

Boundary Matrices and the Marcus-de Oliveira Determinantal Conjecture

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ABSTRACT

We present notes on the Marcus-de Oliveira conjecture. The conjecture concerns the region in the complex plane covered by the determinants of the sums of two normal matrices with prescribed eigenvalues. Call this region Δ . This paper focuses on boundary matrices of Δ . We prove 2 theorems regarding these boundary matrices. This paper uses ideas from [1].

KEYWORDS

determinantal conjecture, Marcus-de Oliveira, determinants, normal matrices, convex-hull

1. Introduction

Marcus [2] and de Oliveira [3] made the following conjecture. Given two normal matrices A and B with prescribed eigenvalues $a_1, a_2 \dots a_n$ and $b_1, b_2 \dots b_n$ respectively, $\det(A + B)$ lies within the region:

$$co\left\{ \prod (a_i + b_{\sigma(i)}) \right\}$$

where $\sigma \in S_n$. co denotes the convex hull of the $n!$ points in the complex plane. As described in [1], the problem can be restated as follows. Given two diagonal matrices, $A_0 = \text{diag}(a_1, a_2 \dots a_n)$ and $B_0 = \text{diag}(b_1, b_2 \dots b_n)$, let:

$$\Delta = \{ \det(A_0 + UB_0U^*) : U \in U(n) \} \quad (1)$$

where $U(n)$ is the set of $n \times n$ unitary matrices. Then we can write the conjecture as:

Conjecture 1.1 (Marcus-de Oliveira Conjecture).

$$\Delta \subseteq \text{co}\left\{\prod (a_i + b_{\sigma(i)})\right\} \quad (2)$$

Let

$$M(U) = \det(A_0 + UB_0U^*). \quad (3)$$

The paper is organized as follows. In section 2 we define terms and functions that will be used in the rest of the paper. These definitions are necessary to state our results. In section 3, we state 3 lemmas and 2 theorems that form the bulk of the paper. We state them in the order they are proved.

2. Preparatory definitions

2.1. Terms

Given a unitary matrix U and square, diagonal matrices A_0 and B_0 all of dimension $n \times n$,

- If $M(U)$ is a point on $\partial\Delta$ (the boundary of Δ), we call $M(U)$ a boundary point of Δ and we call U a **boundary matrix** of Δ . See eq. (1) and eq. (3).
- We define the **B-matrix** of U as UB_0U^* .
- We define the **C-matrix** of U as $A_0 + UB_0U^*$.
- We define the **F-matrix** of U as $C^{-1}A_0 - A_0C^{-1}$ where C is the C-matrix of U . Note that the F-matrix is only defined when C is invertible, or equivalently when $\det(C) = M(U) \neq 0$. See eq. (3). Also note that since A_0 is diagonal, the F-matrix is a zero-diagonal matrix. The idea for using the F-matrix comes from [1], Theorem 4, p.27.

Throughout the rest of the paper, we'll assume A_0 and B_0 are defined, even if we don't explicitly mention them.

2.2. Functions given a unitary matrix U

Given a unitary matrix U with B-matrix B , C-matrix C and F-matrix F . For every skew-hermitian matrix Z , we define the following functions

let

$$U_Z(t) = (e^{Zt})U \quad (4)$$

where t is any real number.

Since the exponential of a skew-hermitian matrix is unitary, $U_Z(t)$ is a function of unitary matrices.

let

$$B_Z(t) = U_Z(t)B_0U_Z^*(t) \tag{5}$$

let $C_Z(t) = A_0 + B_Z(t)$

We note that $B_Z(0) = B$ and $C_Z(0) = C$.

let

$$R_Z(t) = \det(C_Z(t)) \tag{6}$$

We can see by eq. (1) that $R_Z(t) \subseteq \Delta$.

$$R_Z(0) = A_0 + UB_0U^*$$

So by eq. (3) we see that $R_Z(0) = M(U)$.

So all the $R_Z(t)$ functions go through $M(U)$ at $t = 0$.

We shall refer to these functions in the rest of the paper with the same notation (for example $R_Z(t)$ for a skew-hermitian matrix Z . $R_{Z_1}(t)$ for a skew-hermitian matrix Z_1). Note that $R_Z(t)$ requires A_0, B_0, U and Z in order to be defined. But we won't explicitly mention A_0 and B_0 . All the results in this paper assume there are two diagonal matrices A_0 and B_0 defined in the background.

2.3. Skew-Hermitian matrices Z^{ab} and $Z^{ab,i}$

Given two integers a, b where $1 \leq a, b \leq n$ and $a \neq b$.

