DOES EINSTEIN’S FIELD EQUATION(S) PROVIDE AN “EXACT” DESCRIPTION OF CLASSICAL SPACE-TIME?

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Abstract: It will be showed based on the ground of pure reasoning, that classical space-time as derived from Einstein’s Field Equation(s) is not in conformity with an exact relativistic theory of Gravitation (surprisingly!), contrary to usual belief. Also it signs for a distance-scale, for which the exact nature of space-time, and as found from Field-Equations must have a significant discrepancy between them. And most importantly, this will be showed WITHOUT using any ad-hoc basis assumption or hypothesis or untested theories. Only known and unambiguous laws of Physics will be used in a coherent and consistent manner to arrive at this conclusion. The author strongly believes in the fact that all deep and beautiful things in nature must be simple enough! Therefore simplicity has been tried at best to preserve everywhere. First this conjecture will be proved for Schwarzschild space-time, one of the simplest solutions of Einstein’s Field Equation(s). Later same will be shown for other known space-times, pointing to the fact that the trouble is not inherent in solution(s), but in the key-equation (EFE) itself. Finally experimental support towards this proposition will also be presented; it will be showed that, the key-ingredient (of a relativistic-theory of gravity) which is proposed to be missing in General Relativity, if we include its effect correctly to the solutions of the field-equations, it can properly describe some gravitational anomaly quantitatively (not describable by GR), long been superseded from Scientific Community.

In 1915, Albert Einstein published his General Theory of Relativity\(^{(1)}\), which is the most accurate theory of gravitation known still. Certainly a great deal of work has been spent for extending his theory for incorporating additional facilities, like in Kaluza-Klein theory, Scalar-Tensor-Vector theory, Supergravity etc., though (so far as I know) there’s still not any enough evidence to consider any of them superior to that of Einstein. Even works are in progress to quantize General Relativity from different approaches like Causal dynamic triangulation\(^{(2)}\), Loop Quantum Gravity\(^{(3)}\) etc. Though a “true” quantized version of GR still appears to be outside the scope of present research.

A subtle property of a Relativistic Gravitation:

One crucial property of a gravitational field is that it must be a self-interactive field (We shall argue below). This is a feature which is not shared with electromagnetic field, but certain classes of non-abelian gauge fields\(^{(4)}\). Roughly speaking, by self-interacting field we mean, a field which also interacts with self, or in other words field itself may act as its source. Even outside the regime of GR, one can easily guess why gravitational field should be self-interactive. Before the advent of special relativity, mass and energy were separate entities. But special relativity reconciled them in a unified framework stating them to be equivalent. In Newton’s law of gravitation, mass is coupled to gravity. But if energy is equivalent to mass, then all forms of energy should also be coupled with gravity, including energy of gravitational field itself! Since GR is a relativistic theory of gravity, its solution therefore must show self-interaction. Now given this fact, is it possible that in a relativistic theory of gravitation, gravitational potential (or field) should fall off exactly in a similar way just like a non-relativistic theory?
From a priori reasoning, it’s not. Since as we move toward the gravitating object, there is more and more field to interact with itself, making the field growing in a faster rate than the non self-interactive (or non-relativistic) field, making the functional dependence very different. Hence we conclude the following:

It is obvious that a gravitating object in GR should never give birth of a potential which grows (as we move towards the source) at the same spatial rate as a Newtonian one, rather it’s obvious to grow rapidly, if GR is a correct relativistic theory of Gravitation at an arbitrary scale. Thus potential can be considered as a fingerprint of self-interaction!

So far we have provided linguistic argument of the above proposition we arrived. Now we shall check if our intuition is correct from mathematical viewpoint.

†Apart from its property of finite propagation velocity as that of light

Making Newton’s Gravitation Self-Interactive:

The Lagrangian-density for Newton’s Gravitation is given by,

\[ L = -\rho(x,t)\varphi(x,t) - \frac{1}{8\pi G}(\nabla\varphi(x,t))^2 \]

Here \( \rho(x,t) \) is some source of Gravitation. Now if we consider a gravitational-field, which is self-coupled and source function due to matter is zero everywhere (except the origin), then the Lagrangian can be written as,

\[ L = -\frac{1}{8\pi Gc^2}(\nabla\varphi(x,t))^2\varphi(x,t) - \frac{1}{8\pi G}((\nabla\varphi(x,t))^2) \]

Now the EL equations give,

\[ \left( \frac{\varphi}{c^2} + 1 \right)\nabla^2\varphi = -\frac{1}{2c^2}(\nabla\varphi)^2 \]

Using suitable boundary-conditions, we can write the solution \( \varphi(r) \) in a nice form:

\[ \varphi(r) = c^2\left( 1 - \frac{3GM}{rc^2} + \frac{9G^2M^2}{4r^2c^4} \right) - c^2 \]

Clearly this potential agrees well with Newtonian-potential \( \varphi = -\frac{GM}{r} \), at \( r \gg \frac{GM}{c^2} \). But we plot this potential together with the Newtonian-potential (with \( c=1, GM=1 \)) we can clearly find the difference which becomes more and more prominent at the vicinity of \( \frac{GM}{c^2} \):
This graph clearly shows that the potential of a self-interactive gravitational field must grow at a faster spatial rate, more and more we move towards the gravitating-source. Clearly, any relativistic-theory of gravitation must produce this feature of the potential we just observed. The validity of the above potential will be found when the experimental consequences of our proposition will be discussed.

