BOUNDARY MATRICES AND THE MARCUS-DE OLIVEIRA DETERMINANTAL CONJECTURE*

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Abstract. We present notes on the Marcus-de Oliveira conjecture. The conjecture concerns the region in the complex plane covered by the determinants of the sums of two normal matrices with prescribed eigenvalues. Call this region Δ. This paper focuses on boundary matrices of Δ. We prove 3 theorems regarding these boundary matrices. We propose 2 conjectures related to the Marcus-de Oliveira conjecture and prove a theorem related to these 2 conjectures. This paper uses ideas from [1].

Key words. determinantal conjecture, Marcus-de Oliveira, determinants, normal matrices, convex-hull

AMS subject classifications. 15A15, 15A16

1. Introduction.

Marcus [4] and de Oliveira [2] made the following conjecture. Given two normal matrices A and B with prescribed eigenvalues $a_1, a_2 ... a_n$ and $b_1, b_2 ... b_n$ respectively, $det(A + B)$ lies within the region:

$$co\left\{ \prod (a_i + b_{\sigma(i)}) \right\}$$

where $\sigma \in S_n$. co denotes the convex hull of the $n!$ points in the complex plane. As described in [1], the problem can be restated as follows. Given two diagonal matrices, $A_0 = diag(a_1, a_2 ... a_n)$ and $B_0 = diag(b_1, b_2 ... b_n)$, let:

$$\Delta = \left\{ det(A_0 + UB_0U^*) : U \in U(n) \right\} \tag{1.1}$$

where $U(n)$ is the set of $n \times n$ unitary matrices. Then we can write the conjecture as:

Conjecture 1.1 (Marcus-de Oliveira Conjecture).

$$\Delta \subseteq co\left\{ \prod (a_i + b_{\sigma(i)}) \right\} \tag{1.2}$$

Let

$$M(U) = det(A_0 + UB_0U^*). \tag{1.3}$$

Then the points forming the convex hull are at $M(P_0), M(P_1) ... M(P_{n!-1})$, where the P's are the $n \times n$ permutation matrices. We will refer to these as **permutation points** from now on.

Note that $U(n)$ is a compact set. A continuous image of a compact set is compact. Therefore Δ is compact. And so Δ is a closed set, because a compact subset of any metric space (in this case the complex numbers) is closed.

*Submitted to the editors June 6th, 2018.
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The paper is organized as follows. In section 2 we define terms and functions that will be used in the rest of the paper. These definitions are necessary to state our results. In section 3, we state 3 theorems, 1 lemmas and 1 corollary that form the bulk of the paper. We state them in the order they are proved. The lemma is an intermediate tools for proving the 3 theorems. The corollary is an interesting consequence of the third theorem. In sections 5-10 we prove all of these. In section 9, we state 2 conjectures and prove a theorem related to these conjectures. In section 10, we conclude.

2. Preparatory definitions.

2.1. Terms.

Given a unitary matrix U and square, diagonal matrices A_0 and B_0 all of dimension $n \times n$,

- If $M(U)$ is a point on $\partial\Delta$ (the boundary of Δ), we call U a **boundary matrix** of Δ. See (1.1) and (1.3).
- We define the **B-matrix** of U as UB_0U^*.
- We define the **C-matrix** of U as $A_0 + UB_0U^*$.
- We define the **F-matrix** of U as $C^{-1}A_0 - A_0C^{-1}$ where C is the C-matrix of U. Note that the F-matrix is only defined when C is invertible, or equivalently when $det(C) = M(U) \neq 0$. See (1.3). Also note that since A_0 is diagonal, the F-matrix is a zero-diagonal matrix. The idea for using the F-matrix comes from [1], Theorem 4, p.27.

Throughout the rest of the paper, we'll assume A_0 and B_0 are defined, even if we don't explicitly mention them.

2.2. Multidirectional Unitary Matrix.

Given a unitary matrix U with $M(U) \neq 0$ and F-matrix F. If there exist two skew-hermitian matrices Z_1 and Z_2 such that $tr(Z_1F)$ and $tr(Z_2F)$ are both non-zero, non-collinear vectors in the complex-plane, we say U is **multidirectional**.

2.3. Functions given a unitary matrix U.

Given a unitary matrix U with B-matrix B, C-matrix C and F-matrix F. Given $M(U) \neq 0$. For every skew-hermitian matrix Z, we define the following functions

let

$$U_Z(t) = (e^{Zt})U \tag{2.1}$$

where t is any real number.

