Quantum Gravity Field

Malik Matwi

Department of Physics, Damascus, Syria, 2017.
malik.matwi@hotmail.com

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Abstract

We study the dynamics of the gravity field according to the quantum fields theory on arbitrary spacetime x^μ. Therefore, we suggest a canonical momentum π_I as a momentum conjugate to the canonical gravity field $\tilde{e}^I = e e^I_\mu n^\mu$. We derive both the canonical gravity field and its conjugate momentum from the holonomy $U(\gamma, A)$ of the complex selfdual connection A^i_a. The canonical momentum π_I is represented in the Lorentz group. We use it in deriving the path integral of the gravity field according to the quantum fields theory. Then, we discuss the situation of the free gravity field (like the electromagnetic field). We find that this situation takes place only in the background spacetime approximation, the situation of low matter density(weak gravity). Then, we derive the Lagrange of the Plebanski two form complex field Σ^i, which is represented in selfdual representation $\left|\Sigma^i\right\rangle$.

We try to combine the two fields(gravity and Plebanski) in one field: K^i_μ. Finally, we derive the static potential of exchanging gravitons in scalar and spinor fields, the Newtonian gravitational potential.

key words: Conjugate momentum, path integral, free gravity propagator, Plebanski field Lagrange, Plebanski and gravity fields combination.

1 The Lagrange of the quantum gravity field

We search for conditions allow us to consider the gravity as a dynamical field. The problem of the dynamics in the general relativity is that spacetime is itself a dynamical. It interacts with matter, it is an operator $d\hat{x}^\mu$. So we have to consider it as a quantum field as the other fields. Both the dynamical curved spacetime x^μ and the gravity field e^I have the same entity, it is the gravity. Therefore, we study only one of them as a gravity field: $\hat{e}^I$. Then the dynamical spacetime x^μ is implicitly included in the gravity field e^I via $\hat{e}^I = \hat{e}^I_\mu dx^\mu$. We will see that it is substantially different in the background spacetime, where the gravity field becomes like the usual quantum fields in flat spacetime.

To study the dynamics of the gravity field according to the quantum fields theory, we need to find the canonical conjugate momentum $\pi^I(x)$(represented in the Lorentz group). We find that it acts canonically on the local-Lorentz vectors $V^I(x)$ on a closed 3D surface δM immersed in arbitrary curved spacetime x^μ of a manifold M. The closed surface δM is parameterized by three parameters X^1, X^2 and X^3. In a certain gauge, we consider that they carry the spatial indices of the local-Lorentz inertial frame. The local-Lorentz frame($X^I : X^0, X^1, X^2, X^3$) is considered as a tangent-space on the curved spacetime x^μ. We will see that the path integral of the gravity field is independent on this gauge.

Therefore, the exterior derivative operator, on the surface δM, leads to a change along the norm of that surface, so it causes the change in the time dX^0 direction. That allows the 3D surface δM to extend, thus we get the 4D local-Lorentz frame X^I, which, in our gauge, parameterize the four dimensions x^μ coordinates of the curved spacetime x^μ in the manifold M. With that the gravity field propagates from one surface to another by the extension of those surfaces.

To study the gravity field propagation, we suggest canonical states $\left|\tilde{e}^I\right\rangle$ and $\left|\pi^I\right\rangle$ represented in Lorentz group. We use them in deriving the path integral. We find that there is no propagation on the dynamical spacetime x^μ. But in the background spacetime, the gravity field propagates freely like the electromagnetic field.

The holonomy of the complex connection A^i in the quantum gravity is[1, 2]

$$U(\gamma, A) = TrPe^{i\oint_\gamma A}, \tag{1.1}$$

where the path ordered P is defined via

$$Pe^{i\oint_\gamma A} = \sum_{n=0}^{\infty} \int_0^1 ds_1 \int_0^{s_1} ds_2 ... \int_0^{s_{n-1}} ds_n iA\left(\gamma\left(s_n\right)\right) ... iA\left(\gamma\left(s_1\right)\right) : \dot{\gamma}^\mu\left(s\right) = \frac{dx^\mu}{ds},$$

where $\gamma(s)$ is a closed path in the curved spacetime x^μ. In irreducible self-dual representation of the Lorentz group, we write $A = A^i\tau^i$, where τ^i are Pauli matrices. The element $U(\gamma, A)$ is invariant under local-Lorentz transformation $V^I \to L^I_J(x)V^J$ and under arbitrary changing of the spacetime coordinates $dx^\mu \to \Lambda^\mu{}_\nu(x)dx^\nu$, therefore the quantum gravity is studied using it[1, 2].

The complex connection A^i is selfdual of the real local-Lorentz spin connection $\omega(x)$[1, 2]:

$$A^i_\mu\left(x\right) = \left(P^i\right)_{IJ} \omega^{IJ}_\mu\left(x\right),$$

where P^i are the selfdual projectors.

Let us rewrite the holonomy $U(\gamma, A)$ using the real spin connection $\omega^{IJ}_\mu dx^\mu$ of the local-Lorentz frame:

$$U(\gamma, \omega) = TrPe^{i\oint_\gamma \omega^I{}_J}.$$

We expect that it has the same properties of $U(\gamma, A)$; satisfies the symmetries of GR.

The connection ω^{IJ} transforms under local Lorentz transformation $L(x)_I{}^J$ as:

$$\omega' = L\omega L^{-1} + LdL^{-1} \ \ or \ \ \omega'_\mu dx^\mu = L\omega_\mu L^{-1}dx^\mu + L\partial_\mu L^{-1}dx^\mu,$$

therefore

$$\oint_\gamma \omega'_\mu dx^\mu = \oint_\gamma L\omega_\mu L^{-1}dx^\mu + \oint_\gamma L\partial_\mu L^{-1}dx^\mu.$$

The connection $L\partial_\mu L^{-1}$ is polar: it reverses sign when $x^\mu \to -x^\mu$, where $L(x^\mu) = L(-x^\mu)$, so its integral over a closed curve γ vanishes. So

$$\oint_\gamma \omega'_\mu dx^\mu = \oint_\gamma L\omega_\mu L^{-1}dx^\mu.$$

Therefore, the connection ω transforms covariantly inside the integration over a closed curve. Let us write it as

$$\omega_\mu dx^\mu = \Omega_\mu dx^\mu + B_\mu dx^\mu \ with \ \oint_\gamma B_\mu dx^\mu = 0,$$

where the tensor $\Omega^{IJ}_\mu dx^\mu$ transforms covariantly under local Lorentz transformation, while the polar connection B_μ does not. Therefore

$$L(\omega - B)L^{-1} = \omega' - B' \ ; \ \Omega' = L\Omega L^{-1}$$

which yields

$$B' = LBL^{-1} + LdL^{-1} \ with \ \omega' = L\omega L^{-1} + LdL^{-1}, \tag{1.2}$$

for $B = \left(B^{IJ}\right)$ and $\omega = \left(\omega^{IJ}\right)$.
And yields

$$B' = LBL^{-1} - (dL)L^{-1} \ with \ \omega' = L\omega L^{-1} - (dL)L^{-1}, \tag{1.3}$$

for $B = \left(B^I{}_J\right)$ and $\omega = \left(\omega^I{}_J\right)$.

With that, the condition $\oint_\gamma B'_\mu dx^\mu = \oint_\gamma B_\mu dx^\mu = 0$ is satisfied under local Lorentz transformation, because the connection B_μ remains polar under it.

The loop $\oint_\gamma \omega^{IJ} = \oint_\gamma \Omega^{IJ}$ transforms covariantly under local Lorentz transformation. For free gravity field, let us impose the relation:

$$\Omega^{IJ} = \pi_K{}^{IJ} e^K,$$

thus $\pi_K{}^{IJ}(x)$ transforms covariantly under local Lorentz transformation, so we consider it as a conjugate momentum represented in the local Lorentz group and acts on its vectors. Therefore, we consider it as a dynamical operator. The loop becomes

$$\oint_\gamma \omega^{IJ} = \oint_\gamma \pi_K{}^{IJ} e^K. \tag{1.4}$$

Inserting it in the holonomy $U(\gamma, \omega)$, we get

$$U(\gamma, \pi, e) = TrP \exp i \oint_\gamma \left(\pi_K{}^I{}_J\right) e^K_\mu dx^\mu.$$

In the free gravity field, we expect that the momentum π^{IJK} is antisymmetric. Therefore we can write it as

$$\pi^{IJK} = \pi_L \varepsilon^{LIJK}.$$

This is our starting point in studying the dynamics of the quantum gravity. By that, the last holonomy becomes

$$U(\gamma, \pi, e) = TrP \exp i \oint_\gamma \left(\pi^{KI}{}_J\right) e_{K\mu} dx^\mu = TrP \exp i \oint_\gamma \left(\varepsilon^{LKI}{}_J\right) \pi_L e_{K\mu} dx^\mu.$$

Let us expand it as

$$\sum_{n=0}^{\infty} \int_0^1 ds_1 \int_0^{s_1} ds_2 ... \int_0^{s_{n-1}} ds_n \left(i\varepsilon^{LKI}{}_J \pi_L e_{K\mu} \dot{\gamma}^\mu\right)(s_n) \left(i\varepsilon^{L_1 K_1 J}{}_{J_1} \pi_{L_1} e_{K_1 \mu_1} \dot{\gamma}^{\mu_1}\right)(s_{n-1})$$
$$... \left(i\varepsilon^{L_{n-1} K_{n-1} J_{n-2}}{}_I \pi_{L_{n-1}} e_{K_{n-1}\mu_{n-1}} \dot{\gamma}^{\mu_{n-1}}\right)(s_1),$$

where $\left(i\varepsilon^{LKI}{}_J \pi_L e_{K\mu} \dot{\gamma}^\mu\right)(s_n) = i\varepsilon^{LKI}{}_J \pi_L(s_n) e_{K\mu}(s_n) \dot{\gamma}^\mu(s_n)$, with the tangent $\dot{\gamma}^\mu(s) = \frac{dx^\mu}{ds}$ on the closed path $\gamma(s)$ in the manifold M.