**Connection between metric of a given space-time and gravitational potential:**

We can make the following conclusions based on our proof:

1) GR must be a self-interactive theory of gravitation, if it’s an exact relativistic theory (in classical domain)

2) A self-interacting potential of gravitation can never be Newtonian-one, rather should grow at a faster rate as one moves to gravitating object, owing to the fact that field of gravitation is coupled to itself.

Now an exact solution of GR, must therefore give birth of a self-interacting potential (Fortunately we can calculate potential from the metric†, as we shall see). Otherwise it must lead to the fact that it’s not in conformity with the properties of relativistic gravitation.

Now let’s take a general spherically symmetric static metric, which can be written as,

\[ ds^2 = e^{\nu(t)} dt^2 - e^{\lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

Now we consider equation(s) of geodesic,

\[ \frac{d^2 x^i}{d\tau^2} = -\Gamma^i_{jk} \frac{dx^j}{d\tau} \frac{dx^k}{d\tau} \]

We are using here \( c=1 \). So \( ds \) is \( d\tau \). Now we imagine an object freely falling in a spherically symmetric gravitational field from infinity. If we can calculate the work done by the field when
the object is at some point \( r = r \), then clearly it will be the potential at that field, which will be simply the integral of the proper acceleration over radial co-ordinate. So we get,

\[
\varphi = \int_{\infty}^{r} \frac{d^2r}{d\tau^2} \, dr
\]

Therefore we are to calculate the integral,

\[
\int_{\infty}^{r} \Gamma_{jk} \frac{dx^j}{d\tau} \frac{dx^k}{d\tau} \, dr
\]

With \( J \to 0 \), the integral becomes,

\[
\frac{1}{2} \int_{\infty}^{r} (s^2 f(r) + g(r)) \, dr
\]

Where \( f(r) = \nu' + \lambda' \), \( g(r) = \nu' e^{-\lambda} \) and \( s = \frac{dr}{d\tau} \). This integral is equal to,

\[
\int_{\infty}^{r} \frac{d^2r}{d\tau^2} \, dr = \int_{\infty}^{r} s \frac{ds}{dr} \, dr = -\varphi
\]

Connecting these equations, we get,

\[
\varphi = \frac{1}{2} s^2
\]

And

\[
\frac{ds}{dr} + \frac{1}{2} s f(r) = -\frac{1}{2s} g(r)
\]

This is a Bernoulli’s equation, which can be readily solved to yield,

\[
s = \left\{ -e^{-\int g(r) e^i \, dr + de^{-i}} \right\}^{1/2}
\]

Here

\[
I = \int f(r) \, dr
\]

Hence we get,
Concerning Potential of Gravitation in various space-time's:

Exterior Schwarzschild Solution:

The Schwarzschild (ext.) metric is given by,

\[ ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \]

Hence we get,

\[ e^\lambda = \left(1 - \frac{2GM}{r}\right)^{-1} \]

\[ e^\nu = \left(1 - \frac{2GM}{r}\right) \]

Substituting these in expression of \( \varphi \), we get finally

\[ \varphi = -\frac{GM}{r} + k(k = \frac{d}{2}) \]

This is simply the Newtonian potential, with an additive constant!(which can be always eliminated having no observable effect) All our calculations simply lead to the fact that Schwarzschild metric involves no self-interaction in the gravitational field, following the argument we made. The potential doesn’t clearly show any special character of gravitational interaction at short distance, which is expected due to its self-interactive nature, as we proved before.

Interior Schwarzschild Solution:

As is well known, there is another kind of Schwarzschild Solution valid in the interior of a non-rotating body consisting some incompressible fluid(5):
\[ ds^2 = \left( \frac{3}{2} \sqrt{1 - \frac{r^2}{X^2}} - \frac{1}{2} \sqrt{1 - \frac{r^2}{X^2}} \right)^2 dt^2 - \left( 1 - \frac{r^2}{X^2} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

with \( X^2 = \frac{3}{8\pi G \rho} \), \( \rho \) being density of matter. By slightly rearranging, we can also write this as,

\[ ds^2 = \left( \frac{3}{2} \sqrt{1 - \frac{r}{r_g}} - \frac{1}{2} \sqrt{1 - \frac{r^2 r_s}{r_g^3}} \right)^2 dt^2 - \left( 1 - \frac{r^2 r_s}{r_g^3} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

\( r_g \) and \( r_s \) being value of “r” coordinate on body’s surface and Schwarzschild radius respectively.

The Gravitational Potential corresponding to interior metric becomes,

\[ \varphi = -\frac{1}{2} \left( 1 - \frac{r^2 r_s}{r_g^3} \right) + 2d \left( 1 - \frac{r^2 r_s}{r_g^3} \right) \left( k - \sqrt{1 - \frac{r^2 r_s}{r_g^3}} \right)^{-2} \]

where

\[ k = 3 \sqrt{1 - \frac{r^2 r_s}{r_g^3}} \]

The potential from exterior metric is

\[ \varphi = -\frac{1}{2} \left( 1 - \frac{r_s}{r} \right) + \frac{d}{2} \]

Now equality of these two \( \varphi \)'s (potential must be single valued) at \( r = r_g \) clearly demands that “d” must be zero (since “r” is a variable, whereas \( r_s, r_g \) are arbitrary parameters). Hence, the potential in the interior of a mass distribution becomes (neglecting the constant -1/2),

\[ \varphi = \frac{1}{2} \frac{r^2 r_s}{r_g^3} \]

Now this is simply the Newtonian-Potential formula for the interior of a spherical mass of constant density of radius \( r_g \). Hence again, no difference is being found in the potential from that of Newton’s theory, even in the presence of matter, which could show up any self-interaction effect, the key ingredient of a relativistic theory of Gravitation.

