given by the third derivative

$$C_{\mu\nu\sigma}(\mathbf{x},\mathbf{y}) \;=\; \frac{1}{4}\; \frac{\partial^3 F^2}{\partial y^\mu \partial y^\nu \partial y^\sigma} \tag{2.3}$$

If $C_{\mu\nu\sigma}(\mathbf{x},\mathbf{y}) = 0$ everywhere in the tangent space, the Finsler space becomes a metric space with $g_{\mu\nu}(\mathbf{x})$ independent on the tangent space coordinates $\mathbf{y}$ (velocities). The geodesic equation on a Finsler manifold is given by

$$\frac{d^2 x^\mu}{d\tau^2} \;+2\; G^\mu \;=\; 0 \tag{2.4}$$

where the geodesic spray coefficients G^μ (dropping the $\mathbf{x},\mathbf{y}$ dependence for convenience) are given by

$$G^\mu \;=\; \frac{1}{4}\; g^{\mu\nu}\; \left(y^\sigma \frac{\partial^2 F^2}{\partial x^\nu \partial y^\sigma} \;-\; \frac{\partial F^2}{\partial x^\nu} \right) \tag{2.5}$$

The corresponding *nonlinear* connection $N^\mu_\nu(\mathbf{x},\mathbf{y})$ associated to the geodesic spray coeffcients is defined by

$$2\; G^\mu(\mathbf{x},\mathbf{y}) \;=\; N^\mu_\nu(\mathbf{x},\mathbf{y})\; y^\nu \;=\; \frac{1}{2}\; g^{\mu\nu}\; \left(y^\sigma \frac{\partial^2 F^2}{\partial x^\nu \partial y^\sigma} \;-\; \frac{\partial F^2}{\partial x^\nu} \right) \tag{2.6}$$

The nonlinear connection allows to decompose the tangent space to the tangent bundle $T_{(\mathbf{x},\mathbf{y})}TM$ at the point $(\mathbf{x},\mathbf{y})$ into a vertical space spanned by $\frac{\partial}{\partial y^\mu}$, and a horizontal space spanned by the so-called elongated derivatives $\frac{\delta}{\delta x^\mu} \equiv \frac{\partial}{\partial x^\mu} - N^\nu_\mu \frac{\partial}{\partial y^\nu}$. The nonlinear curvature derived from N^μ_ν is

$$\mathcal{R}^\mu_{\nu\sigma} \;\equiv\; \frac{\delta N^\mu_\nu}{\delta x^\sigma} \;-\; \frac{\delta N^\mu_\sigma}{\delta x^\nu},\quad \frac{\delta}{\delta x^\mu} \equiv \frac{\partial}{\partial x^\mu} - N^\nu_\mu \frac{\partial}{\partial y^\nu} \tag{2.7}$$

Regarding the tangent bundle T_M as an $2(n+1)$-dim manifold of its own, one can construct linear covariant derivatives which are compatible with the structure induced by the nonlinear connection and which preserves the horizontal-vertical split of the tangent bundle TM with basis $\frac{\delta}{\delta x^\mu}, \frac{\partial}{\partial y^\mu}$.

The nonlinear connection is unique, however the linear connections are *not*, and many different choices are possible [28]. For example, the *horizontal* part of a torsionless linear connection is given in terms of the elongated derivatives $\delta/\delta x^\mu$ as

$$\Gamma^\mu_{\nu\sigma}(\mathbf{x},\mathbf{y}) \;=\; \frac{1}{2}\; g^{\mu\lambda}(\mathbf{x},\mathbf{y})\; (\; \delta_\nu g_{\sigma\lambda}(\mathbf{x},\mathbf{y}) \;+\; \delta_\sigma g_{\nu\lambda}(\mathbf{x},\mathbf{y}) \;-\; \delta_\lambda g_{\nu\sigma}(\mathbf{x},\mathbf{y})\;) \tag{2.8}$$