We define the $n \times n$ skew-hermitian matrix Z^{ab} as follows. $Z_{ab}^{ab} = -1$ (the element at the a th row and b th column is -1.) $Z_{ba}^{ab} = 1$ (the element at the b th row and a th column is 1.) And all other elements are 0. Note that $Z^{ab} = -Z^{ba}$.

We define the $n \times n$ skew-hermitian matrix $Z^{ab,i}$ as follows. $Z_{ab}^{ab,i} = i$ and $Z_{ba}^{ab,i} = i$. All other elements are zero. Note that $Z^{ab,i} = Z^{ba,i}$.

It is straightforward to verify that Z^{ab} and $Z^{ab,i}$ are skew-hermitian.

3. Main Results

Lemma 3.1. *Given a unitary matrix U with $M(U) \neq 0$. Let F be its F -matrix. Then $R'_Z(0) = M(U)\text{tr}(ZF)$ for any skew-hermitian matrix Z .*

Lemma 3.2. *Given an $n \times n$ zero-diagonal matrix W . Given $\text{tr}(Z^{ab}W) = 0$ and $\text{tr}(Z^{ab,i}W) = 0$ for all pairs (a,b) where $1 \leq a, b \leq n$ and $a \neq b$. Then W is the zero-matrix.*

Lemma 3.3. *Given a boundary matrix U with $M(U) \neq 0$ and with F -matrix $F \neq 0$. Given there's a unique tangent line L to Δ at $M(U)$ with direction vector v . Then for every skew-hermitian matrix Z , $\text{tr}(ZF) = cv$ where c is some real number.*

Theorem 3.4. *Given a boundary matrix U with $M(U) \neq 0$ and with F -matrix $F \neq 0$. Given there's a unique tangent line to Δ at $M(U)$. Then F can be written uniquely in the form $F = e^{i\theta}H$ where H is a zero-diagonal hermitian matrix and $0 \leq \theta < \pi$.*

Theorem 3.5. *Given a boundary matrix U with $M(U) \neq 0$ and with F -matrix $F \neq 0$. Given there's a unique tangent line L to Δ at $M(U)$. By the previous theorem we know that $F = e^{i\theta}H$ for some real $0 \leq \theta < \pi$. Then L makes an angle $\arg(M(U)) + \theta + \pi/2$ with the positive real axis.*

4. Proof of lemma 3.1

The proof given here uses ideas from [1], Theorem 4, p.26-27. But the proof given here is complete on its own.

Proof. We're given a unitary matrix U where $M(U) \neq 0$. So its F -matrix is well-defined and we call it F . Let B be its B -matrix, and C be its C -matrix. Given an arbitrary skew-hermitian matrix Z .

We can use Jacobi's formula [4] on eq. (6) to find $R'_Z(t)$

$$R'_Z(t) = \text{tr}(\det(C_Z(t))C_Z^{-1}(t)C'_Z(t)) \quad (7)$$

$$R'_Z(0) = \text{tr}(\det(C_Z(0))C_Z^{-1}(0)C'_Z(0))$$

We can substitute C for $C_Z(0)$.

$$R'_Z(0) = \text{tr}(\det(C)C^{-1}C'_Z(0))$$

$$R'_Z(0) = \det(C)\text{tr}(C^{-1}C'_Z(0))$$

We know that $C'_Z(t) = B'_Z(t)$ so

$$R'_Z(0) = \det(C)\text{tr}(C^{-1}B'_Z(0))$$

By section 2.1 and eq. (3) we know that $\det(C) = M(U)$

$$R'_Z(0) = M(U)\text{tr}(C^{-1}B'_Z(0)) \quad (8)$$

Using eq. (5),

$$B'_Z(t) = \frac{dU_Z(t)}{dt} B_0 U_Z^*(t) + U_Z(t) B_0 \frac{dU_Z^*(t)}{dt} \quad (9)$$

Using eq. (4),

$$\frac{dU_Z(t)}{dt} = Z e^{Zt} U$$

$$U_Z^*(t) = (U^*) e^{-Zt}$$

$$\frac{dU_Z^*(t)}{dt} = -(U^*) Z e^{-Zt}$$

Substitute these and eq. (4) into eq. (9)

$$B'_Z(t) = Z e^{Zt} U B_0 (U^*) e^{-Zt} - (e^{Zt} U B_0 (U^*) Z e^{-Zt}$$

$$B'_Z(0) = Z U B_0 U^* - U B_0 (U^*) Z$$

Using the definition of the C-matrix in section 2.1

$$B'_Z(0) = Z(C - A_0) - (C - A_0)Z$$

$$C^{-1} B'_Z(0) = C^{-1} Z C - C^{-1} Z A_0 - Z + C^{-1} A_0 Z$$

$$\text{tr}(C^{-1} B'_Z(0)) = \text{tr}(C^{-1} Z C) - \text{tr}(C^{-1} Z A_0) - \text{tr}(Z) + \text{tr}(C^{-1} A_0 Z)$$