Using the properties

$$\varepsilon_{IJKL} \varepsilon^{IJK_1L_1} = -2\left(\delta_K^{K_1} \delta_L^{L_1} - \delta_K^{L_1} \delta_L^{K_1}\right) \ \textit{and} \ \varepsilon_{IJKL} \varepsilon^{I_1 JKL} = -6\delta_I^{I_1},$$

the integrals of the holonomy $U(\gamma, \pi, e)$ become over terms like

$$...\pi_I(s_j)e^I_\mu(s_i)\dot{\gamma}^\mu(s_i)ds_i...\pi_J(s_i)e^J_\nu(s_k)\dot{\gamma}^\nu(s_k)ds_k...\ with\ i \neq j\ and\ i \neq k.$$

This holonomy satisfies the general relativity symmetries; invariance under local Lorentz transformation $V^I \to L^I{}_J(x)V^J$ and under arbitrary changing of the coordinates $dx^\mu \to \Lambda^\mu{}_\nu(x)dx^\nu$. Therefore, we use it in the quantum gravity.

Let us suggest another term: $\oint_\gamma \pi_K e^K_\mu dx^\mu$. We expect that it also satisfies the general relativity symmetries if it is integrated over a closed 3D surface δM instead of the closed path $\gamma(s)$. This is because

$$ed^4x = \frac{1}{4}d^3x_\mu \wedge dx^\mu = \frac{1}{4}e\varepsilon_{\mu\nu\rho\sigma}dx^\nu \wedge dx^\rho \wedge dx^\sigma \wedge dx^\mu/3!$$

is invariant element, so we can replace $\pi_K e^K_\mu dx^\mu$ with

$$\pi_K e^{K\mu} d^3x_\mu = \pi_K e^{K\mu} e\varepsilon_{\mu\nu\rho\sigma}dx^\nu \wedge dx^\rho \wedge dx^\sigma/3!.$$

With integrating it over a three dimensions closed surface δM, it becomes invariant under GR transformations because in the free gravity there are no sources for the gravity field. Therefore, the flux of the Lorentz vectors is invariant under arbitrary changing of the closed surface δM.

The determinant e of the gravity field e^I_μ is defined in $e = \sqrt{-g}$ with writing the metric $g_{\mu\nu}(x)$ on the curved spacetime x^μ as

$$g_{\mu\nu}(x) = \eta_{IJ}e^I_\mu e^J_\nu.$$

Under arbitrary transformations, we have invariant element:

$$\sqrt{g}\varepsilon_{i_1...i_n} = \sqrt{g'}\varepsilon'_{i_1...i_n}.$$

Therefore, the element

$$e\varepsilon_{\mu\nu\rho\sigma}dx^\nu \wedge dx^\rho \wedge dx^\sigma/3! = d^3x_\mu$$

is a co-vector, as ∂_μ. Thus, the integral

$$U(\delta M, \pi, e) = \exp i \oint_{\delta M} \pi_I e^{I\mu} e\varepsilon_{\mu\nu\rho\sigma}dx^\nu \wedge dx^\rho \wedge dx^\sigma/3! = \exp i \oint_{\delta M} \pi_I e^{I\mu} d^3x_\mu$$

satisfies the same conditions of the holonomy $U(\gamma, A)$; invariant under local Lorentz transformation $V^I \rightarrow L^I_J(x)V^J$ and under arbitrary changing of the coordinates $dx^\mu \rightarrow \Lambda^\mu{}_\nu(x)dx^\nu$. This relates to the fact that the integrals of free vectors over a closed surface δM, in a manifold M, are invariant under arbitrary changing of that surface if there are no sources for those vectors. It is the conservation. The spin connection ω^μ and so $\pi_K e^{K\mu}$, as vectors, satisfy this fact in the free gravity.

The equation of motion of the gravity field e^I is

$$De^I = de^I + \omega^I{}_J \wedge e^J = 0.$$

Replacement $\omega^{IJ} \rightarrow \pi_K{}^{IJ} e^K$, we get

$$de^I \rightarrow -\pi_N{}^I{}_J e^N \wedge e^J.$$

But the tensor

$$e^N \wedge e^J = e^N_\mu e^J_\nu dx^\mu \wedge dx^\nu = \frac{1}{2}\left(e^N_\mu e^J_\nu - e^N_\nu e^J_\mu\right) dx^\mu \wedge dx^\nu$$

measures the area in the manifold M. Therefore, the changes of the gravity field around a closed path (rotation) relate to the flux of the momentum π through the area determined by that closed path. It is like the magnetic field generated by straight electric current. Thus we have

$$e^N \wedge e^J \rightarrow area,$$

$$de^I \equiv -\pi_N{}^I{}_J e^N \wedge e^J \rightarrow flux\ through\ this\ area.$$

For this reason, we suggested that the conjugate momentum π^{IJK} is anti-symmetric.

Now, in the integral

$$\exp i \oint_{\delta M} \pi_I e^{I\mu} e \varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3!,$$

let us define the canonical gravity field $\tilde{e}^I$ via

$$\tilde{e}^I d^3X = \tilde{e}^I dX^1 dX^2 dX^3 \equiv e^{I\mu} e \varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3!,$$

or

$$\tilde{e}^I = e e^I_\mu n^\mu \left(X^i\right),$$

where $n^\mu(X^i)$ is the norm to the surface δM. Thus, the holonomy $U(\delta M, \pi, e)$ becomes

$$\tilde{U}_{\delta M}(\delta M, \pi, \tilde{e}) = \exp i \oint_{\delta M} \pi_I \tilde{e}^I d^3 X,$$

where the parameters $X^I : I = 1, 2, 3$ parameterize the closed 3D surface δM in the manifold M. As mentioned before, in certain gauge, we consider that the indices $I = 1, 2, 3$ are the spatial indices of the local-Lorentz frame($I = 0, 1, 2, 3$). Therefore, the exterior derivative, on the surface δM, becomes along the time dX^0. The time dX^0 is the direction of the norm on the surface $\delta M(X^1, X^2, X^3)$. We will see that the result of the path integral is independent on this gauge.

As we suggested before, the integral $\exp i \oint_{\delta M} \pi_I \tilde{e}^I d^3 X$ satisfies the same conditions of the holonomy $U(\gamma, A)$; The GR symmetries, therefore we consider it as a canonical dynamical element.

Comparing it with

$$\langle \phi \mid \pi \rangle = \exp i \int d^3 X \phi(X) \pi(X)/\hbar,$$

a canonical relation in the scalar field ϕ theory on flat spacetime, for $\hbar = 1$, we suggest canonical states $\left|\tilde{e}^I\right\rangle$ and $\left|\pi^I\right\rangle$ with

$$\left\langle \tilde{e}^I \mid \pi_I \right\rangle_{\delta M} = \exp i \int_{\delta M} \tilde{e}^I(X) \pi_I(X) d^3 X,$$

where π_I is the canonical momentum conjugate to $\tilde{e}^I$. Let us write this relation on the surface δM as

$$\left\langle \tilde{e}^I \mid \pi_I \right\rangle_{\delta M} = \prod_{n,I} \left\langle \tilde{e}^I \left(x_n + dx_n\right) \mid \pi_I \left(x_n\right) \right\rangle_{\delta M},$$

with

$$\left\langle \tilde{e}^I \left(x_n + dx_n\right) \mid \pi_I \left(x_n\right) \right\rangle_{\delta M} = \exp i \tilde{e}^I(x_n + dx_n) \pi_I(x_n) d^3 X \to \exp i \tilde{e}^I(x_n) \pi_I(x_n) d^3 X.$$

In general, for two points in adjacent surfaces δM_1 and δM_2, let us rewrite it as

$$\left\langle \tilde{e}^I\left(x_n + dx_n\right) \,\middle|\, \pi_I\left(x_n\right)\right\rangle = \exp i\tilde{e}^I(x_n + dx_n)\pi_I(x_n)d^3X. \tag{1.5}$$

Here the variation

$$\tilde{e}^I(x_n + dx_n) - \tilde{e}^I(x_n)$$

is exterior derivative along the time dX^0 direction, the direction of the norm to the surface δM_1. It allows the extension of the surface: $\delta M(X^1, X^2, X^3) \to M(X^0, X^1, X^2, X^3)$. This leads to the propagation of those surfaces.

We need to make $\hat{e}d^4\hat{x}$ commute with $\hat{\tilde{e}}^I d^3X$. For this purpose, we write

$$\begin{aligned} -\hat{e}d^4\hat{x} &= \hat{e}d\hat{x}^\mu \wedge \varepsilon_{\mu\nu\rho\sigma}d\hat{x}^\nu \wedge d\hat{x}^\rho \wedge d\hat{x}^\sigma/4! \\ &= \hat{e}d\hat{x}^\mu \wedge \frac{\varepsilon_{\mu\nu\rho\sigma}}{4!}\frac{\partial\hat{x}^\nu}{\partial X^i}\frac{\partial\hat{x}^\rho}{\partial X^j}\frac{\partial\hat{x}^\sigma}{\partial X^k}\frac{\varepsilon^{ijk}}{3!}d^3X = \frac{1}{4}\hat{e}d\hat{x}^\mu\hat{n}_\mu d^3X. \end{aligned}$$

The indexes i, j *and* k, in our gauge, are the local-Lorentz frame indices for $I = 1, 2, 3$.
We can rewrite it, in our gauge, as:

$$-ed^4x = \frac{1}{4}edx^\mu n_\mu d^3X = \frac{1}{4}e\frac{\partial x^\mu}{\partial X^0}n_\mu d^3X dX^0 = \frac{1}{4}ee_0^\mu n_\mu d^3X dX^0.$$

Comparing it with the term

$$\tilde{e}^I d^3X = e^{I\mu}e\varepsilon_{\mu\nu\rho\sigma}dx^\nu \wedge dx^\rho \wedge dx^\sigma/3! = ee^{I\mu}n_\mu d^3X,$$

we find that it commutes with it:

$$\left[\hat{e}\hat{e}^{I\mu}\hat{n}_\mu d^3X, \hat{e}\hat{e}_0^\mu\hat{n}_\mu d^3X dX^0\right] = 0 \to \left[\hat{\tilde{e}}^I d^3X, \hat{e}d^4\hat{x}\right] = 0,$$

where $\left[\hat{e}_\mu^I, \hat{e}_\nu^J\right] = 0$. Thus, the operator $\hat{e}d^4\hat{x}$ takes eigenvalues when it acts on the states $\left|\tilde{e}^I\right\rangle$.