and the horizontal part of the curvature (dropping $\mathbf{x}, \mathbf{y}$) is

$$\mathcal{R}^{\mu}_{\nu\sigma\lambda} \;=\; \delta_\sigma \Gamma^{\mu}_{\nu\lambda} \;-\; \delta_\lambda \Gamma^{\mu}_{\nu\sigma} \;+\; \Gamma^{\mu}_{\xi\sigma}\,\Gamma^{\xi}_{\nu\lambda} \;-\; \Gamma^{\mu}_{\xi\lambda}\,\Gamma^{\xi}_{\nu\sigma} \;-\; C^{\mu}_{\nu\xi}\,\mathcal{R}^{\xi}_{\sigma\lambda} \tag{2.9}$$

In Finsler geometry there is a geometrical invariant (under coordinate transformations) that only depends on the Finsler structure $F(\mathbf{x}, \mathbf{y})$ and is insensitive to the choices of the linear connection. It is the Ricci scalar defined in terms of the geodesic spray coefficients as

$$\mathcal{R} \;\equiv\; \mathcal{R}^{\mu}_{\mu} \;=\; \frac{1}{F^2} \left(2\frac{\partial G^\mu}{\partial x^\mu} \;-\; y^\lambda \frac{\partial^2 G^\mu}{\partial x^\lambda \partial y^\mu} \;+\; 2G^\lambda \frac{\partial^2 G^\mu}{\partial y^\lambda \partial y^\mu} \;-\; \frac{\partial G^\mu}{\partial y^\lambda}\frac{\partial G^\lambda}{\partial y^\mu} \right) \tag{2.10}$$

With these geometrical ingredients the analog of the Newtonian limit in Finsler spaces based on the notion of "locally Minkowski" spacetime was studied by [30]. A Finsler spacetime is "locally Minkowski" if there is a coordinate system $\mathbf{x}$ with induced tangent space coordinates $\mathbf{y}$, such that F only depends on $\mathbf{y}$, but *not* on $\mathbf{x}$. A "locally Minkowski" spacetime is a solution of the Finslerian vacuum field equations [29] . The authors [30] assumed a very small metric perturbation $h_{\mu\nu}(\mathbf{x}, \mathbf{y})$ to the locally Minkowski one $\eta_{\mu\nu}(\mathbf{y})$

$$g_{\mu\nu}(\mathbf{x}, \mathbf{y}) \;=\; \eta_{\mu\nu}(\mathbf{y}) \;+\; h_{\mu\nu}(\mathbf{x}, \mathbf{y}), \quad |h_{\mu\nu}(\mathbf{x}, \mathbf{y})| << 1 \tag{2.11}$$

and found that to leading order in the post-Newtonian approximation the explicit form of the Finslerian line element which solves the vacuum field equations in a $4D$ Finsler spacetime is given by

$$F^2\;(d\tau)^2 \;=\; (1 - \frac{2GM}{R(r,v)})\;(dt)^2 \;-\; (1 + \frac{2GM}{R(r,v)})\;(dR)^2 \;-\; R^2(r,v)\;(d\Omega)^2 \tag{2.12}$$

where the radial coordinate in the locally Minkowski space-time of the galaxies is defined as $R(r,v) \equiv \sqrt{\eta_{ij}(v)x^i x^j}$. In this spherically symmetric case, the radial function is denoted by $R(r,v)$ (which must *not* be *confused* with the scalar curvature $\mathcal{R}$), and now it depends on both r, *and* the velocity v, due to functional dependence of $g_{\mu\nu}(\mathbf{x}, \mathbf{y})$ on both coordinates and velocities.

Li and Chang [30] have shown that Finsler gravity reduces to MOND if the spatial components of the locally Minkowski metric of galaxies is of the form

$$\eta_{00} \;=\; 1, \quad \eta_{ij}(y) \;=\; \delta_{ij} \left(1 - (\frac{GMa_o(y^0)^4}{(\delta_{mn}y^m y^n)^2})^2 \right) \;=\; \delta_{ij} \left(1 - (\frac{GMa_o}{v^4})^2 \right) \tag{2.13}$$

$$v^i = \frac{dx^i}{dx^0} = \frac{(dx^i/d\tau)}{(dx^0/d\tau)} = \frac{y^i}{y^0} \tag{2.14}$$

where $a_o = 1.2 \times 10^{-10} m/s^2$ is the acceleration constant of MOND [14], and which is of the order of $\frac{c^2}{R_H}$, where R_H is the present-day Hubble scale.