Reissner-Nordstrom Solution:

The Reissner-Nordstrom metric is \( ^{(6)} \).
\[ ds^2 = \left(1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi\varepsilon r^2}\right)dt^2 - \left(1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi\varepsilon r^2}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \]

from which the potential comes out to be,

\[ \varphi = -\frac{GM}{r} + \frac{GQ^2}{4\pi\varepsilon r^2} + k \]

which shows a different behaviour of the potential around \( \sqrt{\frac{GQ}{4\pi\varepsilon}} \). But this is simply due to electromagnetic coupling of gravity, rather than a self-interaction effect we are in search of; that can be easily checked letting the electromagnetic-coupling constant \( \frac{Q^2}{4\pi\varepsilon} \) go to zero (clearly any self-interaction effect of gravity must not vanish if we let any other parameter go to zero other than gravitational one), in which case we get back usual Newtonian Potential.

**Kerr-Solution:**

The Kerr-metric\(^{(7)}\) is written as,

\[ ds^2 = \left(1 - \frac{2GM}{\Sigma} + \frac{\Sigma^2}{\Delta} dr^2 - \Sigma^2 d\theta^2 - \left(\frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2\theta}{\Sigma^2}\right)\sin^2\theta d\phi^2 + \frac{4aGMr \sin^2\theta}{\Sigma^2} dtd\phi \]

where \( \Delta = r^2 - 2GMr + a^2 \), \( \Sigma^2 = r^2 + a^2 \cos^2\theta \) and \( a = \frac{J}{M} \)

Clearly this metric deviates from spherically-symmetric due to the presence of angular momentum. But following our previous argument, as limit \( J \to 0 \) will suffice our purpose, we can take the zero angular-momentum limit of Kerr-Solution:

\[ ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \]

which is simply the Schwarzschild-Solution(external) for a mass \( M \). Hence, proceeding like before, we arrive at the same Newtonian result.

**Kerr-Newman Solution:**

The Kerr-Newman metric\(^{(8)}\) is

\[ ds^2 = -\left(\ln \frac{r^2 + (a \sin\theta \sin d\phi - d\theta)^2}{\Delta} \frac{r^2 + (a^2 + d\phi - a\theta)^2}{(r^2 + a^2)^2} \right) \Delta^2 - \frac{\Delta}{r^2} d\phi - \frac{\Delta}{r} d\theta \]

Here
\[ a = \frac{J}{M}; \quad R^2 = r^2 + a^2 \cos^2 \theta; \quad \Delta = r^2 - 2GMr + a^2 + \frac{GQ^2}{4\pi\varepsilon} \]

Clearly at \( J \to 0 \), this solution merges with Reissner-Nordstrom metric, and we already discussed this case.

**Cosmological Solutions:**

The famous cosmological solutions of EFE e.g. Einstein’s universe, Di-Sitter’s universe, FLRW universe, all can be written in the form\(^9\)

\[ ds^2 = dt^2 - a^2(t) \left( \frac{dr^2}{1-kr^2} + r^2d\Omega^2 \right) \]

with

\[ d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \]

In this metric, though \( a(t) \) is a time varying scale-factor (in general), it is enough to consider the solution at a particular time, since self-interaction, if present, should persist at each moment; it can’t disappear and reappear at random!

The potential becomes,

\[ \varphi = -\frac{1}{2}e^{-\lambda} = -\frac{1}{2a^2(t_0)}(1-kr^2) \]

which is again a Newton-like potential inside a sphere.

From the above consideration we find, no spherically-symmetric solution (or any solution, consequently) of Einstein’s Field Equations gives birth such a potential, which is radically different from Newtonian one, carrying the fingerprint of self-interaction. Hence, we are to arrive at the conclusion that Einstein’s Field Equations cannot provide an exact relativistic description of the Gravitational-Field, the self-interaction being absent in the theory.

**Concerning disturbing-feature of energy-momentum conservation in Einstein’s Field Equation(s):**

The fact that, the same trouble is being manifested in each of the solutions of the field equations convinces that there is some issue in the key equation itself. Clearly all of them are solutions of
\( G_{\mu\nu} = 8\pi T_{\mu\nu} \). Hence, one gets the conservation of energy-momentum when a gravitational field is present,

\[
T_{ik}^{\;\;\;\;\;\;\;\;k} = \frac{1}{\sqrt{-g}} \frac{\partial(T_i^{\;\;k}\sqrt{-g})}{\partial x^k} - \frac{1}{2} \frac{\partial g_{ik}}{\partial x^l} T^{kl} = 0
\]

In this form, this does not express any conservation law \(^{(10)}\). This is because the integral

\[
\int T_i^{\;\;k}\sqrt{-g} \, dS_k
\]

becomes conserved only if the condition,

\[
\frac{\partial(T_i^{\;\;k}\sqrt{-g})}{\partial x^k} = 0
\]

is satisfied, which is not clearly same as the condition above. Such an error is related to the fact that in a gravitational field, the four momentum of the matter alone should not be conserved; the total four momentum of matter plus gravitation field must be conserved, which is not included in the stress-energy tensor of Einstein’s Field Equations. Hence we see, this is the inherent trouble which is giving rise of “incorrect” space-time, in General Relativity.