The first and third terms cancel since similar matrices have the same trace.

$$\text{tr}(C^{-1} B'_Z(0)) = -\text{tr}(C^{-1} Z A_0) + \text{tr}(C^{-1} A_0 Z).$$

Using the idea that $\text{tr}(XY) = \text{tr}(YX)$

$$\text{tr}(C^{-1} B'_Z(0)) = -\text{tr}(Z A_0 C^{-1}) + \text{tr}(Z C^{-1} A_0)$$

$$\text{tr}(C^{-1} B'_Z(0)) = \text{tr}(Z(C^{-1} A_0 - A_0 C^{-1}))$$

$$\text{tr}(C^{-1} B'_Z(0)) = \text{tr}(ZF)$$

Substitute this into eq. (8) to get

$$R'_Z(0) = M(U) \text{tr}(ZF) \quad (10)$$

This proves lemma 3.1. □

5. Proof of lemma 3.2

Proof. Given an $n \times n$ zero-diagonal matrix W . Given that for every pair (a,b) where $1 \leq a, b \leq n$ and $a \neq b$,

$$\text{tr}(Z^{ab}W) = 0.$$

$$\text{tr}(Z^{ab,i}W) = 0$$

(See section 2.3 for definitions of Z^{ab} and $Z^{ab,i}$).

by direct computation we see that

$$\text{tr}(Z^{ab}W) = W_{ab} - W_{ba} = 0$$

$$\text{tr}(Z^{ab,i}W) = (W_{ab} + W_{ba})i = 0$$

Solving these, we get that $W_{ab} = 0$ and $W_{ba} = 0$. So all the off-diagonal elements of W are zero. Hence W is the zero-matrix. \square

6. Proof of lemma 3.3

Proof. Given a boundary matrix U with $M(U) \neq 0$ and with F-matrix $F \neq 0$. Given there's a unique tangent line L to Δ at $M(U)$. Let v be the direction vector of the line L . Note that v is just a non-zero complex number.

Let Z be a skew-hermitian matrix. By lemma 3.1 we know that $R'_Z(0) = M(U)\text{tr}(ZF)$.

Since $R_Z(t) \subseteq \Delta$ and $R_Z(0) = M(U)$, we know that $R'_Z(0) = kv$ for some real number k . (if L is the unique tangent to the region Δ at $M(U)$, then it must be the tangent to every curve that lies in Δ and goes through $M(U)$ and has a well-defined derivative at $M(U)$).

$$\text{So, } M(U)\text{tr}(ZF) = kv$$

$$\text{tr}(ZF) = \left(\frac{k}{M(U)}\right)v \quad \square$$

7. Proof of theorem 3.4

Proof. Given a boundary matrix U with $M(U) \neq 0$ and with F-matrix $F \neq 0$. Given there's a unique tangent line to Δ at $M(U)$.

We pick an arbitrary pair (a,b) such that $1 \leq a, b \leq n$ and $a \neq b$

We have two skew-hermitian matrices Z^{ab} and $Z^{ab,i}$ defined as per section 2.3.

By direct computation we see that

$$\text{tr}(Z^{ab}F) = F_{ab} - F_{ba}$$

$$\text{tr}(Z^{ab,i}F) = (F_{ab} + F_{ba})i$$

Suppose $F_{ab} = F_{ab,r} + iF_{ab,i}$. (note that these are not tensors. $F_{ab,r}$ is just the real component of F_{ab} and $F_{ab,i}$ is just the imaginary component.) We can substitute this in to get

$$\text{tr}(Z^{ab}F) = (F_{ab,r} - F_{ba,r}) + i(F_{ab,i} - F_{ba,i}) \quad (11)$$

$$\text{tr}(Z^{ab,i}F) = (-F_{ab,i} - F_{ba,i}) + i(F_{ab,r} + F_{ba,r}) \quad (12)$$

We know by lemma 3.3 that these are collinear vectors in the complex plane.

So we know that

$$(F_{ab,i} - F_{ba,i})(-F_{ab,i} - F_{ba,i}) = (F_{ab,r} + F_{ba,r})(F_{ab,r} - F_{ba,r})$$

We can simplify this to get:

$$F_{ab,r}^2 + F_{ab,i}^2 = F_{ba,r}^2 + F_{ba,i}^2$$

$$|F_{ab}| = |F_{ba}|$$

We can write:

$$F_{ab} = |F_{ab}| \angle \theta_{ab}$$

$$F_{ba} = |F_{ab}| \angle \theta_{ba}$$

Note that we are given that $F \neq 0$. Note that we already know by section 2.1 that F is zero-diagonal.

