Boundary Matrices and the Marcus-de Oliveira Determinantal Conjecture

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ABSTRACT
We present notes on the Marcus-de Oliveira conjecture. The conjecture concerns the region in the complex plane covered by the determinants of the sums of two normal matrices with prescribed eigenvalues. Call this region Δ. This paper focuses on boundary matrices of Δ. We prove 2 theorems regarding these boundary matrices. This paper uses ideas from [1].

KEYWORDS
determinantal conjecture, Marcus-de Oliveira, determinants, normal matrices, convex-hull

1. Introduction

Marcus [2] and de Oliveira [3] made the following conjecture. Given two normal matrices A and B with prescribed eigenvalues $a_1, a_2...a_n$ and $b_1, b_2...b_n$ respectively, $det(A + B)$ lies within the region:

$$co\Big\{ \prod (a_i + b_{\sigma(i)}) \Big\}$$

where $\sigma \in S_n$. co denotes the convex hull of the $n!$ points in the complex plane. As described in [1], the problem can be restated as follows. Given two diagonal matrices, $A_0 = diag(a_1, a_2...a_n)$ and $B_0 = diag(b_1, b_2...b_n)$, let:

$$\Delta = \Big\{ det(A_0 + UB_0U^*) : U \in U(n) \Big\} \tag{1}$$

where $U(n)$ is the set of $n \times n$ unitary matrices. Then we can write the conjecture as:

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Conjecture 1.1 (Marcus-de Oliveira Conjecture).

$$\Delta \subseteq co\{\prod(a_i + b_{\sigma(i)})\} \tag{2}$$

Let

$$M(U) = det(A_0 + UB_0U^*). \tag{3}$$

Note that the unitary matrices are a compact set. And since the continuous image of a compact set is compact, Δ is compact. Since a compact set in a metric space is closed, Δ is closed. So $\partial\Delta \subseteq \Delta$ where $\partial\Delta$ is the boundary of Δ.

The paper is organized as follows. In section 2 we define terms and functions that will be used in the rest of the paper. These definitions are necessary to state our results. In section 3, we state 3 lemmas and 2 theorems that form the bulk of the paper. We state them in the order they are proved.

2. Preparatory definitions

2.1. Ordinary point of $\partial\Delta$

For the purposes of this paper we call a point $P \in \partial\Delta$ an ordinary point of $\partial\Delta$ if P isn't any kind of singularity of $\partial\Delta$. Formally, we define an ordinary point P of $\partial\Delta$ as one that satisfies the following four conditions:

- $\partial\Delta$ has a unique tangent at P.

To state the rest of the conditions we first replace the real and imaginary axes with the x-y axes. Then we translate Δ so that P coincides with the origin. Now we rotate the resulting figure about the origin so that the tangent to $\partial\Delta$ at P coincides with the x-axis. For simplicity we keep the labels Δ, $\partial\Delta$ and P post translation and rotation. Then if P is an ordinary point of Δ, there exists an open ball B centered on the origin and a function f from $\mathbb{R} \to \mathbb{R}$ such that:

- $(x, y) \in \partial\Delta \cap B \iff f(x) = y$.
  ie: within B, we don't have two different boundary points with the same x-coordinate.
- $\forall(x, y) \in \Delta \cap B$ we have $y \leq f(x)$ OR
  $\forall(x, y) \in \Delta \cap B$ we have $y \geq f(x)$
  ie: within B, Δ lies entirely above the boundary, or entirely below the boundary.
- f is continuous and differentiable at the origin.

Note some of these may be redundant conditions, but we state them for completeness and clarity.

Now suppose P is an ordinary point of $\partial\Delta$ and we have a curve R $\subseteq \Delta$ that interects P and has a unique tangent at P. We wish to demonstrate that the tangent to R at P is the same as the tangent $\partial\Delta$ at P. We translate Δ so that P coincides with the origin, and we rotate Δ so that the tangent coincides with the x-axis. We keep the labels Δ, $\partial\Delta$, P and R post translation and rotation. We know there's an open ball B centered on the origin such that within B we can write the points of the boundary as $(x, f(x))$ for some function f. We can also write the points of R as $(x, g(x))$ for some function g. Note that $f(0) = g(0) = 0$. Let $d(x) = f(x) - g(x)$. We know that within B

$g(x) = f(x) - d(x)$

$g'(x) = f'(x) - d'(x)$

$g'(0) = f'(0) - d'(0).$

Since we know that Δ lies entirely above, or entirely below $\partial\Delta$ within B, we know that $d(0) = 0$ is either a local maximum or a local minimum of $d(x)$. So $d'(0) = 0$. We already know $f'(0) = 0$ by our setup.