The action of the gravity field is[1]

$$S(e,\omega) = \frac{1}{16\pi G}\int \varepsilon_{IJKL}\left(e^I \wedge e^J \wedge R^{KL}(\omega) + \lambda e^I \wedge e^J \wedge e^K \wedge e^L\right).$$

Let us consider only the first term:

$$S(e,\omega) = c\int \varepsilon_{IJKL} e^I \wedge e^J \wedge R^{KL}(\omega),$$

where C is constant. The Riemann curvature here is

$$R^{KL}(\omega) = d\omega^{KL} + \omega^K{}_M \wedge \omega^{ML}.$$

Inserting the relation we suggested before:

$$(\omega)^{IJ} = \Omega^{IJ} + B^{IJ} \ ; \ \Omega^{IJ} = \pi_K{}^{IJ} e^K,$$

the action becomes

$$S(e,\pi) = c\int \varepsilon_{IJKL} e^I \wedge e^J \wedge (d\Omega + dB + \Omega\wedge\Omega + \Omega\wedge B + B\wedge\Omega + B\wedge B)^{KL}. \tag{1.6}$$

As we assumed before that Ω^{IJ} transforms covariantly, so the remaining term $d\Omega + dB + \Omega\wedge B + B\wedge\Omega + B\wedge B$ also transforms covariantly, this is because the action transforms covariantly.
Let us choose a gauge for the polar vector B_μ:

$$d\Omega + dB + \Omega\wedge B + B\wedge\Omega + B\wedge B = 0.$$

This gauge satisfies the conditions (1.2) and (1.3), let us test them:

$$d\left(L\Omega L^{-1}\right) + d\left(LBL^{-1} + LdL^{-1}\right) + L\Omega L^{-1}\wedge\left(LBL^{-1} + LdL^{-1}\right)$$

$$+\left(LBL^{-1} - (dL)L^{-1}\right)\wedge L\Omega L^{-1} + \left(LBL^{-1} - (dL)L^{-1}\right)\wedge\left(LBL^{-1} + LdL^{-1}\right).$$

Expanding it:

$$(dL)\wedge\Omega L^{-1} + L\left(d\Omega\right)L^{-1} - L\Omega\wedge\left(dL^{-1}\right) + (dL)\wedge BL^{-1} + L\left(dB\right)L^{-1} - LB\wedge dL^{-1}$$

$$+\,(dL)\wedge dL^{-1} + L\Omega L^{-1}\wedge LBL^{-1} + L\Omega L^{-1}\wedge LdL^{-1} + LBL^{-1}\wedge L\Omega L^{-1}$$

$$-(dL)L^{-1}\wedge L\Omega L^{-1} + LBL^{-1}\wedge LBL^{-1} - (dL)L^{-1}\wedge LBL^{-1} + LBL^{-1}\wedge LdL^{-1}$$

$$-(dL)L^{-1}\wedge LdL^{-1},$$

so

$$(dL)\wedge\Omega L^{-1}+L\,(d\Omega)\,L^{-1}-L\Omega\wedge(dL^{-1})+(dL)\wedge BL^{-1}+L\,(dB)\,L^{-1}-LB\wedge dL^{-1}$$

$$+\,(dL)\wedge dL^{-1}+L\Omega\wedge BL^{-1}+L\Omega\wedge dL^{-1}+LB\wedge\Omega L^{-1}-(dL)\wedge\Omega L^{-1}$$

$$+LB\wedge BL^{-1}-(dL)\wedge BL^{-1}+LB\wedge dL^{-1}-(dL)\wedge dL^{-1},$$

it becomes

$$L\,(d\Omega)\,L^{-1}+L\,(dB)\,L^{-1}+L\Omega\wedge BL^{-1}+LB\wedge\Omega L^{-1}+LB\wedge BL^{-1},$$

therefore, as expected, it transforms covariantly and satisfies the conditions (1.2) and (1.3), therefore we can choose the gauge

$$d\Omega+dB+\Omega\wedge B+B\wedge\Omega+B\wedge B=0.$$

We need it, because the polar vector B_μ is not included in the loop (1.4), so it must not be included in the Lagrange.

Finally, the action becomes

$$S(e,\pi)=c\int\varepsilon_{IJKL}e^I\wedge e^J\wedge(\Omega\wedge\Omega)^{KL},$$

or

$$S(e,\pi)=c\int\varepsilon_{IJKL}e^I\wedge e^J\wedge\Omega^K{}_M\wedge\Omega^{ML}.$$

Inserting our assumption $\Omega^{IJ}=\pi_K{}^{IJ}e^K$, this action becomes

$$S(e,\pi)=c\int\varepsilon_{IJKL}e^I\wedge e^J\wedge\left(\pi_{K_1}{}^K{}_M\right)e^{K_1}\wedge\left(\pi_{K_2}{}^{ML}\right)e^{K_2}.$$

Making the replacement

$$e^I\wedge e^J\wedge e^{K_1}\wedge e^{K_2}\rightarrow\varepsilon^{IJK_1K_2}e^0\wedge e^1\wedge e^2\wedge e^3,$$

we get

$$S(e,\pi)=c\int\varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right)\varepsilon^{IJK_1K_2}e^0\wedge e^1\wedge e^2\wedge e^3.$$

Inserting the relation $\pi^{IJL} = \pi_K \varepsilon^{KIJL}$ we imposed before, and using $\varepsilon_{IJKL}\varepsilon^{IJK_1K_2} = -2\left(\delta_K^{K_1}\delta_L^{K_2} - \delta_L^{K_1}\delta_K^{K_2}\right)$, the action becomes

$$S(e,\pi) = c\int 2\pi^I \varepsilon_{ILMK}\pi_J\varepsilon^{JLMK} e^0 \wedge e^1 \wedge e^2 \wedge e^3,$$

and using $\varepsilon_{ILMK}\varepsilon^{JLMK} = -6\delta_I^J$, it becomes

$$S(e,\pi) = -12c\int \pi_I\pi^I e^0 \wedge e^1 \wedge e^2 \wedge e^3$$

Or

$$S_0(e,\pi) = -12c\int \pi^2 e d^4x.$$

In the background spacetime, we have $e \to 1 + \delta e$, so this action becomes

$$S_0(\delta e,\pi) \to -12c\int \pi^2 d^4x + ...$$

To find its meaning, we compare it with the scalar field Lagrange in the flat spacetime:

$$Ld^4x = \left(\pi\partial_0\phi - H(\phi,\pi)\right)d^4x \text{ with } H(\phi,\pi)d^4x = \left(\frac{1}{2}\pi^2 + \frac{1}{2}\left(\nabla\phi\right)^2 + \frac{1}{2}m^2\phi^2\right)d^4x.$$

For $\hbar = 1$, we conclude that the term

$$\int 12c\pi^2 d^4x \succ 0$$

is the energy of the gravity field in the background spacetime. The missed term is the kinetic term, like $\pi\partial_0\phi$, we find it using the path integral.

We derive the path integral according to the quantum fields theory. As we saw before, in our gauge, the operator $\hat{e}d^4\hat{x}$ takes eigenvalues when it acts on the states $\left|\tilde{e}^I\right\rangle$. Using eq.(1.2), we get the amplitude

$$\left\langle \tilde{e}^I\left(x+dx\right)\right| e^{iS}\left|\pi_I\left(x\right)\right\rangle \to \left\langle \tilde{e}^I\left(x+dx\right)\right| e^{-i12c\hat{\pi}^2\hat{e}d^4\hat{x}}\left|\pi_I\left(x\right)\right\rangle$$

$$= \exp\left(-i12c\pi^2\left(x\right)e\left(x+dx\right)d^4x + i\tilde{e}^I\left(x+dx\right)\pi_I(x)d^3X\right)$$

$$\to \exp\left(-i12c\pi^2\left(x\right)e\left(x\right)d^4x + i\tilde{e}^I\left(x+dx\right)\pi_I(x)d^3X\right),$$

where we let the momentum π_I acts on the left. The amplitude of the propagation between two points x and $x+dx$ of adjacent surfaces δM_1 and δM_2 is

$$\left\langle \tilde{e}_I\left(x+dx\right)\right| e^{-ic12\hat{\pi}^2\hat{e}d^4\hat{x}} \left|\tilde{e}^I\left(x\right)\right\rangle_{\delta M_1\to\delta M_2}$$

$$=\int\prod_I d\pi^I \left\langle \tilde{e}_I\left(x+dx\right)\right| e^{-ic12\hat{\pi}^2\hat{e}d^4\hat{x}} \left|\pi^I\left(x\right)\right\rangle_{\delta M_1\to\delta M_2} \left\langle \pi_I\left(x\right) \,\middle|\, \tilde{e}^I\left(x\right)\right\rangle_{\delta M_1}$$

$$=\int\prod_I d\pi^I \exp\left[-i12c\pi^2\left(x\right)e\left(x+dx\right)d^4x+i\tilde{e}^I(x+dx)\pi_I(x)d^3X\right]\exp\left(-i\tilde{e}^I(x)\pi_I(x)d^3X\right)$$

$$\to\int\prod_I d\pi^I \exp\left[-i12c\pi^2\left(x\right)e\left(x\right)d^4x+i\left(\tilde{e}^I(x+dx)-\tilde{e}^I(x)\right)\pi_I(x)d^3X\right].$$

The exterior derivative

$$\left(\tilde{e}^I(x+dx)-\tilde{e}^I(x)\right)d^3X=\frac{\partial\tilde{e}^I(x)}{\partial X^0}d^3XdX^0=d\tilde{e}^I(x)d^3X$$

is along the time dX^0 direction, the direction of the norm to the surface $\delta M(X^1,X^2,X^3)$, so it leads to the propagation from one surface to another.

Thus, we write the amplitude as

$$\left\langle \tilde{e}_I\left(x+dx\right)\right| e^{-ic12\hat{\pi}^2\hat{e}d^4\hat{x}} \left|\tilde{e}^I\left(x\right)\right\rangle_{\delta M_1\to\delta M_2}$$
$$=\int\prod_I d\pi^I \exp\left[-i12c\pi^2\left(x\right)e\left(x\right)d^4x+i\pi_I(x)d\tilde{e}^I(x)d^3X\right].$$

The path integral is the integral of ordered product of those amplitudes on all spacetime points(over all ordered 3D surfaces), thus we write it as

$$W_{ST}=\int\prod_I D\tilde{e}^I D\pi_I \exp i\int\left(-12c\pi^2ed^4x+\pi_Id\tilde{e}^Id^3X\right)$$
$$=\int\prod_I D\tilde{e}^I D\pi_I \exp i\int\left(-12c\pi^2e^0\wedge e^1\wedge e^2\wedge e^3+\pi_Id\tilde{e}^Id^3X\right).$$

There is no problem with Lorentz non-invariance in $\frac{\partial\tilde{e}^I(x)}{\partial X^0}d^3XdX^0$, because the equation of motion we find in the result of the path integral is

$$\frac{\partial\tilde{e}^I(x)}{\partial X^0}\propto-\pi^I,$$

thus we have

$$\frac{\partial\tilde{e}^I(x)}{\partial X^0}\pi_I d^3X dX^0 \propto -\pi_I\pi^I d^3X dX^0.$$

This is Lorentz invariant. This is like the equation of motion of the scalar field ϕ; $\pi = \partial_0\phi$ which solves the same problem.
In our gauge, we have

$$\begin{aligned}\pi_I\pi^I d^3X dX^0 \to \pi^2 dX^0\wedge dX^1\wedge dX^2\wedge dX^3 &= \pi^2 e^0_\mu e^1_\nu e^2_\rho e^3_\sigma dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma\\ &= \pi^2 e^0_\mu e^1_\nu e^2_\rho e^3_\sigma \varepsilon^{\mu\nu\rho\sigma}d^4x = \pi^2 e d^4x.\end{aligned}$$

It is invariant element, we find it in the path integral.