Since the exponential of a skew-hermitian matrix is unitary, $U_Z(t)$ is a function of unitary matrices.

let

$$B_Z(t) = U_Z(t)B_0U_Z^*(t) \tag{2.2}$$

let $C_Z(t) = A_0 + B_Z(t)$

We note that $B_Z(0) = B$ and $C_Z(0) = C$.

let

$$R_Z(t) = det(C_Z(t)) \tag{2.3}$$

We can see by (1.1) that $R_Z(t) \subseteq \Delta$.

$R_Z(0) = A_0 + UB_0U^*$

So by (1.3) we see that $R_Z(0) = M(U)$.

So all the $R_Z(t)$ functions go through $M(U)$ at $t = 0$.

We shall refer to these functions in the rest of the paper with the same notation (for example $R_Z(t)$ for a skew-hermitian matrix Z. $R_{Z_1}(t)$ for a skew-hermitian matrix Z_1). Note that $R_Z(t)$ requires A_0, B_0, U and Z in order to be defined. But we won't explicitly mention A_0 and B_0. All the results in this paper assume there are two diagonal matrices A_0 and B_0 defined in the background even if we don't explicitly mention them.

3. Main Results.

Lemma 3.1. *Given a unitary matrix U with $M(U) \neq 0$. Let F be its F-matrix. Then $R'_Z(0) = M(U)tr(ZF)$ for any skew-hermitian matrix Z.*

Theorem 3.2. *Given a unitary matrix U with $M(U) \neq 0$. If U is a boundary matrix then U is not multidirectional.*

Theorem 3.3. *Given a boundary matrix U with $M(U) \neq 0$ with F-matrix F. If $F \neq 0$, F can be written uniquely in the form $F = e^{i\theta}H$ where H is a zero-diagonal hermitian matrix and $0 \leq \theta < \pi$.*

Theorem 3.4. *Given a boundary matrix U with $M(U) \neq 0$ and F-matrix $F \neq 0$. So we know that $F = e^{i\theta}H$ for some real $0 \leq \theta < \pi$. Given $\partial\Delta$ has a tangent line L at $M(U)$.Then L makes an angle $arg(M(U)) + \theta + \pi/2$ with the positive real axis.*

Corollary 3.5. *Given a point P in the complex plane such that $P \neq 0$, $P \in \partial\Delta$ and $\partial\Delta$ has a tangent at P. Given two different unitary matrices U_1 and U_2, with corresponding non-zero F-matrices F_1 and F_2, such that $M(U_1) = M(U_2) = P$. Then $F_1 = e^{i\theta}H_1$ and $F_2 = e^{i\theta}H_2$ for some unique $0 \leq \theta < \pi$.*

4. Proof of Lemma 3.1.

The proof given here uses ideas from [1], Theorem 4, p.26-27. But the proof given here is complete on its own.

Proof. We're given a unitary matrix U where $M(U) \neq 0$. So its F-matrix is well-defined and we call it F. Let B be its B-matrix, and C be its C-matrix. Given an arbitrary skew-hermitian matrix Z.

We can use Jacobi's formula [5] on (2.3) to find $R'_Z(t)$

$$R'_Z(t) = tr(det(C_Z(t))C_Z^{-1}(t)C'_Z(t)) \tag{4.1}$$

$R'_Z(0) = tr(det(C_Z(0))C_Z^{-1}(0)C'_Z(0))$

We can substitute C for $C_Z(0)$.

$R'_Z(0) = tr(det(C)C^{-1}C'_Z(0))$

$R'_Z(0) = det(C)tr(C^{-1}C'_Z(0))$

We know that $C'_Z(t) = B'_Z(t)$ so

$R'_Z(0) = det(C)tr(C^{-1}B'_Z(0))$

By subsection 2.1 and (1.3) we know that $det(C) = M(U)$

$$R'_Z(0) = M(U)tr(C^{-1}B'_Z(0)) \tag{4.2}$$

Using (2.2),

$$B'_Z(t) = \frac{dU_Z(t)}{dt}B_0U^*_Z(t) + U_Z(t)B_0\frac{dU^*_Z(t)}{dt} \tag{4.3}$$

Using (2.1),

$\frac{dU_Z(t)}{dt} = Ze^{Zt}U$

$U^*_Z(t) = (U^*)e^{-Zt}$

$\frac{dU^*_Z(t)}{dt} = -(U^*)Ze^{-Zt}$

Substitute these and (2.1) into (4.3)