We can estimate a distance-scale at which one can expect a significantly different result between space-time given by EFE and actual space-time. In the case of interior-Schwarzschild solution, we take \( \rho(r) = \rho \) as stress-tensor component \( T_{00}^{\;\;0} \). Clearly we should expect a large discrepancy, when this \( \rho \) will be comparable to gravitational energy density. The gravitational energy density around a mass “m” can be estimated as (in vacuum),

\[
u = \frac{1}{8\pi G} \left( \frac{GM}{r^2} \right)^2
\]

Now if at \( r = r_g \), the energy-density of matter \( \frac{3M}{4\pi r^3 c^2} \) is same as the density of the gravitational field, we get \( r_g \sim \frac{GM}{c^2} \). Thus at distances of order Schwarzschild-Radius, the solutions of EFE should be significantly incorrect.

**Concerning experimental verifications:**

From the Schwarzschild metric, the usual geodesic equation of motion for planets can be written as,

\[
u'' + u = \mu + \frac{3}{2} u^2
\]
with \( u = \frac{r_s}{r} \) and \( \mu = \frac{r_s^2 c^2}{2 J^2} \), where \( r_s \) is Schwarzschild radius of the Sun and \( J \) is angular-momentum per unit mass of the planet. Here a “prime” denotes the differentiation with respect to azimuthal-angle \( \psi \).

As is well known, the second-term at the r.h.s. of the above equation gives the second-approximation to the Newtonian-solution \( u_0 = \mu (1 + e \cos \psi) \). The complete solution can be written as,

\[
u = u_0 + u_1 = \mu (1 + e \cos \psi) + \frac{3}{2} \mu^2 \epsilon \psi \sin \psi \approx \mu \left( 1 + e \cos \left( 1 - \frac{3\mu}{2} \psi \right) \right)
\]

Hence the perihelion precession per revolution turns out to be \( 3\pi \mu \) or \( 5.01866 \times 10^{-7} \) radians per revolution or 42.98 arc seconds per century.

It is widely believed by Scientific-Community that this amount of perihelion precession due to space-time curvature caused by Sun is in complete conformity with experimental data. The perihelion precession of Mercury per Julian Century is found to be \( 574.10 \pm 0.91 \) arc seconds\(^{(11)}\). Precession due to gravitational effect of other planets are calculated to be 531.9 arc seconds\(^{(12)}\) and General Relativity gives 42.98 arc seconds. Summing we get, 574.88 arc seconds which is in well conformity with experimental data.

But there’s an important fallacy in the game! We are to emphasize the fact that the *only* observable number related to perihelion precession is the number \( 574.10 \pm 0.91 \). We can’t observe separately the precession due to planets and precession due to General relativity by our instruments; instead we calculate those numbers to make ourselves convinced that their sum matches with observation. So we shall attempt to review those calculations once more.

The calculation of the number 42.98 arc seconds originated from GR, we already discussed. What is left, is the calculation of the number 531.9 arc seconds, which comes from Newtonian effect of other planets. The obvious approximation which is used to calculate the perturbations due to the other planets, can be said “ring-planet model approximation”\(^{(13)}\).

If \( \delta \) is the angle between two consecutive apsis in the orbit (at which radius vector assumes an extremum value), we can write\(^{(14)}\)

\[
\delta = \pi \left( 1 - \frac{F(a)}{F_0} - \frac{G \pi m S}{2 F_0} \right)
\]

Here \( F_0 \) is the contribution from the Sun and \( F(a) \) is the contribution from other planets. \( F(a) \) is written as,

\[
F(a) = G \pi m \sum_{i=2}^{8} \frac{\lambda_i}{R_i^2} \frac{a}{R_i^2 - a^2}
\]
With \[ S = \sum_{i=2}^{8} \lambda_i \frac{R_i^2 + a^2}{(R_i^2 - a^2)^3}. \]

(Here "m" and "a" is the mass and radius of the Planet concerned, \( R_i \) is the radius of i-th planet contributing to precession and \( \lambda_i \) is its linear mass density)

The rate of precession is obtained as,

\[ \omega = \frac{2 \delta - 2\pi}{P} = 531.9\text{arc s/century} \]

\( P \) being the orbital period of mercury.

In the above calculation, actually contributions up to Saturn have been taken, the outer planets being less massive and more distant, giving no significant contribution. However, if we add the contributions from Uranus and Neptune, the value becomes close to 532.08 arc seconds/century.

Invalidity of the Ring-Model:

There is no doubt in the fact that, taking all planets as rings, we made a strange approximation, to be able to do the calculation by hand. But fortunately, at present time, using powerful softwares, we can well accomplish all such calculations very exactly, without any requirement of strange approximation. It is possible to determine the perihelion precession rate directly from solutions of the differential equations of the system, as we shall see below.