In this particular case the radial coordinate in the locally Minkowski space-time of the galaxies becomes

$$R(r,v) \ \equiv \ \sqrt{\eta_{ij}(v) \ x^i \ x^j} \ = \ r \ f(v), \quad f(v) \ \equiv \ \sqrt{1 - \ (\frac{GMa_o}{v^4})^2}, \quad i,j = 1,2,3. \quad (2.15)$$

and the *modified* Newtonian equations of motion associated with the Finslerian line element (2.12) are given by

$$\frac{GM}{R^2} \ = \ \frac{v^2}{R} \ \Rightarrow \ \frac{GM}{r^2 f^2(v)} \ = \ \frac{v^2}{rf(v)} \ \Rightarrow \ \frac{GM}{r^2} \ = \ \frac{v^2}{r} f(v) \ = \ \frac{v^2}{r} \sqrt{1 - \ (\frac{GMa_o}{v^4})^2} \quad (2.16)$$

One may recover the MONDian behavior from eq-(2.16) if $v(r)$ satisfies the following relation

$$\frac{GMa_o}{v^4(r)} \ = \ \frac{1}{\sqrt{1 \ + \ (\frac{v^2(r)}{ra_o})^2}} \quad (2.17)$$

upon inserting (2.17) into the last term of eq-(2.16) it allows to rewrite the scaling factor $f(v)$ in terms of v and r, leading finally to the desired result of MOND

$$\frac{GM}{r^2} \ = \ \frac{v^2}{r} f(v) \ = \ \frac{v^2}{r} \ \frac{(\frac{v^2}{ra_o})}{\sqrt{1 \ + \ (\frac{v^2}{ra_o})^2}} \ = \ \frac{v^2}{r} \ \mu(\frac{v^2}{ra_o}) \quad (2.18)$$

where

$$\mu(x) \ \equiv \ \frac{x}{\sqrt{1+x^2}}, \quad x \equiv \frac{v^2}{ra_o} \quad (2.19)$$

is the *interpolating* function in MOND. From eqs-(2.18, 2.19) one learns

$$x << 1, \ \mu(x) \sim x; \quad x >> 1, \ \mu(x) \sim 1 \quad (2.20a)$$

The deep-MOND regime is characterized by $x << 1, \mu(x) \sim x$, such that

$$r \to \infty, \quad R(r) \to \frac{\sqrt{GMa_o}}{a_o}, \quad v^4(r) \to GMa_o, \quad f(v) \to 0 \quad (2.20b)$$

and one recovers the Tully-Fisher relation $GMa_o \sim v^4$.

Introducing the following definitions in the deep infrared

$$R_c \equiv R(r \to \infty), \quad v_c \equiv v(r \to \infty) \quad (2.21a)$$

one can then rewrite (2.20b) in the following more familiar form

$$\frac{GM}{R_c^2} \ = \ \frac{v_c^2}{R_c} \ = \ a_o \ \simeq \ \frac{c^2}{R_H} \quad (2.21b)$$

Given that $2GM/c^2 \equiv R_S$ is the definition of the Schwarzschild radius associated to a point mass M gravitational source, from eq-(2.21b) one arrives at the following scaling relations involving the Hubble radius R_H, R_S and R_c

$$\frac{1}{2}\frac{R_S}{R_H} = (\frac{v_c}{c})^4, \quad \frac{1}{2} R_S R_H = (R_c)^2 \tag{2.22}$$

Below we shall see the importance of these scaling relations (2.22) within the context of Black Hole Cosmology [32] and Born's Reciprocal Relativity Theory [31].