We will divide the possible values of F into multiple cases. First we split all cases into two. The first is when only one pair of elements of the F -matrix, F_{ab} and F_{ba} is nonzero. The second case is when multiple pairs of elements of the F -matrix are nonzero. We shall further subdivide the second case using the fact that all $\text{tr}(ZF)$ values are collinear. We can divide these cases into 3 possibilities: 1. All nonzero $\text{tr}(ZF)$ values are imaginary. 2. All nonzero $\text{tr}(ZF)$ values are real. 3. All nonzero $\text{tr}(ZF)$ values are not real or imaginary. (note that since F is nonzero, we don't have to deal with the possibility that $\text{tr}(ZF)$ is 0 for all skew-hermitian matrices Z . see lemma 3.2).

So we have 4 cases to deal with.

Case 1: $|F_{ab}|$ is non-zero for only one pair $\{a, b\}$ where $a \neq b$

In this case,

$H = e^{-(\theta_{ab} + \theta_{ba})/2} F$ is a hermitian matrix, and we're finished.

Case 2: $|F_{ab}|$ is non-zero for multiple pairs $\{a, b\}$ where $a \neq b$. For any skew-hermitian Z , when $\text{tr}(ZF)$ is non-zero, it is imaginary.

If $|F_{ab}| \neq 0$, then by eq. (11) and eq. (12), $\theta_{ab} = -\theta_{ba}$. This holds for all distinct pairs $\{a, b\}$, so our F -matrix is already hermitian, and we're done.

Case 3: $|F_{ab}|$ is non-zero for multiple pairs $\{a, b\}$ where $a \neq b$. For any skew-hermitian Z , when $\text{tr}(ZF)$ is non-zero, it is real.

If $|F_{ab}| \neq 0$, then by eq. (11) and eq. (12), $\theta_{ab} = \pi - \theta_{ba}$. This holds for all distinct

pairs $\{a,b\}$

$H = e^{-i\frac{\pi}{2}}F$ is hermitian and we're done.

Case 4: $|F_{ab}|$ is non-zero for multiple pairs $\{a,b\}$ where $a \neq b$. For any skew-hermitian matrix Z , when $\text{tr}(ZF)$ is non-zero, it isn't real or imaginary.

Suppose $|F_{ab}| \neq 0$ and $|F_{cd}| \neq 0$

if $\text{tr}(Z^{ab}F) \neq 0$, then

$$\text{slope of } \text{tr}(Z^{ab}F) = \frac{\sin(\theta_{ab}) - \sin(\theta_{ba})}{\cos(\theta_{ab}) - \cos(\theta_{ba})} = -\cot\left(\frac{\theta_{ab} + \theta_{ba}}{2}\right)$$

if $\text{tr}(Z^{ab,i}F) \neq 0$:

$$\text{slope of } \text{tr}(Z^{ab,i}F) = \frac{\cos(\theta_{ab}) + \cos(\theta_{ba})}{-\sin(\theta_{ab}) - \sin(\theta_{ba})} = -\cot\left(\frac{\theta_{ab} + \theta_{ba}}{2}\right)$$

We know that since $|F_{ab}| \neq 0$, at least one of $\text{tr}(Z^{ab}F)$ or $\text{tr}(Z^{ab,i}F)$ is non-zero. similarly,

if $\text{tr}(Z^{cd}F) \neq 0$, then

$$\text{slope of } \text{tr}(Z^{cd}F) = -\cot\left(\frac{\theta_{cd} + \theta_{dc}}{2}\right)$$

if $\text{tr}(Z^{cd,i}F) \neq 0$:

$$\text{slope of } \text{tr}(Z^{cd,i}F) = -\cot\left(\frac{\theta_{cd} + \theta_{dc}}{2}\right)$$

We know that since $|F_{cd}| \neq 0$, at least one of $\text{tr}(Z^{cd}F)$ or $\text{tr}(Z^{cd,i}F)$ is non-zero.