$B'_Z(t) = Ze^{Zt}UB_0(U^*)e^{-Zt} - (e^{Zt})UB_0(U^*)Ze^{-Zt}$

$B'_Z(0) = ZUB_0U^* - UB_0(U^*)Z$

Using the definition of the C-matrix in subsection 2.1

$B'_Z(0) = Z(C - A_0) - (C - A_0)Z$

$B'_Z(0) = ZC - ZA_0 - CZ + A_0Z$

$C^{-1}B'_Z(0) = C^{-1}ZC - C^{-1}ZA_0 - Z + C^{-1}A_0Z$

$tr(C^{-1}B'_Z(0)) = tr(C^{-1}ZC) - tr(C^{-1}ZA_0) - tr(Z) + tr(C^{-1}A_0Z)$

The first and third terms cancel since similar matrices have the same trace.

$tr(C^{-1}B'_Z(0)) = -tr(C^{-1}ZA_0) + tr(C^{-1}A_0Z)$.

Using the idea that $tr(XY) = tr(YX)$

$tr(C^{-1}B'_Z(0)) = -tr(ZA_0C^{-1}) + tr(ZC^{-1}A_0)$

$tr(C^{-1}B'_Z(0)) = tr(ZC^{-1}A_0) - tr(ZA_0C^{-1})$

$tr(C^{-1}B'_Z(0)) = tr(Z(C^{-1}A_0 - A_0C^{-1}))$

$tr(C^{-1}B'_Z(0)) = tr(ZF)$

Substitute this into (4.2) to get

$$R'_Z(0) = M(U)tr(ZF) \tag{4.4}$$

This proves Lemma 3.1. ∎

5. Proof of Theorem 3.2.

We will prove the contrapositive. ie: We'll start with a multidirectional matrix U, and prove that it is not a boundary matrix.

Proof. Given a unitary matrix U with $M(U) \neq 0$. Let F be its F-matrix and C be its C-matrix.

Assume U is multidirectional. See subsection 2.2.

Then there exist two skew-hermitian matrices Z_1 and Z_2 such that

$$T_1 = tr(Z_1F) \tag{5.1}$$

$$T_2 = tr(Z_2F) \tag{5.2}$$

are both non-zero and non-collinear.

We know by Lemma 3.1 that:

$R'_{Z_1}(0) = M(U)tr(Z_1F)$

$R'_{Z_2}(0) = M(U)tr(Z_2F)$

substitute in (5.1) and (5.2),

$R'_1(0) = M(U)T_1$

$R'_2(0) = M(U)T_2$

Since we know T_1 and T_2 are non-collinear, $R'_{Z_1}(0)$ and $R'_{Z_2}(0)$ are non-collinear. They are also non-zero since $T_1, T_2, M(U) \neq 0$. Therefore they form a linear basis for all the complex numbers over the real numbers. Let Q be an arbitrary non-zero complex number.

$Q = aR'_{Z_1}(0) + bR'_{Z_2}(0)$ where a and b are real.

$Q = a(M(U))T_1 + b(M(U))T_2$

$Q = M(U)(aT_1 + bT_2)$

substitute in (5.1) and (5.2),

$Q = M(U)(tr(aZ_1F) + tr(bZ_2F))$

$Q = M(U)tr((aZ_1 + bZ_2)F)$

let $Z_3 = aZ_1 + bZ_2$

$Q = M(U)tr(Z_3F)$

Note that Z_3 is also a skew-hermitian matrix.

Again by Lemma 3.1, we know that,

$R'_{Z_3}(0) = M(U)tr(Z_3F) = Q$

$R'_{Z_3}(0) \neq 0$ since we chose Q to be non-zero.

Therefore since $R'_{Z_3}(0) \neq 0$, $R_{Z_3}(t)$ goes through $M(U)$ in a direction parallel to Q. But Q was chosen arbitrarily. So through $M(U)$ there exist curves $R_{Z_3}(t) \subseteq \Delta$ going in all possible directions. Therefore $M(U)$ is an internal point of Δ. So it's not a boundary point. Therefore U is not a boundary matrix. That gives us Theorem 3.2.□

6. Proof of Theorem 3.3. For n = 3, we define the following 12 skew-hermitian matrices with zero diagonal:

$$Z_{12} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad Z_{13} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad Z_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Z_{21} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad Z_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad Z_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$Z_{12,i} = Z_{21,i} = \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad Z_{13,i} = Z_{31,i} = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \qquad Z_{23,i} = Z_{32,i} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$$

Note that the commas do not indicate tensors. They're just used here as a label to distinguish imaginary and real matrices.