Given a galaxy of size L_g, from eq-(2.17) one can solve for $v = v(r)$ and obtain the rotational velocities for test masses in the region $r > L_g$

$$v^4(r) = \frac{(\frac{GM}{r})^2 + \sqrt{(\frac{GM}{r})^4 + 4(GMa_o)^2}}{2}, \quad r > L_g \tag{2.23}$$

Eq-(2.23) is equivalent to the following expression

$$r^2(v) = \frac{(v^4/a_o^2)}{(\frac{v^4}{GMa_o})^2 - 1}, \quad r > L_g \tag{2.24}$$

and

$$\frac{v^2}{r} = \frac{v^4}{GM}\sqrt{1 - \frac{a_o^2}{(\frac{v^4}{GM})^2}} \Leftrightarrow \frac{v^4}{GM} = a_o\sqrt{1 + \frac{(v^2/r)^2}{a_o^2}}, \quad r > L_g \tag{2.25}$$

To sum up, given that the range of scales is

$$R_S < L_g < R(r, v(r)) < R_c < R_H \tag{2.26}$$

in the region $R(r, v(r)) > L_g$ one has $\frac{GM}{R^2} = \frac{v^2}{R}$ leading then to the rotational velocities associated to MONDian dynamics. In the interior region of the galaxy, ordinary Newtonian gravity is assumed to be valid, and for spherical symmetric mass distributions one has

$$\frac{GM(r)}{r^2} = \frac{v^2}{r}, \quad r < L_g \tag{2.27}$$

which just follows from Gauss theorem when the mass enclosed $M(r)$ inside the spherical region of radius $r < L_g$ is given by $M(r) = \int_0^r \rho(r')4\pi r'^2 dr'$.

Having gone through this Finsler geometric tour underlying MONDian dynamics, we can finally relate the results of this section with the previous one by noticing that in the regime $(2GM/R) << 1$, a Taylor expansion yields

$$(1 - \frac{2GM}{R})^{-1} \sim (1 + \frac{2GM}{R}) \tag{2.28}$$

and the Finslerian line element (2.12) in this regime reduces to

$$F^2 (d\tau)^2 = (1 - \frac{2GM}{R(r,v)}) (dt)^2 - (1 - \frac{2GM}{R(r,v)})^{-1} (dR)^2 - R(r,v)^2 (d\Omega)^2 \tag{2.29}$$

leading then to a metric which is *$diffeomorphic$* (but *not* isometric) to the Hilbert-Schwarzchild one (*$after$* inserting the functional relation $v = v(r)$ given explicitly by eq-(2.23)) into the areal radial function $R(r, v) = R(r, v(r))$. In fact, the metric (2.29) *is* a solution to the vacuum Einstein field equations in a $4D$ spacetime for *any* functional form [3] $v = v(r)$, as shown explicitly in the Appendix [4].

The particular form of $v(r)$ in eq-(2.23) was dictated to us by the empirical astronomical observations. Furthermore, from eq-(2.15) one learns that under the transformations

$$r \to -r, \quad M \to -M, \quad a_o \to -a_o \;\Rightarrow\; R(r, v) \to -R(r, v) \tag{2.30}$$

the areal radial function changes sign as it should be in order for the metric (2.29) to remain *invariant*.

Despite that the metric (2.29) is diffeomorphic (but not isometric) to the Hilbert-Schwarzschild one, it is *not* asymptotically flat. The Kretschmann invariant $\mathcal{R}_{\mu\nu\sigma\lambda}\mathcal{R}^{\mu\nu\sigma\lambda} \sim (\frac{2GM}{R^3(r,v(r))})^2 \neq 0$ is non-vanishing at $r = \infty$ due to the fact that $R(r = \infty) = R_c = \sqrt{GMa_o}/a_o \neq \infty$ as shown in eq-(2.20b). In the limit that $a_o = 0$, the areal radial function becomes the trivial one $R = r$, and one recovers the asymptotically flat Hilbert-Schwarzschild metric associated with ordinary Newtonian mechanics (in the weak field and slow moving bodies limit).