The path integral

$$W_{ST} = \int\prod_I D\tilde{e}^I D\pi_I \exp i\int\left(-12c\pi^2 e^0\wedge e^1\wedge e^2\wedge e^3 + \pi_I d\tilde{e}^I d^3X\right)$$

vanishes unless

$$\frac{\delta S(\pi,e)}{\delta\pi_I} = \frac{\delta}{\delta\pi_I}\left(-12c\pi^2 e^0\wedge e^1\wedge e^2\wedge e^3 + \pi_I d\tilde{e}^I d^3X\right) = -24c\pi^I e^0\wedge e^1\wedge e^2\wedge e^3 + d\tilde{e}^I d^3X = 0.$$

Therefore we get the path(equation of motion):

$$\hat{\pi}^I = \frac{1}{24c}\left(\hat{e}^0\wedge\hat{e}^1\wedge\hat{e}^2\wedge\hat{e}^3\right)^{-1} d\hat{\tilde{e}}^I d^3X, \tag{1.7}$$

or

$$\pi^I\pi^J = \frac{1}{(24c)^2}\left(e^0\wedge e^1\wedge e^2\wedge e^3\right)^{-2} d\tilde{e}^I d^3X d\tilde{e}^J d^3X. \tag{1.8}$$

So,

$$\begin{aligned}-12c\pi^2 e^0\wedge e^1\wedge e^2\wedge e^3 + \pi_I d\tilde{e}^I d^3X &= \frac{-1}{48c}\left(e^0\wedge e^1\wedge e^2\wedge e^3\right)^{-1}\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right)\\ &+\frac{1}{24c}\left(e^0\wedge e^1\wedge e^2\wedge e^3\right)^{-1}\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right).\end{aligned}$$

Inserting it in the path integral, thus we get

$$W_{ST} = \int\prod_I D\tilde{e}^I exp\frac{i}{48c}\int\left(e^0\wedge e^1\wedge e^2\wedge e^3\right)^{-1}\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right).$$

The canonical field $\tilde{e}^I$ is defined in

$$\tilde{e}^K d^3X = e^{K\mu} e \varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3!.$$

Applying the exterior derivative, we get

$$\left(d\hat{\tilde{e}}^K\right) d^3X = \left(\hat{D}_{\mu_1} \hat{e}^{K\mu}\right) \hat{e} \varepsilon_{\mu\nu\rho\sigma} d\hat{x}^{\mu_1} \wedge d\hat{x}^\nu \wedge d\hat{x}^\rho \wedge d\hat{x}^\sigma / 3!,$$

where D is the co-variant derivative defined in

$$DV^I = dV^I + \omega^I{}_J \wedge V^J.$$

Thus, the term

$$\left(e^0 \wedge e^1 \wedge e^2 \wedge e^3\right)^{-1} \left(d\tilde{e}_I d^3X\right) \left(d\tilde{e}^I d^3X\right) = \frac{\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right)}{e^0 \wedge e^1 \wedge e^2 \wedge e^3},$$

in the path integral, becomes

$$\frac{\left(\hat{D}_{\mu_1} \hat{e}_I^\mu\right) \hat{e} \varepsilon_{\mu\nu\rho\sigma} d\hat{x}^{\mu_1} \wedge d\hat{x}^\nu \wedge d\hat{x}^\rho \wedge d\hat{x}^\sigma \left(\hat{D}_{\mu_2} \hat{e}^{I\mu'}\right) \hat{e} \varepsilon_{\mu'\nu'\rho'\sigma'} d\hat{x}^{\mu_2} \wedge d\hat{x}^{\nu'} \wedge d\hat{x}^{\rho'} \wedge d\hat{x}^{\sigma'}}{3!3! \hat{e}^0_{\mu_3} \hat{e}^1_{\nu_3} \hat{e}^2_{\rho_3} \hat{e}^3_{\sigma_3} d\hat{x}^{\mu_3} \wedge d\hat{x}^{\nu_3} \wedge d\hat{x}^{\rho_3} \wedge d\hat{x}^{\sigma_3}}.$$

Let us define the inverse:

$$\left(e^0_\mu e^1_\nu e^2_\rho e^3_\sigma dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma\right)^{-1} = E_0^{\mu'} E_1^{\nu'} E_2^{\rho'} E_3^{\sigma'} \frac{\partial}{\partial x^{\sigma'}} \wedge \frac{\partial}{\partial x^{\rho'}} \wedge \frac{\partial}{\partial x^{\nu'}} \wedge \frac{\partial}{\partial x^{\mu'}}.$$

We can rewrite:

$$e^0_\mu e^1_\nu e^2_\rho e^3_\sigma dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = \frac{1}{4} e d^3x_\mu \wedge dx^\mu.$$

(Actually, we have to rewrite the tensors $\varepsilon^{\mu\nu\rho\sigma}$ and $\varepsilon_{\mu\nu\rho\sigma}$ as $e^{-1}\varepsilon^{\mu\nu\rho\sigma}$ and $e\varepsilon_{\mu\nu\rho\sigma}$, but here we neglect this because we get the same results).

Also, we can rewrite:

$$E_0^{\mu'} E_1^{\nu'} E_2^{\rho'} E_3^{\sigma'} \partial_{\sigma'} \wedge \partial_{\rho'} \wedge \partial_{\nu'} \wedge \partial_{\mu'} = E \partial_\nu \wedge \partial^{3\nu},$$

with inner product like

$$\left(E\partial_\nu \wedge \partial^{3\nu}\right) \left(\frac{1}{4} e d^3x_\mu \wedge dx^\mu\right) = \frac{1}{4} E e \partial_\nu \wedge \partial^{3\nu} d^3x_\mu \wedge dx^\mu = \frac{1}{4} E e \left(\delta^\nu_\mu\right) \partial_\nu dx^\mu = Ee = 1.$$

In general, we can write it as

$$\left(E\partial_\nu \wedge \partial^{3\nu}\right)\left(ed^3x_{\mu'} \wedge dx^\mu\right) = Ee\partial_\nu \wedge \partial^{3\nu} d^3x_{\mu'} \wedge dx^\mu = Ee\delta^\nu_{\mu'}\partial_\nu dx^\mu = \delta^\mu_{\mu'}.$$

In the path integral term, let us make the replacements:

$$\left(D_{\mu_1}e^\mu_I\right) e\varepsilon_{\mu\nu\rho\sigma} dx^{\mu_1} \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma/3! \to \left(D_{\mu_1}e^\mu_I\right) edx^{\mu_1} \wedge d^3x_\mu = -\left(D_{\mu_1}e^\mu_I\right) ed^3x_\mu \wedge dx^{\mu_1},$$

and

$$\left(D_{\mu_2}e^{I\mu'}\right) e\varepsilon_{\mu'\nu'\rho'\sigma'} dx^{\mu_2} \wedge dx^{\nu'} \wedge dx^{\rho'} \wedge dx^{\sigma'}/3! \to -\left(D_{\mu_2}e^{I\mu'}\right) ed^3x_{\mu'} \wedge dx^{\mu_2}.$$

Let us assume the following replacing:

$$d^3x_\mu \wedge dx^\mu = -dx_\mu \wedge d^3x^\mu \to d^3x_\mu \wedge dx^{\mu_1} = -dx_\mu \wedge d^3x^{\mu_1}.$$

There is no problem with this trick because in any 4D spacetime we have the contraction $(d^3x_\mu \wedge dx^\nu) = \delta^\nu_\mu d^4x$.
Therefore, we make the replacement:

$$-\left(D_{\mu_1}e^\mu_I\right) ed^3x_\mu \wedge dx^{\mu_1} \to \left(D_{\mu_1}e^\mu_I\right) edx_\mu \wedge d^3x^{\mu_1}.$$

By that, the term

$$\frac{\left(\hat{D}_{\mu_1}\hat{e}^\mu_I\right) \hat{e}\varepsilon_{\mu\nu\rho\sigma} d\hat{x}^{\mu_1} \wedge d\hat{x}^\nu \wedge d\hat{x}^\rho \wedge d\hat{x}^\sigma \left(\hat{D}_{\mu_2}\hat{e}^{I\mu'}\right) \hat{e}\varepsilon_{\mu'\nu'\rho'\sigma'} d\hat{x}^{\mu_2} \wedge d\hat{x}^{\nu'} \wedge d\hat{x}^{\rho'} \wedge d\hat{x}^{\sigma'}}{3!3!\hat{e}^0_{\mu_3}\hat{e}^1_{\nu_3}\hat{e}^2_{\rho_3}\hat{e}^3_{\sigma_3} d\hat{x}^{\mu_3} \wedge d\hat{x}^{\nu_3} \wedge d\hat{x}^{\rho_3} \wedge d\hat{x}^{\sigma_3}},$$

in the path integral, becomes

$$-\left(E\partial_\nu \wedge \partial^{3\nu}\right)\left(\left(D_{\mu_1}e^\mu_I\right) edx_\mu \wedge d^3x^{\mu_1}\right)\left(\left(D_{\mu_2}e^{I\mu'}\right) ed^3x_{\mu'} \wedge dx^{\mu_2}\right)$$

$$= \left(D^{\mu_1}e_{I\mu}\right)\left(D_{\mu_2}e^{I\mu'}\right) e\left(\partial_\nu \wedge \partial^{3\nu}\right)\left(d^3x_{\mu_1} \wedge dx^\mu\right)\left(d^3x_{\mu'} \wedge dx^{\mu_2}\right),$$

where we used

$$-dx_\mu \wedge d^3x^{\mu_1} = d^3x^{\mu_1} \wedge dx_\mu \ then\ d^3x_{\mu_1} \wedge dx^\mu.$$