First let’s check, how far the ring-planet assumption works in the calculation of perihelion precession of a planet. To do this, we shall first numerically solve the differential equations of a toy solar system in two dimensions containing “Sun” and two planets “Mercury” and “Venus”, and shall find perihelion positions of “Mercury” over time using codes showed below (clearly, no approximation will be involved to find the perihelion-precession rate). We shall use “Wolfram-Mathematica” to solve the problem. We attach the script below with reference of the parameters:

The parametric plot gives the orbits:
The orbit of “Mercury” is clearly showing here a perihelion precession. The rate of perihelion precession can be obtained if we attach a “perihelion-finder” code with the above script, which exploits the fact that the “toward velocity” of a planet suffers a change in sign (from negative to positive) when it just crosses the perihelion position. For that we first write a code of “toward velocity” of the planet together with the angle of revolution in arc seconds:

\[ \text{Ve}lTud[t_\cdot] := \frac{\sqrt{x_1[t]^2 + y_1[t]^2}}{\text{result}[1]}; \text{AngArcsec}[t_\cdot] := \text{ArcTan}[x_1[t], y_1[t]]/\pi 180 \cdot 3600 / \text{result}[1]; \]
The perihelion finder code we use, is as follows:

```plaintext
PeriAng = {}; StdPeriod = 2 \pi \sqrt{(R1/(1-ec1))^3/(GM)};
prevMidT = 0; firstAng = True; prevAng = 0; startAng = 1/3600 AngArcsec[0];
tt = 0; divn = 0.001;
While[tt < period - divn,
  While[tt > divn && VelTwd[tt - divn] > 0, tt -= divn];
  While[tt < period - divn && VelTwd[tt + divn] < 0, tt += divn];
  If[firstAng == False, lowerT = tt - divn; upperT = tt + divn, lowerT = 0; upperT = 2 divn];
  middleT = lowerT + (upperT - lowerT) (VelTwd[lowerT] / (VelTwd[lowerT] - VelTwd[upperT]));
  If[middleT >= 0 && middleT < period,
    thisAng = AngArcsec[middleT];
    thisAng = AngArcsec[lowerT] + (AngArcsec[upperT] - AngArcsec[lowerT]) (middleT - lowerT) / (upperT - lowerT);
  ];
  thisAng /= 3600;
  AppendTo[PeriAng, {middleT/60, thisAng - startAng}];
  tt = middleT;
  If[firstAng == False, StdPeriod = middleT - prevMidT];
  prevMidT = middleT;
  firstAng = False;
  tt += StdPeriod;
]
```

The validity of the above code can be easily found, if we solve the system with only “Sun” and “Mercury” with same parameters and find the slope of perihelion line, which gives a slope of very nearly zero:

Now we plot the perihelion position vs. time of our original system, we get like following:
The line of best fit gives the average perihelion precession 4.28 (degrees/minute).

Now we shall construct a second system with almost exactly same with the previous one, but this time the mass of “Sun” will be seven units. Then according to “Ring planet model” we should clearly get again perihelion advance of Mercury with quite an increased rate; (In the expression of $\delta$ in the previous section, will change to some extent due to change in $F_0$)

Now we get the orbit of mercury as following:
Hence we see, over a sufficient amount of time, the perihelion is recessing instead of getting advanced in an increased rate, and this clearly exhibits that ring model approximation is not a valid approximation in general, which clearly can't produce any such effect of recession of perihelion, as may occur in reality, as showed by exact calculation.

Now we shall consider a case, when the orbital plane of “Venus” is not in the same plane of “Mercury” but tilted at a certain angle. For that purpose, we use usual orbit transformation equations in three dimensions to have desired tilted orbit.
We expect to get the same rate of precession from the ring Planet model as in zero inclination (the expression of $\delta$ contains no information regarding inclination of orbit). What we get instead is like following:

Again the perihelion is receding, in an interesting pattern! Clearly we see the orbital inclination is also a very important factor to determine the precession rate, not accounted in the "Ring Model".

**Solving the Real Solar-System:**
Solving numerically our toy systems, we showed why the “Ring-Planet” model should not be considered as a good approximation to calculate non-relativistic precession rate of planets, and hence our expectation to have same amount of precession-rate (non-relativistic) as predicted by the model gets pulverized. Therefore now we attempt to determine the perihelion-precession rate of real Mercury (taking into account all the orbital elements) using real data of the Solar system.
Now first we write the acceleration equations in such a way, so that we can put the differential equations in the form of arrays.

```mathematica
(* Masses *)
Mjupiter = 1.998844 * 10^24;
Msaturn = 5.6834 * 10^23;
MRahab = 1.43353 * 10^{12};
RPhal = 1.35255 * 10^{12};
EMercury = 0.05410609;
EJupiter = 10.759.22;
ESaturn = 2.484466;
EPhal = 113.75084;
ESaturn = 338.7169;
Phal = 8.6813 * 10^9;
RPhal = 2.89726 * 10^{12};
EPhal = 2.74130 * 10^{12};
ESwh = 0.04716771;
EUranus = 30.695.4;
EUranus = 0.769866;
EJupiter = 74.22988;
EJupiter = 96.73436;
MNeptune = 0.02413 * 10^7;
MNeptune = 4.49506 * 10^7;
MNeptune = 4.44445 * 10^7;
MNeptune = 0.00009587;
MNeptune = 60.189;
MNeptune = 1.76917;
MNeptune = 131.72169;
MNeptune = 273.24966;

(* Planets *)
Mplanets = {MMercury, MVenus, MEarth, MMars, MJupiter, MSaturn, MPhal, MNeptune};
PMass = {MMercury, MVenus, MEarth, MMars, MJupiter, MSaturn, MPhal, MNeptune};
Land = {Land, MMercury, MVenus, MEarth, MMars, MJupiter, MSaturn, MPhal, MNeptune};

(* Acceleration and forces *)
rdist[ptx_, pty_, ptz_, pxx_, psy_, pzz_] := Sqrt[(ptx - pxx)^2 + (pty - psy)^2 + (ptz - pzz)^2];

(* Total force *)
Ftot[ptm_, ptx_, pty_, ptz_, psm_, pxx_, psy_, pzz_] := 6 ptm psm / rdist[ptx, pty, pxx, psy, pzz];

(* x/y/z force components *)
Fxx[ptm_, ptx_, pty_, ptz_, psm_, pxx_, psy_, pzz_] := Ftot[ptm, ptx, pty, ptz, psm, pxx, psy, pzz] / rdist[ptx, pty, pxx, psy, pzz];
Fy[ptm_, ptx_, pty_, ptz_, psm_, pxx_, psy_, pzz_] := Ftot[ptm, ptx, pty, ptz, psm, pxx, psy, pzz] / rdist[ptx, pty, pxx, psy, pzz];
Fz[ptm_, ptx_, pty_, ptz_, psm_, pxx_, psy_, pzz_] := Ftot[ptm, ptx, pty, ptz, psm, pxx, psy, pzz] / rdist[ptx, pty, pxx, psy, pzz];

ma = 1; xa = 2; ya = 3; za = 4;
Fx[pt_, ps_] := Fxx[ptm, ptx, pty, ptz, psm, pxx, psy, pzz];
Fy[pt_, ps_] := Fy[ptm, ptx, pty, ptz, psm, pxx, psy, pzz];
Fz[pt_, ps_] := Fz[ptm, ptx, pty, ptz, psm, pxx, psy, pzz];

(* Acceleration components *)
Axx[pt_, ps_] := Fx[pt, ps] / pm[ma];
Ayy[pt_, ps_] := Fy[pt, ps] / pm[ma];
Azz[pt_, ps_] := Fz[pt, ps] / pm[ma];
```