So we have:

$$\cot\left(\frac{\theta_{cd} + \theta_{dc}}{2}\right) = \cot\left(\frac{\theta_{ab} + \theta_{ba}}{2}\right) \text{ (lemma 3.3)}$$

therefore:

$$\frac{\theta_{cd} + \theta_{dc}}{2} = \frac{\theta_{ab} + \theta_{ba}}{2} + n\pi \text{ for some integer } n.$$

We can freely adjust θ_{cd} by $-2n\pi$. It makes no difference since $|F_{cd}| \angle \theta_{cd} = |F_{cd}| \angle (\theta_{cd} - 2n\pi)$

So after the adjustment we have:

$$\frac{\theta_{cd} + \theta_{dc}}{2} = \frac{\theta_{ab} + \theta_{ba}}{2}.$$

We make the same adjustment for any pair $\{c,d\} \neq \{a,b\}$ where $|F_{cd}| \neq 0$

We set $\beta = \frac{\theta_{ab} + \theta_{ba}}{2}$

let $H = e^{-i\beta}F$

For some pair (x,y) where $x \neq y$ and $|H_{xy}| \neq 0$,

$$H_{xy} = |H_{xy}| \angle \alpha_{xy}$$

$$\alpha_{xy} = -\left(\frac{\theta_{ab} + \theta_{ba}}{2}\right) + \theta_{xy}$$

$$\alpha_{yx} = -\left(\frac{\theta_{ab} + \theta_{ba}}{2}\right) + \theta_{yx}$$

But because of our adjustments,

$$\frac{\theta_{ab} + \theta_{ba}}{2} = \frac{\theta_{xy} + \theta_{yx}}{2}$$

Plugging this into the above two formulas we have

$$\alpha_{xy} = \frac{\theta_{xy} - \theta_{yx}}{2}$$

$$\alpha_{yx} = -\left(\frac{\theta_{xy} - \theta_{yx}}{2}\right)$$

Therefore H is zero-diagonal, with transpositional elements of equal magnitude and opposite arguments. Therefore H is hermitian.

So in all 4 cases we can write $F = e^{i\beta} H$ for some hermitian matrix H and some real β . But we've not arrived at a unique representation for F yet.

Suppose

$$F = e^{i\beta_1} H_1 = e^{i\beta_2} H_2$$

$$e^{i(\beta_1 - \beta_2)} H_1 = H_2$$

$$e^{i(\beta_1 - \beta_2)} H_1 = H_2 = H_2^* = e^{i(\beta_2 - \beta_1)} H_1^* = e^{i(\beta_2 - \beta_1)} H_1$$

So

$$(e^{i(\beta_1 - \beta_2)} - e^{i(\beta_2 - \beta_1)}) H_1 = 0$$

Since $F \neq 0$, we know $H_1 \neq 0$ so

$$e^{i(\beta_1 - \beta_2)} - e^{i(\beta_2 - \beta_1)} = 0$$

$$e^{i(\beta_1 - \beta_2)} = e^{i(\beta_2 - \beta_1)}$$

Then

$$\beta_1 - \beta_2 = \beta_2 - \beta_1 + 2k\pi, \text{ for any integer } k$$

$$\beta_1 = \beta_2 + k\pi$$

So if we restrict all β to $0 \leq \beta < \pi$, we have a unique representation since k is forced to 0.

This completes our proof of theorem 3.4. □

8. Proof of theorem 3.5

Given a boundary matrix U with $M(U) \neq 0$ and F-matrix $F \neq 0$. Given $\partial\Delta$ has the unique tangent line L at $M(U)$.

Proof. By theorem 3.4 we know that

$$F = e^{i\theta} H \tag{13}$$

for some real $0 \leq \theta < \pi$ and some zero-diagonal hermitian matrix H .

We can substitute eq. (13) into eq. (11) and eq. (12) and simplify to get:

$$\text{tr}(Z^{ab}F) = 2H_{ab,i}e^{i(\theta+\pi/2)} \quad (14)$$

$$\text{tr}(Z^{ab,i}F) = 2H_{ab,r}e^{i(\theta+\pi/2)} \quad (15)$$

By lemma 3.2 we know that at least one of the above equations is nonzero for some pair (a,b). So then using lemma 3.1 we know that $R'_Z(0) = M(U)\text{tr}(ZF) \neq 0$ for some skew-hermitian matrix Z .

So by eq. (14) and eq. (15) we see that for some skew-hermitian matrix Z , $\text{tr}(ZF)$ forms an angle of $(\theta + \pi/2)$ or $(\theta + 3\pi/2)$ with the positive real axis (depending on whether the coefficient is negative or not). Therefore $R'_Z(0)$ forms an angle $\text{arg}(M(U)) + \theta + \pi/2$ or $\text{arg}(M(U)) + \theta + 3\pi/2$ with the positive real axis.

Therefore the line L forms an angle $\text{arg}(M(U)) + \theta + \pi/2$ with the positive real axis (since this is a line as opposed to a vector, a rotation of π makes no difference).

This completes our proof of theorem 3.5. □

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