Thus we can write

$$\frac{\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right)}{e^0 \wedge e^1 \wedge e^2 \wedge e^3} \to \left(D^{\mu_1}e_{I\mu}\right)\left(D_{\mu_2}e^{I\mu'}\right) e\left(\partial_\nu \wedge \partial^{3\nu}\right)\left(d^3x_{\mu_1} \wedge dx^\mu\right)\left(d^3x_{\mu'} \wedge dx^{\mu_2}\right).$$

Let us choose the contraction:

$$\left(\partial_\nu \wedge \partial^{3\nu}\right)\left(d^3x_{\mu_1} \wedge dx^\mu\right)\left(d^3x_{\mu'} \wedge dx^{\mu_2}\right) = \left(\partial_\nu \wedge \partial^{3\nu} d^3x_{\mu_1} \wedge dx^\mu\right)\left(d^3x_{\mu'} \wedge dx^{\mu_2}\right)$$
$$= \delta^\nu_{\mu_1}\left(\partial_\nu \wedge dx^\mu\right)\left(d^3x_{\mu'} \wedge dx^{\mu_2}\right) = \delta^\nu_{\mu_1}\left(-dx^\mu \wedge \partial_\nu\right)\left(-dx^{\mu_2} \wedge d^3x_{\mu'}\right)$$
$$= \delta^\nu_{\mu_1} dx^\mu \wedge \partial_\nu dx^{\mu_2} \wedge d^3x_{\mu'} = \delta^\nu_{\mu_1}\delta^{\mu_2}_\nu dx^\mu \wedge d^3x_{\mu'}.$$

Thus we can write the term in the path integral as

$$\frac{\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right)}{e^0 \wedge e^1 \wedge e^2 \wedge e^3} \to \left(D^{\mu_1} e_{I\mu}\right)\left(D_{\mu_2} e^{I\mu'}\right) e\delta^\nu_{\mu_1}\delta^{\mu_2}_\nu dx^\mu \wedge d^3x_{\mu'}$$
$$= \left(D_\nu e_{I\mu}\right)\left(D^\nu e^{I\mu'}\right) e dx^\mu \wedge d^3x_{\mu'} = -\left(D_\nu e_{I\mu}\right)\left(D^\nu e^{I\mu'}\right) e d^3x_{\mu'} \wedge dx^\mu$$
$$= -\left(D_\nu e_{I\mu}\right)\left(D^\nu e^{I\mu'}\right) e\delta^\mu_{\mu'} d^4x = -\left(D_\nu e_{I\mu}\right)\left(D^\nu e^{I\mu}\right) e d^4x.$$

We can also choose another contraction:

$$\left(D^{\mu_1} e_{I\mu}\right)\left(D_{\mu_2} e^{I\mu'}\right) e\left(\partial_\nu \wedge \partial^{3\nu}\right)\left(d^3x_{\mu_1} \wedge dx^\mu\right)\left(d^3x_{\mu'} \wedge dx^{\mu_2}\right) \to$$
$$\left(D^{\mu_1} e_{I\mu}\right)\left(D_{\mu_2} e^{I\mu'}\right) e\left(\partial_\nu \wedge \partial^{3\nu} d^3x_{\mu_1} \wedge dx^\mu\right)\left(d^3x_{\mu'} \wedge dx^{\mu_2}\right)$$
$$= \left(D^{\mu_1} e_{I\mu}\right)\left(D_{\mu_2} e^{I\mu'}\right) e\left(\delta^\nu_{\mu_1}\partial_\nu dx^\mu\right)\left(d^3x_{\mu'} \wedge dx^{\mu_2}\right)$$
$$= \delta^\nu_{\mu_1}\delta^\mu_\nu\left(D^{\mu_1} e_{I\mu}\right)\left(D_{\mu_2} e^{I\mu'}\right) e\left(d^3x_{\mu'} \wedge dx^{\mu_2}\right).$$

Thus, we get

$$\frac{\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right)}{e^0 \wedge e^1 \wedge e^2 \wedge e^3} \to \left(D^\mu e_{I\mu}\right)\left(D_{\mu'} e^{I\mu'}\right) e d^4x.$$

By the two possible contractions, we can write the final result as

$$-\left(e^0 \wedge e^1 \wedge e^2 \wedge e^3\right)^{-1}\left(d\tilde{e}_I d^3X\right)\left(d\tilde{e}^I d^3X\right) = \frac{1}{2}\left(D_\mu e^\nu_I D^\mu e^I_\nu - D_\mu e^\nu_I D_\nu e^{I\mu}\right) e d^4x.$$

This Lagrange is like the Lagrange of the electromagnetic field, but with opposite sign. It is also independent on the gauge we chose for the surface δM: It is invariant under local Lorentz transformation $V^I \to L^I{}_J(x)V^J$ and under any coordinate transformation $V^{\mu} \to \frac{\partial x^{\mu}}{\partial x'^{\nu}} V'^{\nu}$.

The path integral of the gravity field

$$W_{ST} = \int \prod_{I} D\tilde{e}^{I} exp \frac{i}{48c} \int \left(e^{0} \wedge e^{1} \wedge e^{2} \wedge e^{3}\right)^{-1} \left(d\tilde{e}_{I} d^{3}X\right) \left(d\tilde{e}^{I} d^{3}X\right)$$

becomes

$$W_{ST} = \int \prod_{I} De^{I} \exp \frac{i}{48c} \frac{1}{2} \left(-D_{\mu} e_{I}^{\nu} D^{\mu} e_{\nu}^{I} + D_{\mu} e_{I}^{\nu} D_{\nu} e^{I\mu}\right) ed^{4}x,$$

with the free gravity field Lagrange:

$$Ld^{4}x = \frac{1}{48c} \frac{1}{2} \left(-D_{\mu} e_{I}^{\nu} D^{\mu} e_{\nu}^{I} + D_{\mu} e_{I}^{\nu} D_{\nu} e^{I\mu}\right) ed^{4}x. \tag{1.9}$$

We determine the constant c in the Newtonian gravitational potential $c \succ 0$.

In the background spacetime, weak gravity; $D_{\mu} \to \partial_{\mu}$ and $e \to 1 + \delta e$, we get

$$L \to \frac{1}{48c} \frac{1}{2} \left(-\partial_{\mu} e_{I}^{\nu} \partial^{\mu} e_{\nu}^{I} + \partial_{\mu} e_{I}^{\nu} \partial_{\nu} e^{I\mu}\right),$$

or

$$L_{0} = \frac{1}{48c} \frac{1}{2} \eta_{IJ} e_{\mu}^{I} \left(g^{\mu\nu} \partial^{2} - \partial^{\mu} \partial^{\nu}\right) e_{\nu}^{J}.$$

Without background spacetime approximation, in strong gravity field, we have a problem with the determinant e in the path integral

$$W_{ST} = \int \prod_{I} De^{I} exp \frac{i}{48c} \int \frac{1}{2} \left(-D_{\mu} e_{I}^{\nu} D^{\mu} e_{\nu}^{I} + D_{\mu} e_{I}^{\nu} D_{\nu} e^{I\mu}\right) e_{\mu_{1}}^{0} e_{\nu_{1}}^{1} e_{\rho}^{2} e_{\sigma}^{3} \varepsilon^{\mu_{1}\nu_{1}\rho\sigma} d^{4}x,$$

with $\eta_{0123} = -1$, we rewrite it as

$$\int \prod_{I} De^{I} exp \frac{i}{48c} \int \frac{1}{2} \left(-D_{\mu} e_{I}^{\nu} D^{\mu} e_{\nu}^{I} + D_{\mu} e_{I}^{\nu} D_{\nu} e^{I\mu}\right) \left(-\eta_{I_{1}JKL}\right) e_{\mu_{1}}^{I_{1}} e_{\nu_{1}}^{J} e_{\rho}^{K} e_{\sigma}^{L} \varepsilon^{\mu_{1}\nu_{1}\rho\sigma} d^{4}x/4!.$$

The path integral is independent on arbitrary changing of the coordinates x^μ: consider $e^K_\rho \to e^K_\rho + \delta e^K_\rho$, then we have

$$\frac{\delta S(e)}{\delta e^K_\rho} = 0,$$

it yields Dirac delta:

$$\delta\left(-D_\mu e^\nu_I D^\mu e^I_\nu + D_\mu e^\nu_I D_\nu e^{I\mu}\right), \quad so \ - D_\mu e^\nu_I D^\mu e^I_\nu + D_\mu e^\nu_I D_\nu e^{I\mu} = 0.$$

Thus, we get

$$\pi^2 = 0, \quad S\left(\pi, e\right) = -12c \int \pi^2 e d^4x = 0 \ then \ H\left(\pi, e\right) = 0.$$

This path integral is trivial; there is no propagation because there is no gravity energy: $H\left(\pi, e\right) = 0$, as the Wheeler-DeWitt equation $\hat{H}\psi = 0$. The reason is that the gravity field e^I_μ has the entity of spacetime. It is impossible for spacetime to be a dynamical on itself, to propagate over itself.

But if we write $e^I_\mu(x) \to \delta^I_\mu + h^I_\mu(x)$, the path integral exists. Then the propagation is possible. Thus, the gravity field propagates freely only on the background spacetime. This is the situation of the weak gravity (low energy densities).

The path integral of the weak gravity field, in the background spacetime, becomes

$$W = \int \prod_I De^I \exp i \int \frac{1}{48c}\frac{1}{2} e^I_\mu \left(\eta_{IJ} g^{\mu\nu}\partial^2 - \eta_{IJ}\partial^\mu\partial^\nu\right) e^J_\nu d^4x. \qquad (1.10)$$

Thus, the gravity field propagator, $g = \eta$ and $k_\mu e^{\mu I} = 0$, is

$$\Delta^{\mu\nu}_{IJ}(x_2 - x_1) = 48c \int \frac{d^4k}{(2\pi)^4} \frac{\eta_{IJ} g^{\mu\nu} e^{ik(x_2 - x_1)}}{k^2 - i\varepsilon},$$

or

$$\Delta^{\mu\nu}_{\rho\sigma}(x_2 - x_1) = 48c \int \frac{d^4k}{(2\pi)^4} \frac{g_{\rho\sigma} g^{\mu\nu} e^{ik(x_2 - x_1)}}{k^2 - i\varepsilon}. \qquad (1.11)$$

We will use this propagation in the gravity interaction with the scalar and spinor fields.