Therefore

$g'(0) = 0.$

Therefore the tangent to $g(x)$ at the origin is the x-axis. ie: it coincides with tangent to the boundary. And this holds true of the curve and the boundary before translation and rotation.

2.2. Terms

Given a unitary matrix U and square, diagonal matrices A_0 and B_0 all of dimension $n \times n$,

- If $M(U)$ is a point on $\partial\Delta$ (the boundary of Δ), we call $M(U)$ a boundary point of Δ and we call U a **boundary matrix** of Δ. See eq. (1) and eq. (3).
- We define the **B-matrix** of U as UB_0U^*.
- We define the **C-matrix** of U as $A_0 + UB_0U^*$.
- We define the **F-matrix** of U as $C^{-1}A_0 - A_0C^{-1}$ where C is the C-matrix of U. Note that the F-matrix is only defined when C is invertible, or equivalently when $det(C) = M(U) \neq 0$. See eq. (3). Also note that since A_0 is diagonal, the F-matrix is a zero-diagonal matrix. The idea for using the F-matrix comes from [1], Theorem 4, p.27.

Throughout the rest of the paper, we'll assume A_0 and B_0 are defined, even if we don't explicitly mention them.

2.3. Functions given a unitary matrix U

Given a unitary matrix U with B-matrix B, C-matrix C and F-matrix F. For every skew-hermitian matrix Z, we define the following functions

let

$$U_Z(t) = (e^{Zt})U \tag{4}$$

where t is any real number.

Since the exponential of a skew-hermitian matrix is unitary, $U_Z(t)$ is a function of unitary matrices.

let

$$B_Z(t) = U_Z(t)B_0U_Z^*(t) \tag{5}$$

let $C_Z(t) = A_0 + B_Z(t)$

We note that $B_Z(0) = B$ and $C_Z(0) = C$.

let

$$R_Z(t) = det(C_Z(t)) \tag{6}$$

We can see by eq. (1) that $R_Z(t) \subseteq \Delta$.

$R_Z(0) = A_0 + UB_0U^*$

So by eq. (3) we see that $R_Z(0) = M(U)$.

So all the $R_Z(t)$ functions go through $M(U)$ at $t = 0$.

We shall refer to these functions in the rest of the paper with the same notation (for example $R_Z(t)$ for a skew-hermitian matrix Z. $R_{Z_1}(t)$ for a skew-hermitian matrix Z_1). Note that $R_Z(t)$ requires A_0, B_0, U and Z in order to be defined. But we won't explicitly mention A_0 and B_0. All the results in this paper assume there are two diagonal matrices A_0 and B_0 defined in the background.

2.4. Skew-Hermitian matrices Z^{ab} and $Z^{ab,i}$

Given two integers a,b where $1 \leq a, b \leq n$ and $a \neq b$.

We define the $n \times n$ skew-hermitian matrix Z^{ab} as follows. $Z^{ab}_{ab} = -1$ (the element at the ath row and bth column is -1.) $Z^{ab}_{ba} = 1$ (the element at the bth row and ath column is 1.) And all other elements are 0. Note that $Z^{ab} = -Z^{ba}$.

We define the $n \times n$ skew-hermitian matrix $Z^{ab,i}$ as follows. $Z^{ab,i}_{ab} = i$ and $Z^{ab,i}_{ba} = i$.

All other elements are zero. Note that $Z^{ab,i} = Z^{ba,i}$.

It is straightforward to verify that Z^{ab} and $Z^{ab,i}$ are skew-hermitian.