Unfortunately, while solving the equations, Mathematica doesn't allow us to have the required precision in the calculation. To overcome this, we add a "ghost planet" H in our system, which by no means affect the other objects in our calculation (as one can clearly find in the code containing the differential equations given below), but it turns out that it only increases the
accuracy of the whole calculation to a very good amount, as we decrease its distance from the Sun. To include that, we write the acceleration code for it as above:

\[
\begin{align*}
Hrdist[pdx_, pdy_, pdz_] &= \sqrt{(pdx^2 + pdy^2 + pdz^2)}; \\
HFTot[ptm_, psm_, pdx_, pdy_, pdz_] &= -G \frac{ptm \cdot psm}{Hrdist[pdx, pdy, pdz]^3} \\
HFxx[ptm_, psm_, pdx_, pdy_, pdz_] &= HFTot[ptm, psm, pdx, pdy, pdz] \frac{pdx}{Hrdist[pdx, pdy, pdz]}; \\
HFyy[ptm_, psm_, pdx_, pdy_, pdz_] &= HFTot[ptm, psm, pdx, pdy, pdz] \frac{pdy}{Hrdist[pdx, pdy, pdz]}; \\
HFzz[ptm_, psm_, pdx_, pdy_, pdz_] &= HFTot[ptm, psm, pdx, pdy, pdz] \frac{pdz}{Hrdist[pdx, pdy, pdz]}; \\
\end{align*}
\]

\[\text{\textcopyright}\] since when bodies are closer to one another, gravity changes more rapidly in smaller distance due to its inverse square law, and Mathematica automatically decreases its time-step.

Next we create arrays containing the planets’ masses and positions:

\[
\begin{align*}
\{M0, M1, M2, M3, M4, M5, M6, M7, M8\} &= \{\text{M}0, \text{M}1, \text{M}2, \text{M}3, \text{M}4, \text{M}5, \text{M}6, \text{M}7, \text{M}8\}; \\
Pc0 &= \{M0, x0[t], y0[t], z0[t]\}; \\
Pc1 &= \{M1, x1[t], y1[t], z1[t]\}; \\
Pc2 &= \{M2, x2[t], y2[t], z2[t]\}; \\
Pc3 &= \{M3, x3[t], y3[t], z3[t]\}; \\
Pc4 &= \{M4, x4[t], y4[t], z4[t]\}; \\
Pc5 &= \{M5, x5[t], y5[t], z5[t]\}; \\
Pc6 &= \{M6, x6[t], y6[t], z6[t]\}; \\
Pc7 &= \{M7, x7[t], y7[t], z7[t]\}; \\
Pc8 &= \{M8, x8[t], y8[t], z8[t]\}; \\
PcH &= \{M1, xH[t], yH[t], zH[t]\};
\end{align*}
\]

Then we write the initial conditions of position and velocity of the planets (including the “Ghost Planet”), in proper orientation (first we choose the initial positions of the planets at their perihelion; other cases will also be considered later):
Then we write the differential equations:

\[
G = G_0; \{R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8\} = \{R_{\text{Mercury}}, R_{\text{Venus}}, R_{\text{Earth}}, R_{\text{Mars}}, R_{\text{Jupiter}}, R_{\text{Saturn}}, R_{\text{Uranus}}, R_{\text{Neptune}}\}; \text{velint} = \text{Array}\{0,8\};
\]

\[
\text{For} \{i = 1, i \leq 8, \text{velint} = \lfloor \sqrt{1 + \text{Peccei}^2} + \frac{G R_{\text{Sun}}}{\text{Pperi}^2}\rfloor; \{V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8\} = \text{velint};
\]

\[
\{P_{x1}, P_{y1}, P_{z1}\} = \text{OrbTransXYdeg}\{R_1, 0, 0, \text{Mercury}, \text{Mercury}, \text{Mercury}\}; \{V_{x1}, V_{y1}, V_{z1}\} = \text{OrbTransXYdeg}\{0, V_1, 0, \text{Mercury}, \text{Mercury}, \text{Mercury}\};
\]