2 The Lagrange of the Plebanski two form field

The Plebanski two form complex field Σ^i, in selfdual representation $|\Sigma^i\rangle$, is defined in $\Sigma^i = P^i_{IJ}\Sigma^{IJ}$, where $\Sigma^{IJ} = e^I \wedge e^J$ is real anti-symmetric two form and P^i is the selfdual projector given in[3, 4]:

$$\left(P^i\right)_{jk} = \frac{1}{2}\varepsilon^i{}_{jk}\ ,\ \left(P^i\right)_{0j} = \frac{i}{2}\delta^i_j : i = I \text{ for } I = 1,2,3.$$

We start with the Lagrange of the gravity field (1.6):

$$S(e,\omega) = c\int \varepsilon_{IJKL}e^I\wedge e^J\wedge(d\Omega + dB + \Omega\wedge\Omega + \Omega\wedge B + B\wedge\Omega + B\wedge B)^{KL}.$$

As we did before, we try to find the Lagrange $L(e,\Sigma,B)$, which transforms covariantly under Local Lorentz transforms. For that, we separate the action $S(e,\omega)$ to $S(e,\Omega)+S(e,\Omega,B)$. Because the actions $S(e,\omega)$ and $S(e,\Omega)$ transform covariantly the action $S(e,\Omega,B)$ also transforms covariantly. Therefore, we can choose $S(e,\Omega,B) = 0$.
Let us write the action as

$$S(e,\omega) = c\int \varepsilon_{IJKL}e^I\wedge e^J\wedge(d\Omega + \Omega\wedge\Omega)^{KL} + S(e,\Omega,B).$$

We start with

$$S(e,\Omega) = c\int \varepsilon_{IJKL}e^I\wedge e^J\wedge(d\Omega + \Omega\wedge\Omega)^{KL}.$$

Using the assumption $\Omega^{IJ} = \pi_M{}^{IJ}e^M$, the action becomes

$$S(e,\pi) = c\int \left[\varepsilon_{IJKL}e^I\wedge e^J\wedge d\left(\pi_M{}^{KL}e^M\right) + \varepsilon_{IJKL}e^I\wedge e^J\wedge\left(\pi_{K_1}{}^K{}_M\right)e^{K_1}\wedge\left(\pi_{K_2}{}^{ML}\right)e^{K_2}\right].$$

We assume that the integral of

$$\varepsilon_{IJKL}d\left(e^I\wedge e^J\wedge\left(\pi_M{}^{KL}e^M\right)\right) = \varepsilon_{IJKL}d\left(\Sigma^{IJ}\wedge\left(\pi_M{}^{KL}e^M\right)\right)$$

is zero at the infinities. Using

$$d\Sigma^{IJ}\wedge\left(\pi_M{}^{KL}\right)e^M + e^I\wedge e^J\wedge d\left(\pi_M{}^{KL}e^M\right) = -\left(\pi_M{}^{KL}\right)e^M\wedge d\Sigma^{IJ} + e^I\wedge e^J\wedge d\left(\pi_M{}^{KL}e^M\right),$$

the Action becomes

$$S(e,\pi) = c\int \left[\varepsilon_{IJKL}\left(\pi_M{}^{KL}\right)e^M \wedge d\Sigma^{IJ} + \varepsilon_{IJKL}\Sigma^{IJ} \wedge \left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right)e^{K_1} \wedge e^{K_2}\right],$$

or

$$S(e,\pi) = c\int \left[\varepsilon_{IJKL}\left(\pi_M{}^{KL}\right)e^M \wedge d\Sigma^{IJ} + \varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right)\Sigma^{IJ} \wedge \Sigma^{K_1K_2}\right].$$

Inserting our assumption:

$$\pi^{IJK} = \pi_L\varepsilon^{LIJK},$$

we get

$$\varepsilon_{IJKL}\left(\pi_M{}^{KL}\right)e^M = \varepsilon_{IJKL}\pi^{MKL}e_M = \varepsilon_{IJKL}\pi_N\varepsilon^{NMKL}e_M = -2\left(\pi_I e_J - \pi_J e_I\right).$$

Making the replacement:

$$\Sigma^{IJ} \wedge \Sigma^{K_1K_2} \to \varepsilon^{IJK_1K_2}\Sigma^{01} \wedge \Sigma^{23},$$

we get

$$\begin{aligned}\varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right)\Sigma^{IJ} \wedge \Sigma^{K_1K_2} &= \varepsilon_{IJKL}\left(\pi_{K_1}{}^K{}_M\right)\left(\pi_{K_2}{}^{ML}\right)\varepsilon^{IJK_1K_2}\Sigma^{01} \wedge \Sigma^{23}\\ &= 2\left(\pi_L{}^K{}_M\right)\left(\pi_K{}^{ML}\right)\Sigma^{01} \wedge \Sigma^{23} = 2\left(\pi_{LKM}\right)\left(\pi^{KML}\right)\Sigma^{01} \wedge \Sigma^{23}\\ &= 2\left(\pi_{KML}\right)\left(\pi^{KML}\right)\Sigma^{01} \wedge \Sigma^{23} = 2\pi^I\varepsilon_{IKML}\pi_J\varepsilon^{JKML}\Sigma^{01} \wedge \Sigma^{23}\\ &= -12\pi^2\Sigma^{01} \wedge \Sigma^{23}.\end{aligned}$$

Therefore, the Action becomes

$$S(e,\pi,\Sigma) = c\int \left[-2\left(\pi_I e_J - \pi_J e_I\right) \wedge d\Sigma^{IJ} - 12\pi_I\pi^I\Sigma^{01} \wedge \Sigma^{23}\right].$$

Because the real Plebanski two form $\Sigma^{IJ} = e^I \wedge e^J$ is anti-symmetric, we can rewrite:

$$S(e,\pi,\Sigma) = c\int \left[-4\pi_I e_J \wedge d\Sigma^{IJ} - 12\pi_I\pi^I\Sigma^{01} \wedge \Sigma^{23}\right],$$

then using $\varepsilon_{0123} = -1$, we can rewrite it as

$$S(e,\pi,\Sigma) = c\int \left[-4\pi_I e_J \wedge d\Sigma^{IJ} + 12\pi_I\pi^I\varepsilon_{IJKL}\Sigma^{IJ} \wedge \Sigma^{KL}/4!\right],$$

or

$$S(e,\pi,\Sigma)=c\int\left[-4\pi_I e_J\wedge d\Sigma^{IJ}+\frac{1}{2}\pi^2\varepsilon_{IJKL}\Sigma^{IJ}\wedge\Sigma^{KL}\right].$$

The integral over the momentum π^I vanishes unless (the equation of motion)

$$\frac{\delta S(e,\pi,\Sigma)}{\delta\pi_I}=\frac{\delta}{\delta\pi_I}\int\left[-4\pi_I e_J\wedge d\Sigma^{IJ}+\frac{1}{2}\pi^2\varepsilon_{IJKL}\Sigma^{IJ}\wedge\Sigma^{KL}\right]=0.$$

But, it is not easy to separate Σ from e. It is like the gravity field, it is separable only in weak gravity(background spacetime). Therefore, we solve it in the background spacetime. On arbitrary spacetime, we get the integral:

$$\int\left(-4\pi_I e_J\wedge d\Sigma^{IJ}+\frac{1}{2}\pi^2\varepsilon_{IJKL}\Sigma^{IJ}\wedge\Sigma^{KL}\right)$$

$$\to\int\left(-4\pi_I e_{\mu J}\partial_\nu\Sigma^{IJ}_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}+\frac{1}{2}\pi^2\varepsilon_{IJKL}\Sigma^{IJ}_{\mu\nu}\Sigma^{KL}_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right)d^4x.$$

The background spacetime approximation is

$$e^I_\mu(x)\to\delta^I_\mu+h^I_\mu(x)\ ,\quad e\to 1+\delta e,$$

thus we get

$$\Sigma^{IJ}_{\mu\nu}=\frac{1}{2}\left(e^I_\mu e^J_\nu-e^I_\nu e^J_\mu\right)\to\frac{1}{2}\left(\delta^I_\mu\delta^J_\nu-\delta^I_\nu\delta^J_\mu\right)+\frac{1}{2}\left(h^I_\mu\delta^J_\nu-h^I_\nu\delta^J_\mu\right)+\frac{1}{2}\left(\delta^I_\mu h^J_\nu-\delta^I_\nu h^J_\mu\right).$$

Inserting it in the action

$$S\left(e,\Sigma\right)=c\int\left(-4\pi_I e_{\mu J}\partial_\nu\Sigma^{IJ}_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}+\frac{1}{2}\pi^2\varepsilon_{IJKL}\Sigma^{IJ}_{\mu\nu}\Sigma^{KL}_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right)d^4x,$$

it becomes

$$S\left(e,\Sigma\right)\to S\left(h,\delta\Sigma\right)=c\int\left(-4\pi_I\partial_\nu\Sigma^{IJ}_{\rho\sigma}\varepsilon_J{}^{\nu\rho\sigma}+\frac{1}{2}\pi^2\left(-24\right)+\dots\right)d^4x.$$

Therefore, the condition(equation of motion)

$$\frac{\delta}{\delta\pi_I}\int\left[-4\pi_I e_J\wedge d\Sigma^{IJ}+\frac{1}{2}\pi^2\varepsilon_{IJKL}\Sigma^{IJ}\wedge\Sigma^{KL}\right]=0$$

approximates to

$$\frac{\delta}{\delta\pi_I}\int\left(-4\pi_I\partial_\nu\Sigma^I{}_{J\rho\sigma}\varepsilon^{J\nu\rho\sigma}+\frac{1}{2}\pi^2\left(-24\right)\right)d^4x=0.$$

Its solution is

$$\pi^I=-\frac{1}{6}\partial_\nu\Sigma^I_{J\rho\sigma}\varepsilon^{J\nu\rho\sigma}=-\frac{1}{6}\partial^\nu\Sigma^{IJ\rho\sigma}\varepsilon_{J\nu\rho\sigma}.$$

Thus, the action in the background spacetime is approximated to

$$S(\Sigma)\to c\int\left[\frac{2}{3}\partial^{\nu_1}\Sigma^{IJ_1\rho_1\sigma_1}\varepsilon_{J_1\nu_1\rho_1\sigma_1}\partial_\nu\Sigma_{IJ\rho\sigma}\varepsilon^{J\nu\rho\sigma}+...\right]d^4x.$$

Defining inner product via $\Sigma^{IJ_1\rho_1\sigma_1}\Sigma_{IJ\rho\sigma}=\Sigma^2\delta_J^{J_1}\delta_\rho^{\rho_1}\delta_\sigma^{\sigma_1}$, we get

$$S(\Sigma)\to c\int\left(-4\partial_\mu\Sigma_{IJ}^{\nu\rho}\partial^\mu\Sigma_{\nu\rho}^{IJ}+...\right)d^4x\ \ \text{with}\quad\partial_\mu\Sigma_{IJ}^{\mu\rho}=0.$$