3. Main Results

Lemma 3.1. *Given a unitary matrix U with $M(U) \neq 0$. Let F be its F-matrix. Then $R'_Z(0) = M(U)tr(ZF)$ for any skew-hermitian matrix Z.*

Lemma 3.2. *Given an $n \times n$ zero-diagonal matrix W. Given $tr(Z^{ab}W) = 0$ and $tr(Z^{ab,i}W) = 0$ for all pairs (a,b) where $1 \leq a, b \leq n$ and $a \neq b$. Then W is the zero-matrix.*

Lemma 3.3. *Given a boundary matrix U with $M(U) \neq 0$ and with F-matrix $F \neq 0$. Given $M(U)$ is an ordinary point of $\partial\Delta$. Then there exists a complex number v such that for every skew-hermitian matrix Z, $tr(ZF) = cv$ where c is some real number.*

Theorem 3.4. *Given a boundary matrix U with $M(U) \neq 0$ and with F-matrix $F \neq 0$. Given $M(U)$ is an ordinary point of $\partial\Delta$. Then F can be written uniquely in the form $F = e^{i\theta}H$ where H is a zero-diagonal hermitian matrix and $0 \leq \theta < \pi$.*

Theorem 3.5. *Given a boundary matrix U with $M(U) \neq 0$ and with F-matrix $F \neq 0$. Given $M(U)$ is an ordinary point of $\partial\Delta$. Let L be the tangent line to Δ at $M(U)$. By the previous theorem we know that $F = e^{i\theta}H$ for some real $0 \leq \theta < \pi$. Then L makes an angle $arg(M(U)) + \theta + \pi/2$ with the positive real axis.*

4. Proof of lemma 3.1

The proof given here uses ideas from [1], Theorem 4, p.26-27. But the proof given here is complete on its own.

Proof. We're given a unitary matrix U where $M(U) \neq 0$. So its F-matrix is well-defined and we call it F. Let B be its B-matrix, and C be its C-matrix. Given an arbitrary skew-hermitian matrix Z.

We can use Jacobi's formula [4] on eq. (6) to find $R'_Z(t)$

$$R'_Z(t) = tr(det(C_Z(t))C_Z^{-1}(t)C'_Z(t)) \tag{7}$$

$R'_Z(0) = tr(det(C_Z(0))C_Z^{-1}(0)C'_Z(0))$

We can substitute C for $C_Z(0)$.

$R'_Z(0) = tr(det(C)C^{-1}C'_Z(0))$

$R'_Z(0) = det(C)tr(C^{-1}C'_Z(0))$

We know that $C'_Z(t) = B'_Z(t)$ so

$R'_Z(0) = det(C)tr(C^{-1}B'_Z(0))$

By section 2.2 and eq. (3) we know that $det(C) = M(U)$

$$R'_Z(0) = M(U)tr(C^{-1}B'_Z(0)) \tag{8}$$

Using eq. (5),

$$B'_Z(t) = \frac{dU_Z(t)}{dt}B_0U^*_Z(t) + U_Z(t)B_0\frac{dU^*_Z(t)}{dt} \tag{9}$$

Using eq. (4),

$\frac{dU_Z(t)}{dt} = Ze^{Zt}U$

$U^*_Z(t) = (U^*)e^{-Zt}$

$\frac{dU^*_Z(t)}{dt} = -(U^*)Ze^{-Zt}$

Substitute these and eq. (4) into eq. (9)

$B'_Z(t) = Ze^{Zt}UB_0(U^*)e^{-Zt} - (e^{Zt})UB_0(U^*)Ze^{-Zt}$

$B'_Z(0) = ZUB_0U^* - UB_0(U^*)Z$

Using the definition of the C-matrix in section 2.2

$B'_Z(0) = Z(C - A_0) - (C - A_0)Z$

$C^{-1}B'_Z(0) = C^{-1}ZC - C^{-1}ZA_0 - Z + C^{-1}A_0Z$

$tr(C^{-1}B'_Z(0)) = tr(C^{-1}ZC) - tr(C^{-1}ZA_0) - tr(Z) + tr(C^{-1}A_0Z)$

The first and third terms cancel since similar matrices have the same trace.

$tr(C^{-1}B'_Z(0)) = -tr(C^{-1}ZA_0) + tr(C^{-1}A_0Z)$.