In a nutshell, simply by rewriting $\frac{GM}{R^2}$ as $\frac{GM_{eff}(r)}{r^2}$, in terms of an effective mass $M_{eff}(r)$ enclosed in a spherical region of radius r, it leads to the relation $M_{eff}(r) = M/f^2(v(r)) \geq M$, since $f(v(r)) \leq 1$, and such that the *enhanced* value of the "effective" mass $M_{eff}(r)$ compared to M would seem as if non-luminous "dark matter" were present in the galaxies.

3 Concluding Remarks : Scale Invariance and Born's Reciprocal Relativity Theory

By simple inspection one can verify that eqs-(2.15-2.26) are scale invariant under

$$t \to \lambda t, \quad r \to \lambda r, \quad R(r, v) \to \lambda R(r, v), \quad M \to \lambda M, \quad a_o \to \lambda^{-1} a_o, \quad v \to v, \quad c \to c \tag{3.1}$$

with $\lambda =$ constant and which implies a flat rotation curve. Note the *anomalous* scaling of the macroscopic galactic mass $M \to \lambda M$ as compared to the scaling $m \to \lambda^{-1} m$ of a fundamental particle (consistent with the scaling of the Compton wavelength $\hbar/mc$). Under these scalings (3.1) $f(v)$ given by eq-(2.15) is invariant and the metric (2.29) scales $(ds)^2 \to \lambda^2 (ds)^2$ as it occurs in Weyl's geometry under conformal transformations.

Another similar scaling occurs in Born's Reciprocal Relativity theory. It was shown in [31] how one can implement a *maximal* proper force principle within the context of Born's Reciprocal Relativity theory, Mach's principle and Black-Hole Cosmology [32], [33]

[3] With the provision that $R(-r, v(-r)) = -R(r, v(r))$
[4] In the appendix a Lorentzian signature is used

by setting the following proper forces to be equal to the maximal proper force value b ("b" stands for Born)

$$M_U \; (\frac{c^2}{R_H}) \; = \; m_P \; (\frac{c^2}{L_P}) \; = \; b \tag{3.2}$$

where M_U is the Universe's total mass inside the present-day Hubble radius R_H; m_P, L_P are the Planck mass, and length, respectively. What (3.2) indicates is that the observed Universe's total mass M_U coincides with the product of the maximal proper force times the Hubble horizon scale (an infrared cutoff), and which in turn, is the black hole horizon radius corresponding to a Universe-mass black hole. The Planck mass is the product of the maximal proper force times the Planck scale (ultraviolet cutoff), and which in turn, is the black hole horizon radius corresponding to a Planck-mass black hole. And so forth, namely a black hole's mass M coincides with the product of the maximal proper force b with its black hole horizon radius R_h.

Eq-(3.2) is also invariant under the scalings (3.1). Milgrom [14] long ago pointed out the importance of scale invariance for MOND phenomenology, it is still impressive how much of that phenomenology, (i.e. flat rotation curves and baryonic Tully Fisher) is a consequence of scale invariance alone.

The recent new hypothesis of [34] is that this scale invariance is due to the dark matter undergoing a second order phase transition in the region normally associated with MONDian behavior. It is based on the idea that dark matter has a super-fluid phase [35] which, if successful, would explain the flattening of the rotation curves, the Tully-Fisher [36] and acceleration relations and the relation between $a_o \sim c^2/R_H$.

In this work we do not have to recur to these hypothesis [34]. Eqs-(2.20, 2.21) lead to these acceleration relations. A proposal that advocates the fall of dark matter can be found in [38]. Scale invariance is assumed in the *empty* regions of space. The Weyl gauge field A_μ of dilatations contributes to modifications of the Christoffel connection leading then to repulsive corrections to the geodesic equations. We have not invoked the role of Quantum Gravity in this work nor what are the asymptotic symmetries (if any). What we find remarkable is how powerful is the diffeomorphism symmetry of Einstein's vacuum field equations to account for the Finsler gravity solution described here, and which is able to model MOND by simply replacing the radial coordinate r with the areal radial function $R(r, v(r))$.