This is the action of the real Plebanski two form in weak gravity field (background spacetime). It is like the electromagnetic field. The corresponding Lagrange is

$$L_0(\Sigma)\to-4c\left(\partial_\mu\Sigma_{IJ}^{\nu\rho}\right)\left(\partial^\mu\Sigma_{\nu\rho}^{IJ}\right)\ \text{with}\quad\partial_\mu\Sigma_{IJ}^{\mu\rho}=0.$$

In curved spacetime, we rewrite it as

$$L_0(\Sigma)d^4x\to-4c'\left(\partial_\mu\Sigma_{IJ}^{\nu\rho}\right)\left(\partial^\mu\Sigma_{\nu\rho}^{IJ}\right)ed^4x. \tag{2.1}$$

It does not transform covariantly, because of the partial derivative ∂. But the total Lagrange $L(e,\omega)$ transforms covariantly, thus $L(e,\omega)=L(e,\Sigma)+L(e,\Sigma,B)$ transforms covariantly. Let us rewrite:

$$L(e,\Sigma)+L(e,\Sigma,B)=L(e,\Sigma)+\Delta L-\Delta L+L(e,\Sigma,B),$$

with $L(e,\Sigma)+\Delta L$ transforms covariantly, and $-\Delta L+L(e,\Sigma,B)=0$, which determines B. We get

$$L(e,\Sigma)+\Delta L\to L(\Sigma)d^4x=-4c'\left(D_\mu\Sigma_{IJ}^{\nu\rho}\right)\left(D^\mu\Sigma_{\nu\rho}^{IJ}\right)ed^4x,$$

which transforms covariantly.

We get the Lagrange of the complex Plebanski two form field $\Sigma^i = P^i_{IJ}\Sigma^{IJ}$ using the selfdual projector P^i, which projects the real Plebanski two form Σ^{IJ} to two states: selfdual $|\Sigma^i\rangle$ and anti-selfdual $\left|\bar{\Sigma}^i\right\rangle$. Thus, the term

$$D_\mu \Sigma^{\nu\rho}_{IJ} D^\mu \Sigma^{IJ}_{\nu\rho},$$

in the Lagrange, becomes

$$D_\mu \Sigma^{\nu\rho}_i D^\mu \Sigma^i_{\nu\rho} + D_\mu \bar{\Sigma}^{\nu\rho}_i D^\mu \bar{\Sigma}^i_{\nu\rho},$$

where the hermitian conjugate $\bar{\Sigma}^{\nu\rho}_i \bar{\Sigma}^i_{\nu\rho}$ is represented in the anti-selfdual $\bar{\Sigma}^i = \bar{P}^i_{IJ}\Sigma^{IJ}$.

We search for conditions allow us to rewrite the complex Plebanski two form field Σ^i as a real field. As done for the left and right spinor fields: In the left spinor field representation, the acting of the right spinor field is zero. While, in the right spinor field representation, the acting of left spinor field is zero[5]. Therefore, in the selfdual representation $|\Sigma^i\rangle$, we assume that the acting of the complex Plebanski field, represented in the anti-selfdual, is zero, thus $\bar{\Sigma}^j |\Sigma^i\rangle = 0$. Like that is in the anti-selfdual representation $\left|\bar{\Sigma}^i\right\rangle$, thus $\Sigma^j \left|\bar{\Sigma}^i\right\rangle = 0$.
Therefore, in the selfdual representation $|\Sigma^i\rangle$, we get

$$\bar{\Sigma}^i = \frac{1}{2}\varepsilon^{ijk}\Sigma_{jk} - i\Sigma^{0i} = 0 \rightarrow \frac{1}{2}\varepsilon^{ijk}\Sigma_{jk} = i\Sigma^{0i}.$$

Thus, the Plebanski two form field, in the selfdual representation, becomes

$$\Sigma^i = \frac{1}{2}\varepsilon^{ijk}\Sigma_{jk} + i\Sigma^{0i} = \varepsilon^{ijk}\Sigma_{jk},$$

which is real as required for satisfying the reality condition. It is equivalent to the replacement $x^0 \rightarrow -ix^0$. Same result we get in the anti-selfdual representation $\left|\bar{\Sigma}^i\right\rangle$: $\Sigma^i = 0 \rightarrow \bar{\Sigma}^i = \varepsilon^{ijk}\Sigma_{jk}$, it is equivalent to the replacement $x^0 \rightarrow ix^0$. That allows the splitting: $SO(3,1) \rightarrow SU(2) \otimes SU(2)$.
In the two representations, the term $\frac{1}{2}\left(\Sigma^{\nu\rho}_{IJ}\Sigma^{IJ}_{\nu\rho}\right) = \frac{1}{2}\left(\Sigma^{\nu\rho}_i \Sigma^i_{\nu\rho} + \bar{\Sigma}^{\nu\rho}_i \bar{\Sigma}^i_{\nu\rho}\right)$ becomes

$$\frac{1}{2}\varepsilon_{ij'k'}\Sigma^{j'k'}_{\nu\rho}\varepsilon^{ijk}\Sigma^{\nu\rho}_{jk} = \Sigma^{\nu\rho}_{jk}\Sigma^{jk}_{\nu\rho}.$$

Therefore, the Lagrange of the Plebanski two form field, in selfdual representation, becomes:

$$L_0(\Sigma)d^4x = -4c'\left(D_\mu \Sigma_i^{\nu\rho}\right)\left(D^\mu \Sigma^i_{\nu\rho}\right) ed^4x. \tag{2.2}$$

Let us combine the gravity and the real Plebanski two form in one field K^i_μ as

$$K^i_\mu = \frac{1}{2}\left(e^i_\mu + \frac{i}{4}\varepsilon^{ijk}\varepsilon_{0\mu\rho\sigma}\Sigma^{\rho\sigma}_{jk}\right),$$

and its hermitian conjugate is

$$\bar{K}^i_\mu = \frac{1}{2}\left(e^i_\mu - \frac{i}{4}\varepsilon^{ijk}\varepsilon_{0\mu\rho\sigma}\Sigma^{\rho\sigma}_{jk}\right),$$

the local-Lorentz frame indices are for $I = i = 1, 2, 3$. Thus we get

$$K^i_\mu + \bar{K}^i_\mu = e^i_\mu \; and \; K^i_\mu - \bar{K}^i_\mu = \frac{i}{4}\varepsilon^{ijk}\varepsilon_{0\mu\rho\sigma}\Sigma^{\rho\sigma}_{jk} = \frac{i}{2}\varepsilon_{0\mu\rho\sigma}\Sigma^{i\rho\sigma},$$

then we get

$$\left(\partial^\nu K^i_\mu + \partial^\nu \bar{K}^i_\mu\right)\left(\partial_\nu K^\mu_i + \partial_\nu \bar{K}^\mu_i\right) = \partial^\nu e^i_\mu \partial_\nu e^\mu_i,$$

and

$$\begin{aligned}\left(\partial^\nu K^i_\mu - \partial^\nu \bar{K}^i_\mu\right)\left(\partial_\nu K^\mu_i - \partial_\nu \bar{K}^\mu_i\right) &= \frac{-1}{4}\varepsilon_{0\mu\rho\sigma}\varepsilon_0^{\mu\rho'\sigma'}\partial^\nu \Sigma^{i\rho\sigma}\partial_\nu \Sigma_{i\rho'\sigma'} \\ &= \frac{1}{4}\varepsilon_{0\mu\rho\sigma}\varepsilon^{0\mu\rho'\sigma'}\partial^\nu \Sigma^{i\rho\sigma}\partial_\nu \Sigma_{i\rho'\sigma'} = -\partial^\nu \Sigma^{i\rho\sigma}\partial_\nu \Sigma_{i\rho\sigma}.\end{aligned}$$

Therefore, we have

$$\partial_\nu e^\mu_i \partial^\nu e^i_\mu + \partial_\nu \Sigma^{i\rho\sigma}\partial^\nu \Sigma_{i\rho\sigma} \to 4\partial_\mu \bar{K}^i \partial^\mu K_i.$$

Using them in the gravity Lagrange (1.9) and in the Plebanski Lagrange (2.2):

$$L(e)\, d^4x = -\frac{1}{48c}\frac{1}{2}\left(\partial_\mu e^\nu_I\right)\left(\partial^\mu e^I_\nu\right) ed^4x,$$

and

$$L\left(\bar{\Sigma}\right)d^4x = -8c'\frac{1}{2}\left(\partial^\nu \Sigma^{i\rho\sigma}\partial_\nu \Sigma_{i\rho\sigma}\right)ed^4x,$$

for the gauge $e^I_\mu = 0$ and $8c' = 1/(48c)$, we get

$$L\left(e\right)d^4x + L\left(\bar{\Sigma}\right)d^4x \to \frac{1}{12c}\frac{-1}{2}\left(\partial^\nu \bar{K}^i_\mu \partial_\nu K^\mu_i\right)ed^4x. \tag{2.3}$$

To make it invariance, we rewrite:

$$L\left(K\right)d^4x = \frac{1}{12c}\frac{-1}{2}\left(D^\nu \bar{K}^i_\mu D_\nu K^\mu_i - D^\nu \bar{K}^i_\mu D^\mu K_{i\nu}\right)ed^4x.$$

It satisfies the reality condition as required, like the Lagrange of the electromagnetic field.

3 The static potential of the weak gravity

We derive the static potential of the scalar and spinor fields interactions with the weak gravity field in the static limit; the Newtonian gravitational potential. We find that this potential has the same structure for the both fields, it depends on the fields energy. By that, we determine the constant $c \succ 0$.