Using the idea that $tr(XY) = tr(YX)$

$tr(C^{-1}B'_Z(0)) = -tr(ZA_0C^{-1}) + tr(ZC^{-1}A_0)$

$tr(C^{-1}B'_Z(0)) = tr(Z(C^{-1}A_0 - A_0C^{-1}))$

$tr(C^{-1}B'_Z(0)) = tr(ZF)$

Substitute this into eq. (8) to get

$$R'_Z(0) = M(U)tr(ZF) \tag{10}$$

This proves lemma 3.1. □

5. Proof of lemma 3.2

Proof. Given an $n \times n$ zero-diagonal matrix W. Given that for every pair (a,b) where $1 \leq a, b \leq n$ and $a \neq b$,

$tr(Z^{ab}W) = 0.$

$tr(Z^{ab,i}W) = 0$

(See section 2.4 for definitions of Z^{ab} and $Z^{ab,i}$).

by direct computation we see that

$tr(Z^{ab}W) = W_{ab} - W_{ba} = 0$

$tr(Z^{ab,i}W) = (W_{ab} + W_{ba})i = 0$

Solving these, we get that $W_{ab} = 0$ and $W_{ba} = 0$. So all the off-diagonal elements of W are zero. Hence W is the zero-matrix. □

6. Proof of lemma 3.3

Proof. Given a boundary matrix U with $M(U) \neq 0$ and with F-matrix $F \neq 0$. Given $M(U)$ is an ordinary point of $\partial\Delta$. Let L be the tangent line to $\partial\Delta$ at $M(U)$. Let h be the direction vector of the line L. Note that h is just a non-zero complex number.

Let Z be a skew-hermitian matrix. By lemma 3.1 we know that $R'_Z(0) = M(U)tr(ZF)$.

Since $R_Z(t) \subseteq \Delta$ and $R_Z(0) = M(U)$, we know that $R'_Z(0) = ch$ for some real number c. (since L is the unique tangent to $\partial\Delta$ at $M(U)$, then it must the tangent to every curve that lies in Δ, goes through $M(U)$ and has a well-defined derivative at $M(U)$). We demonstrated this at the end of section 2.1.

So, $M(U)tr(ZF) = ch$

$tr(ZF) = c(\frac{h}{M(U)})$

We can write $v = \frac{h}{M(U)}$

Then

$tr(ZF) = cv.$

Note that v is fixed since it does not depend on the choice of Z.

□

7. Proof of theorem 3.4

Proof. Given a boundary matrix U with $M(U) \neq 0$ and with F-matrix $F \neq 0$. Given $M(U)$ is an ordinary point of $\partial\Delta$.

We pick an arbitrary pair (a,b) such that $1 \leq a, b \leq n$ and $a \neq b$

We have two skew-hermitian matrices Z^{ab} and $Z^{ab,i}$ defined as per section 2.4.

By direct computation we see that

$tr(Z^{ab}F) = F_{ab} - F_{ba}$

$tr(Z^{ab,i}F) = (F_{ab} + F_{ba})i$

Suppose $F_{ab} = F_{ab,r} + iF_{ab,i}$. (note that these are not tensors. $F_{ab,r}$ is just the real component of F_{ab} and $F_{ab,i}$ is just the imaginary component.) We can substitute this in to get

$$tr(Z^{ab}F) = (F_{ab,r} - F_{ba,r}) + i(F_{ab,i} - F_{ba,i}) \tag{11}$$

$$tr(Z^{ab,i}F) = (-F_{ab,i} - F_{ba,i}) + i(F_{ab,r} + F_{ba,r}) \tag{12}$$

We know by lemma 3.3 that these are collinear vectors in the complex plane.

So we know that

$(F_{ab,i} - F_{ba,i})(-F_{ab,i} - F_{ba,i}) = (F_{ab,r} + F_{ba,r})(F_{ab,r} - F_{ba,r})$

We can simplify this to get:

$F_{ab,r}^2 + F_{ab,i}^2 = F_{ba,r}^2 + F_{ba,i}^2$

$|F_{ab}| = |F_{ba}|$

We can write:

$F_{ab} = |F_{ab}|\angle\theta_{ab}$

$F_{ba} = |F_{ab}|\angle\theta_{ba}$

Note that we are given that $F \neq 0$. Note that we already know by section 2.2 that F is zero-diagonal.