We define Z_{ab} and $Z_{ab,i}$ similarly for all $n > 3$, where $a \neq b$. For a given n we have $n(n-1)$ real matrices and $n(n-1)$ imaginary matrices.

Proof. Given a boundary matrix U with $M(U) \neq 0$. Let F be its F-matrix. We know that F is zero-diagonal by subsection 2.1.

Suppose $F_{ab} = F_{ab,r} + iF_{ab,i}$ where $F_{ab,r}$ and $F_{ab,i}$ are real numbers.

$tr(Z_{ab}F) = F_{ab} - F_{ba}$

$tr(Z_{ab,i}F) = (F_{ab} + F_{ba})i$

Substitute in for F_{ab} and F_{ba}

$$tr(Z_{ab}F) = (F_{ab,r} - F_{ba,r}) + i(F_{ab,i} - F_{ba,i}) \tag{6.1}$$

$$tr(Z_{ab,i}F) = (-F_{ab,i} - F_{ba,i}) + i(F_{ab,r} + F_{ba,r}) \tag{6.2}$$

(Note that if both of the above are zero, we get that $F_{ab} = F_{ba} = 0$. So if $F_{ab} \neq 0$ at least one of the above is non-zero.)

By Theorem 3.2, we know that U is not multidirectional. So either $tr(Z_{ab}F)$ and $tr(Z_{ab,i}F)$ are collinear as vectors in the complex plane or at least one of them is zero. In either case we know that

$$(F_{ab,i} - F_{ba,i})(-F_{ab,i} - F_{ba,i}) = (F_{ab,r} + F_{ba,r})(F_{ab,r} - F_{ba,r})$$

We can simplify this to get:

$$F_{ab,r}^2 + F_{ab,i}^2 = F_{ba,r}^2 + F_{ba,i}^2$$

$$|F_{ab}| = |F_{ba}|$$

We can write:

$$F_{ab} = |F_{ab}|\angle\theta_{ab}$$

$$F_{ba} = |F_{ab}|\angle\theta_{ba}$$

In the 4 cases below, we will show that when $F \neq 0$, we can get a representation of F as $F = e^{i\theta}H$ where θ is real and H is zero-diagonal and hermitian. After that we will show that restricting θ to $0 \leq \theta < \pi$ gives us a **unique** representation.

Case 1: $|F_{ab}|$ is non-zero for only one pair (a,b) where $a \neq b$

In this case,

$H = e^{-(\theta_{ab}+\theta_{ba})/2}F$ is a hermitian matrix, and we're finished.

Case 2: $|F_{ab}|$ is non-zero for multiple pairs (a,b) where $a \neq b$. For any Z, when tr(ZF) is non-zero, it is imaginary.

If $|F_{ab}| \neq 0$, then by (6.1) and (6.2), $\theta_{ab} = -\theta_{ba}$. So our F-matrix is already hermitian, and we're done.

Case 3: $|F_{ab}|$ is non-zero for multiple pairs (a,b) where $a \neq b$. For any Z, when tr(ZF) is non-zero, it is real.

If $|F_{ab}| \neq 0$, then by (6.1) and (6.2), $\theta_{ab} = \pi - \theta_{ba}$.

$H = e^{-(\frac{\pi}{2})}F$ is hermitian and we're done.

Case 4: $|F_{ab}|$ is non-zero for multiple pairs (a,b) where $a \neq b$. For any Z, when tr(ZF) is non-zero, it isn't real or imaginary.