APPENDIX A : Schwarzschild-like solutions in $D > 3$

In this Appendix we follow closely the calculations of the static spherically symmetric *vacuum* solutions to Einstein's equations in any dimension $D > 3$. Let us start with the line element with the Lorentzian signature $(-, +, +, +, ..., +)$

$$ds^2 = -e^{\mu(r)}(dt)^2 + e^{\nu(r)}(dr)^2 + R^2(r)\tilde{g}_{ij}d\xi^i d\xi^j. \tag{A.1}$$

where the areal radial function $\rho(r)$ is now denoted by $R(r)$ and which must *not* be $confused$ with the scalar curvature $\mathcal{R}$. Here, the metric $\tilde{g}_{ij}$ corresponds to a homogeneous space and $i, j = 3, 4, ..., D-2$ and the temporal and radial indices are denoted by $1, 2$

respectively. In our text we denoted the temporal index by 0. The only non-vanishing Christoffel symbols are given in terms of the following partial derivatives with respect to the r variable and denoted with a prime

$$\begin{aligned}
&\Gamma^1_{21} = \tfrac{1}{2}\mu', && \Gamma^2_{22} = \tfrac{1}{2}\nu', && \Gamma^2_{11} = \tfrac{1}{2}\mu' e^{\mu-\nu}, \\
&\Gamma^2_{ij} = -e^{-\nu} RR'\tilde{g}_{ij}, && \Gamma^i_{2j} = \tfrac{R'}{R}\delta^i_j, && \Gamma^i_{jk} = \tilde{\Gamma}^i_{jk},
\end{aligned} \tag{A.2}$$

and the only nonvanishing Riemann tensor are

$$\begin{aligned}
&\mathcal{R}^1_{212} = -\tfrac{1}{2}\mu'' - \tfrac{1}{4}\mu'^2 + \tfrac{1}{4}\nu'\mu', && \mathcal{R}^1_{i1j} = -\tfrac{1}{2}\mu' e^{-\nu} RR'\tilde{g}_{ij}, \\
&\mathcal{R}^2_{121} = e^{\mu-\nu}(\tfrac{1}{2}\mu'' + \tfrac{1}{4}\mu'^2 - \tfrac{1}{4}\nu'\mu'), && \mathcal{R}^2_{i2j} = e^{-\nu}(\tfrac{1}{2}\nu' RR' - RR'')\tilde{g}_{ij}, \\
&\mathcal{R}^i_{jkl} = \tilde{R}^i_{jkl} - R'^2 e^{-\nu}(\delta^i_k \tilde{g}_{jl} - \delta^i_l \tilde{g}_{jk}).
\end{aligned} \tag{A.3}$$

The vacuum field equations are

$$\mathcal{R}_{11} = e^{\mu-\nu}(\frac{1}{2}\mu'' + \frac{1}{4}\mu'^2 - \frac{1}{4}\mu'\nu' + \frac{(D-2)}{2}\mu'\frac{R'}{R}) = 0, \tag{A.4}$$

$$\mathcal{R}_{22} = -\frac{1}{2}\mu'' - \frac{1}{4}\mu'^2 + \frac{1}{4}\mu'\nu' + (D-2)(\frac{1}{2}\nu'\frac{R'}{R} - \frac{R''}{R}) = 0, \tag{A.5}$$

and

$$\mathcal{R}_{ij} = \frac{e^{-\nu}}{R^2}(\frac{1}{2}(\nu' - \mu')RR' - RR'' - (D-3)R'^2)\tilde{g}_{ij} + \frac{k}{R^2}(D-3)\tilde{g}_{ij} = 0, \tag{A.6}$$

where $k = \pm 1$, depending if $\tilde{g}_{ij}$ refers to positive or negative curvature. From the combination $e^{-\mu+\nu}\mathcal{R}_{11} + \mathcal{R}_{22} = 0$ we get

$$\mu' + \nu' = \frac{2R''}{R'}. \tag{A.7}$$

The solution of this equation is

$$\mu + \nu \;=\; \ln R'^2 \;+\; C, \tag{A.8}$$

where C is an integration constant that one sets to *zero* if one wishes to recover the flat Minkowski spacetime metric in spherical coordinates in the asymptotic region $r \to \infty$.