We will divide the possible values of F into multiple cases. First we split all cases into two. The first is when only one pair of elements of the F-matrix, F_{ab} and F_{ba} is nonzero. The second case is when multiple pairs of elements of the F-matrix are nonzero. We shall further subdivide the second case using the fact that all tr(ZF) values are collinear. We can divide these cases into 3 possibilities: 1. All nonzero $tr(ZF)$ values are imaginary. 2. All nonzero $tr(ZF)$ values are real. 3. All nonzero $tr(ZF)$ values are not real or imaginary. (note that since F is nonzero, we don't have to deal with the possibility that tr(ZF) is 0 for all skew-hermitian matrices Z. see lemma 3.2).

So we have 4 cases to deal with.

Case 1: $|F_{ab}|$ is non-zero for only one pair $\{a,b\}$ where $a \neq b$

In this case,

$H = e^{-(\theta_{ab}+\theta_{ba})/2}F$ is a hermitian matrix, and we're finished.

Case 2: $|F_{ab}|$ is non-zero for multiple pairs $\{a,b\}$ where $a \neq b$. For any skew-hermitian Z, when tr(ZF) is non-zero, it is imaginary.

If $|F_{ab}| \neq 0$, then by eq. (11) and eq. (12), $\theta_{ab} = -\theta_{ba}$. This holds for all distinct pairs {a,b}, so our F-matrix is already hermitian, and we're done.

Case 3: $|F_{ab}|$ is non-zero for multiple pairs $\{a,b\}$ where $a \neq b$. For any skew-hermitian Z, when tr(ZF) is non-zero, it is real.

If $|F_{ab}| \neq 0$, then by eq. (11) and eq. (12), $\theta_{ab} = \pi - \theta_{ba}$. This holds for all distinct pairs {a,b}

$H = e^{-(\frac{\pi}{2})}F$ is hermitian and we're done.

Case 4: $|F_{ab}|$ is non-zero for multiple pairs $\{a,b\}$ where $a \neq b$. For any skew-hermitian matrix Z, when tr(ZF) is non-zero, it isn't real or imaginary.

Suppose $|F_{ab}| \neq 0$ and $|F_{cd}| \neq 0$

if $tr(Z^{ab}F) \neq 0$, then

slope of $tr(Z^{ab}F) = \frac{\sin(\theta_{ab})-\sin(\theta_{ba})}{\cos(\theta_{ab})-\cos(\theta_{ba})} = -\cot(\frac{\theta_{ab}+\theta_{ba}}{2})$

if $tr(Z^{ab,i}F) \neq 0$:

slope of $tr(Z^{ab,i}F) = \frac{\cos(\theta_{ab})+\cos(\theta_{ba})}{-\sin(\theta_{ab})-\sin(\theta_{ba})} = -\cot(\frac{\theta_{ab}+\theta_{ba}}{2})$

We know that since $|F_{ab}| \neq 0$, at least one of $tr(Z^{ab}F)$ or $tr(Z^{ab,i}F)$ is non-zero.

similarly,

if $tr(Z^{cd}F) \neq 0$, then

slope of $tr(Z^{cd}F) = -\cot(\frac{\theta_{cd}+\theta_{dc}}{2})$

if $tr(Z^{cd,i}F) \neq 0$:

slope of $tr(Z^{cd,i}F) = -\cot(\frac{\theta_{cd}+\theta_{dc}}{2})$

We know that since $|F_{cd}| \neq 0$, at least one of $tr(Z^{cd}F)$ or $tr(Z^{cd,i}F)$ is non-zero.

So we have:

$\cot(\frac{\theta_{cd}+\theta_{dc}}{2}) = \cot(\frac{\theta_{ab}+\theta_{ba}}{2})$ (lemma 3.3)

therefore:

$\frac{\theta_{cd}+\theta_{dc}}{2} = \frac{\theta_{ab}+\theta_{ba}}{2} + n\pi$ for some integer n.

We can freely adjust θ_{cd} by $-2n\pi$. It makes no difference since $|F_{cd}|\angle\theta_{cd} = |F_{cd}|\angle(\theta_{cd} - 2n\pi)$

So after the adjustment we have:

$\frac{\theta_{cd}+\theta_{dc}}{2} = \frac{\theta_{ab}+\theta_{ba}}{2}$.