Suppose $|F_{ab}| \neq 0$ and $|F_{cd}| \neq 0$

if $tr(Z_{ab}F) \neq 0$, then

slope of $tr(Z_{ab}F) = \frac{\sin(\theta_{ab})-\sin(\theta_{ba})}{\cos(\theta_{ab})-\cos(\theta_{ba})} = -\cot(\frac{\theta_{ab}+\theta_{ba}}{2})$

if $tr(Z_{ab,i}F) \neq 0$:

slope of $tr(Z_{ab,i}F) = \frac{\cos(\theta_{ab})+\cos(\theta_{ba})}{-\sin(\theta_{ab})-\sin(\theta_{ba})} = -\cot(\frac{\theta_{ab}+\theta_{ba}}{2})$

We know that since $|F_{ab}| \neq 0$, at least one of $tr(Z_{ab}F)$ or $tr(Z_{ab,i}F)$ is non-zero.

similarly,

if $tr(Z_{cd}F) \neq 0$, then

slope of $tr(Z_{cd}F) = -\cot(\frac{\theta_{cd}+\theta_{dc}}{2})$

if $tr(Z_{cd,i}F) \neq 0$:

slope of $tr(Z_{cd,i}F) = -\cot(\frac{\theta_{cd}+\theta_{dc}}{2})$

We know that since $|F_{cd}| \neq 0$, at least one of $tr(Z_{cd}F)$ or $tr(Z_{cd,i}F)$ is non-zero.

So we have:

$\cot(\frac{\theta_{cd}+\theta_{dc}}{2}) = \cot(\frac{\theta_{ab}+\theta_{ba}}{2})$ (since U is not multidirectional)

therefore:

$\frac{\theta_{cd}+\theta_{dc}}{2} = \frac{\theta_{ab}+\theta_{ba}}{2} + n\pi$ for some integer n.

We can freely adjust θ_{cd} by $-2n\pi$. It makes no difference since $|F_{cd}|\angle\theta_{cd} = |F_{cd}|\angle(\theta_{cd} - 2n\pi)$

So after the adjustment we have:

$\frac{\theta_{cd}+\theta_{dc}}{2} = \frac{\theta_{ab}+\theta_{ba}}{2}$.

We make the same adjustment for any pair $(c, d) \neq (a, b)$ where $|F_{cd}| \neq 0$

We set $\beta = \frac{\theta_{ab}+\theta_{ba}}{2}$

let $H = e^{-i\beta}F$

For some pair (x, y) where $x \neq y$ and $|H_{xy}| \neq 0$,

$H_{xy} = |H_{xy}|\angle\alpha_{xy}$

$\alpha_{xy} = -(\frac{\theta_{ab}+\theta_{ba}}{2}) + \theta_{xy}$

$\alpha_{yx} = -(\frac{\theta_{ab}+\theta_{ba}}{2}) + \theta_{yx}$

But because of our adjustment,

$\frac{\theta_{ab}+\theta_{ba}}{2} = \frac{\theta_{xy}+\theta_{yx}}{2}$

Plugging this into the above two formulas we have

$\alpha_{xy} = \frac{\theta_{xy}-\theta_{yx}}{2}$

$\alpha_{yx} = -(\frac{\theta_{xy}-\theta_{yx}}{2})$

Therefore H is zero-diagonal, with transpositional elements of equal magnitude and opposite arguments. Therefore H is hermitian.

So in all 4 cases we can write $F = e^{i\beta}H$ for some hermitian matrix H and some real β. But we've not arrived at a unique representation for F yet.

Suppose

$F = e^{i\beta_1}H_1 = e^{i\beta_2}H_2$

$e^{i(\beta_1-\beta_2)}H_1 = H_2$

$e^{i(\beta_1-\beta_2)}H_1 = H_2 = H_2^* = e^{i(\beta_2-\beta_1)}H_1^* = e^{i(\beta_2-\beta_1)}H_1$

So

$(e^{i(\beta_1-\beta_2)} - e^{i(\beta_2-\beta_1)})H_1 = 0$

Since $F \neq 0$, we know $H_1 \neq 0$ so

$e^{i(\beta_1-\beta_2)} - e^{i(\beta_2-\beta_1)} = 0$

$e^{i(\beta_1-\beta_2)} = e^{i(\beta_2-\beta_1)}$

Then

$\beta_1 - \beta_2 = \beta_2 - \beta_1 + 2k\pi$, for any integer k

$\beta_1 = \beta_2 + k\pi$

So if we restrict β to $0 \leq \beta < \pi$, we have a unique representation since k is forced to 0.

This completes our proof of Theorem 3.3. ∎

7. Proof of Theorem 3.4. Given a boundary matrix U with $M(U) \neq 0$ with F-matrix $F \neq 0$. Given $\partial\Delta$ has a tangent line L at $M(U)$.

Proof. By Theorem 3.3 we know that

$$F = e^{i\theta}H \tag{7.1}$$

for some real $0 \leq \theta < \pi$ and some zero-diagonal hermitian matrix H.