Substituting (A.7) into the equation (A.6) we find

$$e^{-\nu}\;(\;\nu' RR' - 2RR'' - (D-3)R'^2\;) \;=\; -\,k(D-3) \tag{A.9}$$

or

$$\gamma' RR' + 2\gamma RR'' + (D-3)\gamma R'^2 = k(D-3), \tag{A.10}$$

where

$$\gamma = e^{-\nu}. \tag{A.11}$$

The solution of (A.10) for an ordinary D-dim spacetime (one temporal dimension) corresponding to a $D-2$-dim sphere for the homogeneous space can be written as

$$\gamma = (1 - \frac{16\pi G_D M}{(D-2)\Omega_{D-2} R^{D-3}})\ (\frac{dR}{dr})^{-2} \Rightarrow$$

$$g_{rr} = e^{\nu} = (1 - \frac{16\pi G_D M}{(D-2)\Omega_{D-2} R^{D-3}})^{-1}\ (\frac{dR}{dr})^{2}. \tag{A.12}$$

where Ω_{D-2} is the appropriate solid angle in $D-2$-dim and G_D is the D-dim gravitational constant whose units are $(length)^{D-2}$. Thus $G_D M$ has units of $(length)^{D-3}$ as it should. When $D=4$ as a result that the 2-dim solid angle is $\Omega_2 = 4\pi$ one recovers from eq-(A.12) the 4-dim Schwarzchild solution. The solution in eq-(A.12) is consistent with Gauss law and Poisson's equation in $D-1$ spatial dimensions obtained in the Newtonian limit.

For the most general case of the $D-2$-dim homogeneous space we should write

$$-\nu = \ln(k - \frac{\beta_D G_D M}{R^{D-3}}) - 2\ln R' \tag{A.13}$$

β_D is a constant equal to $16\pi/(D-2)\Omega_{D-2}$, where Ω_{D-2} is the solid angle in the $D-2$ transverse dimensions to r, t and is given by $2\pi^{(D-1)/2}/\Gamma[(D-1)/2]$.

Thus, according to (A.8) we get

$$\mu = \ln(k - \frac{\beta_D G_D M}{R^{D-3}})\ +\ constant. \tag{A.14}$$

we can set the constant to zero, and this means the line element (A.1) can be written as

$$ds^2 = -(k - \frac{\beta_D G_D M}{R^{D-3}})(dt)^2\ +\ \frac{(dR/dr)^2}{(k - \frac{\beta_D G_D M}{R^{D-3}})}(dr)^2\ +\ R^2(r)\tilde{g}_{ij} d\xi^i d\xi^j\ =$$

$$-(k - \frac{\beta_D G_D M}{R^{D-3}})(dt)^2\ +\ \frac{1}{(k - \frac{\beta_D G_D M}{R^{D-3}})}(dR)^2\ +\ R^2(r)\tilde{g}_{ij} d\xi^i d\xi^j \tag{A.15}$$

One can verify, that the equations (A.4)-(A.6),leading to eqs-(A.9)-(A.10), do *not* determine the form $R(r)$. It is also interesting to observe that the only effect of the homogeneous metric $\tilde{g}_{ij}$ is reflected in the $k = \pm 1$ parameter, associated with a positive (negative) constant scalar curvature of the homogeneous $D-2$-dim space. $k=0$ corresponds to a spatially flat $D-2$-dim section. The metric solution in eq-(1.2) is associated to a different signature than the one chosen in this Appendix, and corresponds to $D=4$ and $k=1$.

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