We make the same adjustment for any pair $\{c, d\} \neq \{a, b\}$ where $|F_{cd}| \neq 0$

We set $\beta = \frac{\theta_{ab}+\theta_{ba}}{2}$

let $H = e^{-i\beta}F$

For some pair (x,y) where $x \neq y$ and $|H_{xy}| \neq 0$,

$H_{xy} = |H_{xy}|\angle\alpha_{xy}$

$\alpha_{xy} = -(\frac{\theta_{ab}+\theta_{ba}}{2}) + \theta_{xy}$

$\alpha_{yx} = -(\frac{\theta_{ab}+\theta_{ba}}{2}) + \theta_{yx}$

But because of our adjustments,

$\frac{\theta_{ab}+\theta_{ba}}{2} = \frac{\theta_{xy}+\theta_{yx}}{2}$

Plugging this into the above two formulas we have

$\alpha_{xy} = \frac{\theta_{xy}-\theta_{yx}}{2}$

$\alpha_{yx} = -(\frac{\theta_{xy}-\theta_{yx}}{2})$

Therefore H is zero-diagonal, with transpositional elements of equal magnitude and opposite arguments. Therefore H is hermitian.

So in all 4 cases we can write $F = e^{i\beta}H$ for some hermitian matrix H and some real β. But we've not arrived at a unique representation for F yet.

Suppose

$F = e^{i\beta_1}H_1 = e^{i\beta_2}H_2$

$e^{i(\beta_1-\beta_2)}H_1 = H_2$

$e^{i(\beta_1-\beta_2)}H_1 = H_2 = H_2^* = e^{i(\beta_2-\beta_1)}H_1^* = e^{i(\beta_2-\beta_1)}H_1$

So

$(e^{i(\beta_1-\beta_2)} - e^{i(\beta_2-\beta_1)})H_1 = 0$

Since $F \neq 0$, we know $H_1 \neq 0$ so

$e^{i(\beta_1-\beta_2)} - e^{i(\beta_2-\beta_1)} = 0$

$e^{i(\beta_1-\beta_2)} = e^{i(\beta_2-\beta_1)}$

Then

$\beta_1 - \beta_2 = \beta_2 - \beta_1 + 2k\pi$, for any integer k

$\beta_1 = \beta_2 + k\pi$

So if we restrict all β to $0 \leq \beta < \pi$, we have a unique representation since k is forced to 0.

This completes our proof of theorem 3.4. $\square$

8. Proof of theorem 3.5

Given a boundary matrix U with $M(U) \neq 0$ and F-matrix $F \neq 0$. Given $M(U)$ is an ordinary point of $\partial\Delta$. Let L be the tangent line to $\partial\Delta$ at $M(U)$.

Proof. By theorem 3.4 we know that

$$F = e^{i\theta} H \tag{13}$$

for some real $0 \leq \theta < \pi$ and some zero-diagonal hermitian matrix H.

We can substitute eq. (13) into eq. (11) and eq. (12) and simplify to get:

$$tr(Z^{ab}F) = 2H_{ab,i}e^{i(\theta+\pi/2)} \tag{14}$$

$$tr(Z^{ab,i}F) = 2H_{ab,r}e^{i(\theta+\pi/2)} \tag{15}$$

By lemma 3.2 we know that at least one of the above equations is nonzero for some pair (a,b). So then using lemma 3.1 we know that $R'_Z(0) = M(U)tr(ZF) \neq 0$ for some skew-hermitian matrix Z.

So by eq. (14) and eq. (15) we see that for some skew-hermitian matrix Z, $tr(ZF)$ forms an angle of $(\theta + \pi/2)$ or $(\theta + 3\pi/2)$ with the positive real axis (depending on whether the coefficient is negative or not). Therefore $R'_Z(0)$ forms an angle $arg(M(U)) + \theta + \pi/2$ or $arg(M(U)) + \theta + 3\pi/2$ with the positive real axis.

Therefore the line L forms an angle $arg(M(U)) + \theta + \pi/2$ with the positive real axis (since this is a line as opposed to a vector, a rotation of π makes no difference).

This completes our proof of theorem 3.5. $\square$

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