The action of the scalar field in arbitrary curved spacetime is[1]

$$S(e,\phi) = \int d^4xe\left(\eta^{IJ}e^\mu_I e^\nu_J D_\mu\phi^+ D_\nu\phi - V(\phi)\right).$$

In the weak gravity, the background spacetime approximation is given by

$$e^\mu_I(x) \to \delta^\mu_I + h^\mu_I(x) \quad , \quad e \to 1 + \delta e.$$

Thus, the action is approximated to

$$S(e,\phi) = \int d^4x\left(\partial_\mu\phi^+\partial^\mu\phi + h^{\mu\nu}(x)\partial_\mu\phi^+\partial_\nu\phi + h^{\nu\mu}(x)\partial_\mu\phi^+\partial_\nu\phi - V(\phi) + ...\right).$$

The gravity field is symmetric, so we get

$$S(e,\phi) = \int d^4x \left(\partial_\mu \phi^+ \partial^\mu \phi + 2h^{\mu\nu}(x)\partial_\mu \phi^+ \partial_\nu \phi - V(\phi) + ...\right).$$

The energy-momentum tensor of the scalar field is[5]

$$T_{\mu\nu} = \partial_\mu \phi^+ \partial_\nu \phi + g_{\mu\nu} L,$$

hence

$$\partial_\mu \phi^+ \partial_\nu \phi = T_{\mu\nu} - g_{\mu\nu} L.$$

Inserting it in the Lagrange, it becomes

$$L = \partial_\mu \phi^+ \partial^\mu \phi + 2h^{\mu\nu}(x)\left(T_{\mu\nu} - g_{\mu\nu} L\right) - V(\phi) + ...$$

or

$$L = \partial_\mu \phi^+ \partial^\mu \phi + 2h^{\mu\nu} T_{\mu\nu} - V(\phi) - 2h^{\mu\nu} g_{\mu\nu} L + ...$$

Therefore, in the interaction term, we make the replacement:

$$\partial_\mu \phi^+ \partial_\nu \phi \to T_{\mu\nu} \;\; and \;\; V \to V + 2h^{\mu\nu} g_{\mu\nu} L.$$

Because the gravity field is weak (background spacetime), so $2h^{\mu\nu} g_{\mu\nu} L$ is neglected compared with L.

We find the potential $V(r)$ of the exchanged virtual gravitons by two particles k_1 and k_2, using $M\left(k_1 + k_2 \to k_1' + k_2'\right)$ matrix element (like Born approximation to the scattering amplitude in non-relativistic quantum mechanics [6]).

For one diagram of Feynman diagrams, we have

$$iM\left(k_1 + k_2 \to k_1' + k_2'\right) = i\left(-ik_2'\right)_\mu \left(ik_2\right)_\nu \frac{\bar{\Delta}^{\mu\nu\rho\sigma}(q)}{i} i\left(-ik_1'\right)_\rho \left(ik_1\right)_\sigma,$$

with

$$q = k_1' - k_1 = k_2 - k_2'.$$

The propagator $\Delta^{\mu\nu\rho\sigma}(x_2 - x_1)$ is the gravitons propagator (1.11), we get it from the Lagrange of the free gravity field (in background spacetime) we had before:

$$L_0 = \frac{1}{48c}\frac{1}{2}\eta_{IJ}e^I_\mu\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)e^J_\nu \to \frac{1}{48c}\frac{1}{2}\eta_{IJ}h^I_\mu\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)h^J_\nu.$$

With the gauge $\partial^\mu e^I_\mu = 0$, we get

$$\Delta^{IJ}_{\mu\nu}(y - x) = \int \frac{d^4q}{(2\pi)^4}\bar{\Delta}^{IJ}_{\mu\nu}\left(q^2\right)e^{iq(y-x)} \ : \ \bar{\Delta}^{IJ}_{\mu\nu}\left(q^2\right) = 48c\frac{g_{\mu\nu}\eta^{IJ}}{q^2 - i\varepsilon}.$$

Therefore, the M matrix element becomes

$$iM(k_1 + k_2 \to k_1' + k_2') = i48c\left(-ik_2'\right)_\mu(ik_2)_\rho\frac{g^{\mu\nu}g^{\rho\sigma}}{q^2}\left(-ik_1'\right)_\sigma(ik_1)_\nu,$$

$$where \quad g = \eta \text{ and } q = k_1' - k_1 = k_2 - k_2'.$$

Comparing it with[6]

$$iM(k_1 + k_2 \to k_1' + k_2') = -i\bar{V}(q)\,\delta^4(k_{out} - k_{in}),$$

we get

$$\bar{V}\left(q^2\right) = -48c\left(-ik_2'\right)_\mu(ik_2)_\rho\frac{g^{\mu\nu}g^{\rho\sigma}}{q^2}\left(-ik_1'\right)_\sigma(ik_1)_\nu.$$

Then, comparing this relation with the replacement:

$$\partial_\mu\phi^+\partial_\nu\phi \to T_{\mu\nu},$$

and evaluating the inverse Fourier transform, thus we get

$$V(y - x) = -48cT_{\mu\rho}(y)\,g^{\mu\nu}g^{\rho\sigma}T_{\nu\sigma}(x)\frac{1}{4\pi|y - x|} = -48c\frac{T_{\mu\nu}(y)\,T^{\mu\nu}(x)}{4\pi|y - x|},$$

where $T^{\mu\nu}$ is transferred energy-momentum tensor, it is anti-symmetric, so the summation over the indices μ and ν is repeated twice. Therefore, we divide the right side by 2:

$$V(y - x) = -\frac{48c}{2}\frac{T_{\mu\nu}(y)\,T^{\mu\nu}(x)}{4\pi|y - x|}.$$

In the static limit, for one particle, we approximate T^{00} to m: m is the mass of interacted particles.
Thus, we get the Newtonian gravitational potential:

$$V(y-x) = -\frac{48c}{2}\frac{m^2}{4\pi|y-x|} = -G\frac{m^2}{|y-x|} \rightarrow 48c = 8\pi G.$$

Therefore, the weak gravity Lagrange becomes

$$L_0 = \frac{1}{4\pi G}\frac{1}{4}\eta_{IJ}e_\mu^I\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)e_\nu^J.$$

We do the same thing for the spinor fields interaction with the gravity. The action is[1]

$$S(e,\psi) = \int d^4xe\left(ie_I^\mu\bar{\psi}\gamma^I D_\mu\psi - m\bar{\psi}\psi\right),$$

where the covariant derivative D_μ is

$$D_\mu = \partial_\mu + (\omega_\mu)_J^I L_I^J + A_\mu^a \mathrm{T}^a.$$

In the background spacetime, it becomes

$$S(e,\psi) = \int d^4x\left(i\bar{\psi}\gamma^\mu D_\mu\psi + ih_I^\mu\bar{\psi}\gamma^I D_\mu\psi - m\bar{\psi}\psi + ...\right).$$

Let us consider only the terms:

$$\int d^4x\left(i\bar{\psi}\gamma^\mu\partial_\mu\psi + ih_\nu^\mu\bar{\psi}\gamma^\nu\partial_\mu\psi - m\bar{\psi}\psi\right): \quad g=\eta.$$

The energy-momentum tensor of the spinor field is[5]

$$T^{\mu\nu} = -i\bar{\psi}\gamma^\mu\partial^\nu\psi + g^{\mu\nu}L.$$

Therefore, in the interaction term, we have the replacement

$$i\bar{\psi}\gamma^\mu\partial^\nu\psi \rightarrow -T_{\mu\nu} \quad \mathit{and} \quad L \rightarrow L + h^{\mu\nu}g_{\mu\nu}L.$$

The term $h^{\mu\nu}g_{\mu\nu}L$ is neglected compared with the Lagrange L. We find M matrix element of the exchanged virtual gravitons $p_1 + p_2 \rightarrow p_1' + p_2'$, for one diagram of Feynman diagrams[6]:

$$iM(p_1 + p_2 \rightarrow p_1' + p_2') = i48c\bar{u}(p_1')\gamma^\mu(-ip_1)_\nu u(p_1)\frac{g_{\mu\sigma}g^{\nu\rho}}{q^2}\bar{u}(p_2')\gamma^\sigma(-ip_2)_\rho u(p_2),$$

with

$$q = p_1' - p_1 = p_2 - p_2' \text{ and } \quad g = \eta,$$

we get

$$\bar{V}\left(q^2\right) = -48c\bar{u}\left(p_1'\right)\gamma^\mu\left(-ip_1\right)_\nu u\left(p_1\right)\frac{g_{\mu\sigma}g^{\nu\rho}}{q^2}\bar{u}\left(p_2'\right)\gamma^\sigma\left(-ip_2\right)_\rho u\left(p_2\right).$$

Comparing this relation with the replacement:

$$i\bar{\psi}\gamma^\mu\partial^\nu\psi \rightarrow -T_{\mu\nu},$$

and evaluating the inverse Fourier transform, we get

$$V\left(y-x\right) = -48c\left(-T_{\mu\rho}\left(y\right)\right)g^{\mu\nu}g^{\rho\sigma}\left(-T_{\nu\sigma}\left(x\right)\right)\frac{1}{4\pi\left|y-x\right|} = -48c\frac{T_{\mu\nu}\left(y\right)T^{\mu\nu}\left(x\right)}{4\pi\left|y-x\right|},$$

where $T^{\mu\nu}$ is transferred energy-momentum tensor. Dividing the right side by 2:

$$V\left(y-x\right) = -\frac{48c}{2}\frac{T_{\mu\nu}\left(y\right)T^{\mu\nu}\left(x\right)}{4\pi\left|y-x\right|}.$$

In the static limit, for one particle, we approximate T^{00} to m: m is the mass of interacted particles.
Thus, we get the Newtonian gravitational potential:

$$V\left(y-x\right) = -\frac{48c}{2}\frac{m^2}{4\pi\left|y-x\right|} = -G\frac{m^2}{\left|y-x\right|} \rightarrow 48c = 8\pi G.$$

It is the same potential we found in the scalar field interaction with the gravity field.

References

[1] Carlo Rovelli, Quantum Gravity; Cambridge University Pres, Cambridge, United Kingdom, 2004.

[2] Carlo Rovelli and Francesca Vidotto, Covariant Loop Quantum Gravity, An elementary introduction to Quantum Gravity and Spinfoam Theory; Cambridge University Press, Cambridge, United Kingdom, 2014.

[3] Steffen Gielen and Daniele Oriti, Classical general relativity as BF-Plebanski theory; Classical and Quantum Gravity, IOP Publishing, United Kingdom, 2010.

[4] J. E. Rosales-Quintero, Antiself-dual gravity and supergravity from a pure connection formulation; International Journal of Modern Physics A, World Scientific Publishing Co, 2016.

[5] Mark Srednicki, Quantum Field Theory; Cambridge University Press, Cambridge, United Kingdom, 2007.

[6] Michael E. Peskin and Daniel V. Schroeder, An Introduction to Quantum Field Theory; Westview Press, Colorado, USA, 1995.

[7] Carlo Rovelli, Ashtekar formulation of general relativity and loop space nonperturbative quantum gravity: a report; IOP Publishing, United Kingdom, 1991.

[8] Bryce S. DeWitt, Quantum Theory of Gravity. 1. The Canonical Theory; Phys. Rev, APS, USA, 1967.

[9] Christopher Pope, Geometry and Group Theory; Texas A&M University Press, Texas, USA, 2008.