\[
\{P_{x2}, P_{y2}, P_{z2}\} = \text{OrbTransXYdeg}\{R_2, 0, 0, \text{Venus}, \text{Venus}, \text{Venus}\}; \{V_{x2}, V_{y2}, V_{z2}\} = \text{OrbTransXYdeg}\{0, V_2, 0, \text{Venus}, \text{Venus}, \text{Venus}\};
\]

\[
\{P_{x3}, P_{y3}, P_{z3}\} = \text{OrbTransXYdeg}\{R_3, 0, 0, \text{Earth}, \text{Earth}, \text{Earth}\}; \{V_{x3}, V_{y3}, V_{z3}\} = \text{OrbTransXYdeg}\{0, V_3, 0, \text{Earth}, \text{Earth}, \text{Earth}\};
\]

\[
\{P_{x4}, P_{y4}, P_{z4}\} = \text{OrbTransXYdeg}\{R_4, 0, 0, \text{Mars}, \text{Mars}, \text{Mars}\}; \{V_{x4}, V_{y4}, V_{z4}\} = \text{OrbTransXYdeg}\{0, V_4, 0, \text{Mars}, \text{Mars}, \text{Mars}\};
\]

\[
\{P_{x5}, P_{y5}, P_{z5}\} = \text{OrbTransXYdeg}\{R_5, 0, 0, \text{Jupiter}, \text{Jupiter}, \text{Jupiter}\}; \{V_{x5}, V_{y5}, V_{z5}\} = \text{OrbTransXYdeg}\{0, V_5, 0, \text{Jupiter}, \text{Jupiter}, \text{Jupiter}\};
\]

\[
\{P_{x6}, P_{y6}, P_{z6}\} = \text{OrbTransXYdeg}\{R_6, 0, 0, \text{Saturn}, \text{Saturn}, \text{Saturn}\}; \{V_{x6}, V_{y6}, V_{z6}\} = \text{OrbTransXYdeg}\{0, V_6, 0, \text{Saturn}, \text{Saturn}, \text{Saturn}\};
\]

\[
\{P_{x7}, P_{y7}, P_{z7}\} = \text{OrbTransXYdeg}\{R_7, 0, 0, \text{Uranus}, \text{Uranus}, \text{Uranus}\}; \{V_{x7}, V_{y7}, V_{z7}\} = \text{OrbTransXYdeg}\{0, V_7, 0, \text{Uranus}, \text{Uranus}, \text{Uranus}\};
\]

\[
\{P_{x8}, P_{y8}, P_{z8}\} = \text{OrbTransXYdeg}\{R_8, 0, 0, \text{Neptune}, \text{Neptune}, \text{Neptune}\}; \{V_{x8}, V_{y8}, V_{z8}\} = \text{OrbTransXYdeg}\{0, V_8, 0, \text{Neptune}, \text{Neptune}, \text{Neptune}\};
\]

\[
R_H = R_{\text{Mercury}} \times 0.34; \frac{1}{\sqrt{\frac{G R_H}{R_H}}}
\]

\[
\{P_{xH}, P_{yH}, P_{zH}\} = \{R_H, 0, 0\}; \{V_{xH}, V_{yH}, V_{zH}\} = \{0, V_H, 0\};
\]

Together with assigning the initial conditions:
After doing all these, when we calculate the perihelion-precession rate (in the unit of arc seconds/year) by a linear fit as before, we get the following:

![Plot of the fitted model with data](image)

The line fits very well (which is why we select the planet Mercury), as is manifested in the following plot of the “Fitted-Model” together with data:
We tabulate several other slopes with some different ghost-parameters \(( p = R_{\mu} / R_{\text{mercury}} ):\)

<table>
<thead>
<tr>
<th>Ghost Parameter</th>
<th>Slope (Arcs / year)</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.345</td>
<td>5.29059</td>
<td></td>
</tr>
<tr>
<td>0.344</td>
<td>5.28972</td>
<td></td>
</tr>
<tr>
<td>0.343</td>
<td>5.29017</td>
<td></td>
</tr>
<tr>
<td>0.342</td>
<td>5.29022</td>
<td></td>
</tr>
<tr>
<td>0.341</td>
<td>5.29067</td>
<td>5.29028</td>
</tr>
<tr>
<td>0.340</td>
<td>5.2899</td>
<td></td>
</tr>
<tr>
<td>0.339</td>
<td>5.29057</td>
<td></td>
</tr>
<tr>
<td>0.338</td>
<td>5.29027</td>
<td></td>
</tr>
<tr>
<td>0.337</td>
<td>5.29113</td>
<td></td>
</tr>
<tr>
<td>0.336</td>
<td>5.28973</td>
<td></td>
</tr>
<tr>
<td>0.335</td>
<td>5.29</td>
<td></td>
</tr>
</tbody>
</table>

Now considering some other initial conditions, the precession rate as it comes out is tabulated below:

<table>
<thead>
<tr>
<th>Mercury</th>
<th>Venus</th>
<th>Earth</th>
<th>Mars</th>
<th>Jupiter</th>
<th>Saturn</th>
<th>Uranus</th>
<th>Neptune</th>
<th>Precession-rate (Arcs/year)</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>5.29028</td>
</tr>
<tr>
<td>P</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>5.2959</td>
</tr>
<tr>
<td>P</td>
<td>A</td>
<td>A</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>5.29408</td>
</tr>
<tr>
<td>P</td>
<td>P</td>
<td>P</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>P</td>
<td>5.28701</td>
</tr>
<tr>
<td>P</td>
<td>10°</td>
<td>20°</td>
<td>30°</td>
<td>40°</td>
<td>50°</td>
<td>60°</td>
<td>70°</td>
<td>5.29099</td>
</tr>
</tbody>
</table>
In the last column, the angles are measured from the corresponding perihelia of the planets.