We can substitute (7.1) into (6.1) and (6.2) and simplify to get:

$$tr(Z_{ab}F) = 2H_{ab,i}e^{i(\theta+\pi/2)} \tag{7.2}$$

$$tr(Z_{ab,i}F) = 2H_{ab,r}e^{i(\theta+\pi/2)} \tag{7.3}$$

Assume the above two equations are always 0 for all pairs (a, b). Then $H = 0$ and by (7.1) and $F = 0$. But we are given that $F \neq 0$, so we have a contradiction. So our assumption is false and for some skew-hermitian matrix Z, $tr(ZF) \neq 0$. So by Lemma 3.1 we know that $R'_Z(0) = M(U)tr(ZF) \neq 0$.

By (7.2) and (7.3) we see that $tr(ZF)$ forms an angle of $(\theta+\pi/2)$ with the positive real axis (By Theorem 3.2 U is not multidirectional so any non-zero $tr(ZF)$ forms the same angle with the positive real axis. So the angle is always $\theta + \pi/2$) Therefore $R'_Z(0)$ forms an angle $arg(M(U)) + \theta + \pi/2$ with the positive real axis.

Assume $R'_Z(0)$ is not parallel to L. Then since $R'_Z(0) \neq 0$, $R_Z(t)$ crosses $\partial\Delta$ at $t = 0$. So $R_Z(t) \not\subseteq \Delta$ for some t. But we know by subsection 2.3 that $R_Z(t) \subseteq \Delta$ for all t. We have a contradiction. So our assumption is false and we know that $R'_Z(0)$

is parallel to L. So L also forms an angle $arg(M(U)) + \theta + \pi/2$ with the positive real axis.

This completes our proof of Theorem 3.4. ∎

8. Proof of Corollary 3.5.

This is a simple consequence of Theorem 3.4.

Proof. $\partial\Delta$ has a unique tangent line at P.

So if $F_1 = e^{i\theta_1} H_1$ and $F_2 = e^{i\theta_2} H_2$, then

$arg(M(U_1)) + \theta_1 + \pi/2 = arg(M(U_2)) + \theta_2 + \pi/2$

Since $M(U_1) = M(U_2)$,

$arg(M(U_1)) + \theta_1 + \pi/2 = arg(M(U_1)) + \theta_2 + \pi/2$

giving

$\theta_1 = \theta_2$

9. Conjectures.

Before we state our conjectures we define a region Δ_S which is a restriction of Δ. See (1.1).

$$\Delta_S = \left\{ det(A_0 + O B_0 O^*) : O \in O(n) \right\} \tag{9.1}$$

where $O(n)$ is the set of $n \times n$ real orthogonal matrices.

As proven in [3], p.207, theorem 4.4.7, a matrix is normal and symmetric if and only if it is diagonalizable by a real orthogonal matrix.

Therefore Δ_S is the set of determinants of sums of normal, symmetric matrices with prescribed eigenvalues. We know Δ_S contains all the permutation points.

CONJECTURE 9.1 (Restricted Marcus-de Oliveira Conjecture).

$$\Delta_S \subseteq co\left\{ \prod (a_i + b_{\sigma(i)}) \right\}$$

CONJECTURE 9.2 (Boundary Conjecture).

$$\partial\Delta \subseteq \partial\Delta_S$$

THEOREM 9.3. *If the boundary conjecture is true, the restricted Marcus-de Oliveira conjecture imples the full Marcus-de Oliveira conjecture.*

Proof. Suppose we know Conjecture 9.1 is true. Then Δ_S along with its boundary is within the convex-hull. Suppose we also know that Conjecture 9.2 is true. Then we know that $\partial\Delta$ is inside the convex-hull. Can we have a unitary matrix U such that $M(U)$ is outside the convex-hull? No, because that would mean we have points of Δ on both the inside and outside of $\partial\Delta$. This is impossible since Δ is a closed set (See the paragraph on the compactness of $U(n)$ in section 1). So Δ is within the convex hull proving Conjecture 1.1. ∎

10. Conclusion. We hope that further analysis on boundary matrices of Δ, either by expanding on the results in this paper, or novel research, leads to a proof of the Boundary Conjecture. Then proving the full Marcus-de Oliveira conjecture would amount to proving the restricted conjecture. Whether the restricted conjecture is any easier to prove is unknown, but it's an avenue worth exploring.

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