From the above table, it's quite conclusive that the precession rate is fairly independent to the initial position of the planets at their orbits, being always very close to $5.29\text{arcs/year}^\text{°}$.

This precession-rate is almost $3.08\text{ arcs/century}$ less than predicted by Ring-Planet model (considering all planets). Given the observed precession-rate $574.10\text{ arcs/century}$ to be correct, it must be that due to Sun the perihelion-precession rate is nearly $45.1\text{ arcs/century}$, which certainly can't be explained using General Relativity, which gives a precession rate around $42.98\text{ arcs/century}$, as we saw above. We shall see below, surprisingly, that if the self-interaction effect is added in GR, as I proposed to be missing in it, can almost exactly account for this discrepancy; which shows all our previous discussions were in the direction of truth. Hence, there is neither any reason nor any requirement to doubt the result of Wolfram-Mathematica!

¶While calculating the precession rate from Horizon Ephemeris, one gets precession rate very nearly the rate predicted by ring-planet model, which is clearly not consistent with the conclusion we arrived above, leading the fact that Horizon data is erroneous.

Correcting the Schwarzschild-Geodesic:

We saw before, the Newtonian-Potential of Gravity when it self-interacts, is given by,

$$
\phi(r) = c^2 \left( 1 - \frac{3GM}{rc^2} + \frac{9}{4} \frac{G^2M^2}{r^2c^4} \right) - c^2
$$

After expanding up to second order in $\frac{1}{r^2}$, we get

$$
\phi(r) = -\frac{GM}{r} - \frac{1}{4} \frac{G^2M^2}{r^2c^2}
$$

Since at present we are concerned about gravitational effect at planetary distance, we can use the relation,

$$
g_{\mu\nu} \approx 1 + \frac{2\phi}{c^2}
$$
Whence we can write,

\[ g_{00} \approx 1 - \frac{2GM}{rc^2} - \frac{1}{4} \frac{G^2M^2}{r^2c^4} \]

This second term, together with a nice factor \(-(1/4)\) is originated purely from self-interaction effect of Gravity, and it's worth mentioning that it's never originated from the vacuum solution of the field equations, which gives the value of \( g_{00} \) exactly to be \( \left(1 - \frac{2GM}{rc^2}\right) \).

When we use this new \( g_{00} \), our equation of motion becomes (up to second order)

\[ u'' + u = \mu + \frac{3}{2} u^2 + \frac{1}{4} \mu u \]

We already know the contribution coming from the second term at the right hand side of the equation. To find the effect of the third term, it will be enough and easier to find the second order correction of the equation,

\[ u'' + u = \mu + \frac{1}{4} \mu u \]

Writing \( u = u_0 + \varepsilon(\psi) \) where \( u_0 \) is the familiar Newtonian solution and \( \varepsilon(\psi) \) is a small perturbation coming from the new term, we find, \( \varepsilon(\psi) \) satisfies,

\[ \varepsilon''(\psi) + \varepsilon(\psi) = \frac{1}{4} \mu^2 (1 + e \cos \psi) \]

The particular integral gives the required correction, which is,

\[ \varepsilon_p(\psi) = \frac{e \mu^2}{8} \psi \sin \psi \]

The particular integral from Einstein's equation gives the correction,

\[ \varepsilon_p(\psi) = \frac{3 \mu^2}{2} e \psi \sin \psi \]

Hence the complete solution becomes,
\[ u = \mu \left(1 + e \cos \psi \right) + \frac{13}{8} \mu^2 e \psi \sin \psi \approx \mu \left(1 + e \cos \left(1 - \frac{13 \mu}{8} \right) \psi \right) \]

which gives the perihelion precession rate \( \frac{13 \mu \pi}{4} \), or almost 1.08 times 42.98 arcs per century, which is 46.4 arcs/century, 3.4 arcs/century more than predicted by GR, and explains very well the discrepancy of 3.08 arcs/century, the amount by which the non-relativistic precession rate must differ from the value of Ring-Planet model.

Conclusion:

I proved that description of space-time, as emerged from General Relativity is not in complete conformity as expected from a relativistic theory of Gravitation, since in such a theory, gravity must interact with self, which appears to be missing in GR, and hence the space-time solutions as derived from its field equations, are not correct (though they are correct to a very good approximation, especially at larger distances). It was also showed, at which distance-scale the discrepancy should be significant. Finally, the experimental proof of such a proposition has been presented in which it was showed, the non-relativistic contribution to the precession rate of Mercury is actually around 529 arcs/century (instead of 532arcs/century, as calculated from “Ring Model”), as we get by numerically solving the differential equations of the Solar System in “Wolfram Mathematica”, and in that case the relativistic contribution to the precession rate should be around 45 arcs/century (instead of 42.98arcs/century), and can be described almost exactly if the self-interactive correction is added to the solutions of GR, which shows our proposition is in the direction of truth.

Acknowledgement:

I’m indebted to all those persons who always inspired me in my creative journey. And I convey my cordial thanks to Mr. Bernard Burchell for immensely helping me in the field of “Wolfram Mathematica”.

References:

[1] Preussische Akademie der Wissenschaften, Sitzungsberichte, 1915 (part 2), 844–847


