BOUNDARY MATRICES AND THE MARCUS-DE OLIVEIRA DETERMINANTAL CONJECTURE*

AMEET SHARMA†

Abstract. We present notes on the Marcus-de Oliveira conjecture. The conjecture concerns the region in the complex plane covered by the determinants of the sums of two normal matrices with prescribed eigenvalues. Call this region Δ. This paper focuses on boundary matrices of Δ. We prove 4 theorems regarding these boundary matrices. We propose 2 conjectures related to the Marcus-de Oliveira conjecture.

Key words. determinantal conjecture, Marcus-de Oliveira, determinants, normal matrices, convex-hull

AMS subject classifications. 15A15, 15A16

1. Introduction. Marcus [4] and de Oliveira [2] made the following conjecture. Given two normal matrices A and B with prescribed eigenvalues $a_1, a_2...a_n$ and $b_1, b_2...b_n$ respectively, $det(A + B)$ lies within the region:

$$co\Big\{ \prod (a_i + b_{\sigma(i)}) \Big\}$$

where $\sigma \in S_n$. co denotes the convex hull of the $n!$ points in the complex plane. As described in [1], the problem can be restated as follows. Given two diagonal matrices, $A_0 = diag(a_1, a_2...a_n)$ and $B_0 = diag(b_1, b_2...b_n)$, let:

$$\Delta = \Big\{ det(A_0 + UB_0U^*) : U \in U(n) \Big\} \tag{1.1}$$

where $U(n)$ is the set of $n \times n$ unitary matrices. Then we can write the conjecture as:

CONJECTURE 1.1 (Marcus-de Oliveira Conjecture).

$$\Delta \subseteq co\Big\{ \prod (a_i + b_{\sigma(i)}) \Big\} \tag{1.2}$$

Let

$$R_m(U) = det(A_0 + UB_0U^*). \tag{1.3}$$

Then the points forming the convex hull are at $R_m(P_0), R_m(P_1)...R_m(P_{n!-1})$, where the P's are the $n \times n$ permutation matrices. We will refer to these as **permutation points** from now on.

Note that $U(n)$ is a compact set. A continuous image of a compact set is compact. Therefore Δ is compact. And so Δ is a closed set, because a compact subset of any metric space (in this case the complex numbers) is closed.

*Submitted to the editors June 6th, 2018.

† (ameet_n_sharma@hotmail.com).

The paper is organized as follows. In section 2 we define terms that will be used in the rest of the paper. These terms are necessary to state our main results. In section 3, we state our 4 main theorems. section 4 provides a proof of the first theorem, section 5 provides a proof of the second, section 6 provides a proof of the third and section 7 provides a proof of the fourth. In section 8, we state 2 conjectures. In section 9, we conclude.

2. Terms and definitions.

2.1. Boundary matrix.

- Given a point P on $\partial\Delta$ (the boundary of Δ) and given a unitary matrix U such that $R_m(U) = P$, we call U a **boundary matrix** of Δ. See (1.3).
- Given a boundary matrix U. If $\partial\Delta$ is smooth at $R_m(U)$ and U is not a permutation matrix, we say U is a **regular boundary matrix**.

2.2. Properties of unitary matrices given A_0 and B_0. In this section, we define four properties of unitary matrices that will be very useful when examining boundary matrices of Δ.

The first three of these properties are matrices related to U. These matrices are defined in [1], p.27. They provide a language to talk about unitary matrices within the context of the determinantal conjecture.

B-matrix

$$B = UB_0U^* \tag{2.1}$$

C-matrix

$$C = A_0 + UB_0U^* \tag{2.2}$$

Using (1.3), $R_m(U) = det(C)$

F-matrix

$$F = BC^{-1} - C^{-1}B$$

We can change the F-matrix into a more useful form:

$$F = (C - A_0)C^{-1} - C^{-1}(C - A_0)$$

$$F = C^{-1}A_0 - A_0C^{-1} \tag{2.3}$$

The F-matrix is only defined when C is invertible or equivalently $R_m(U) \neq 0$.

Since A_0 is diagonal, we see that F is a zero-diagonal matrix.

As demonstrated in [1], p.27, the F-matrix is 0 if and only if U is a permutation matrix.

The fourth property is conditional. Given a unitary matrix U with $R_m(U) \neq 0$ and with F-matrix F. Suppose there exist two skew-hermitian matrices Z_1 and Z_2 such that $tr(Z_1F)$ and $tr(Z_2F)$ are both non-zero and non-collinear vectors in the complex plane. Then we say that U is a **multidirectional** matrix. A multidirectional matrix must have a non-zero F-matrix to allow those non-zero traces. So a permutation matrix cannot be multidirectional because its F-matrix is 0.

Note that these properties require an A_0 and B_0 to be defined. Throughout the paper we will assume there's a defined A_0 and B_0 in the background. We will not mention them explicitly in order to simplify our language. For example when we say "the C-matrix of a unitary matrix U", it is clear that there's an unmentioned A_0 and B_0 according to which the C-matrix of U is defined. It is the same thing with the terms "boundary matrix" and "regular boundary matrix". Obviously it is meaningless for a unitary matrix to be a boundary matrix "in general". These terms only make sense in the context of A_0, B_0 and the corresponding Δ. So we'll assume this context has been defined.

3. Main Theorems.

THEOREM 3.1. *Given U is a non-permutation unitary matrix with $R_m(U) \neq 0$ and F-matrix F. Given an arbitrary skew-hermitian matrix Z. There exists a curve $R_f(t) \subseteq \Delta$, where t is real, such that $R_f(0) = R_m(U)$ and $R'_f(0) = R_m(U)tr(ZF)$.*

THEOREM 3.2. *If U is a boundary matrix, then U is not multidirectional.*

THEOREM 3.3. *Given a boundary matrix U such that $R_m(U) \neq 0$. Then its F-matrix has the form $F = e^{i\theta}H$ where H is a zero-diagonal hermitian matrix.*

THEOREM 3.4. *Given a regular boundary matrix U such that $R_m(U) \neq 0$. Let $F = e^{i\theta}H$ be the F-matrix of U. let l be the tangent line to $\partial\Delta$ at the boundary point. Then l makes an angle $arg(R_m(U)) + \theta + \pi/2$ with the positive real axis.*

4. Proof of Theorem 3.1. This theorem is apparent from [1], p.27, but it is not stated explicitly there. It is worth proving explicitly here as it will be used for the other theorems.

Before we can prove the theorem we need to set up some tools. Our aim is to examine boundary matrices of Δ. Towards this aim, it is useful to consider smooth functions of unitary matrices going through these boundary matrices and see how they behave under (1.3). For this reason, we introduce the functional form of (1.3).

$$R_f(t) = \det(A_0 + U_f(t)B_0U_f^*(t)) \tag{4.1}$$

where t is real and $U_f(t)$ is some smooth function of unitary matrices.

Every unitary matrix can be written as an exponential of a skew-hermitian matrix. So we can write:

$$U_f(t) = e^{S_f(t)}. \tag{4.2}$$

where $S_f(t)$ is a smooth function of skew hermitian matrices

For small Δt,

$U_f(t + \Delta t) = (e^{S_f(t+\Delta t)})$

$U_f(t + \Delta t) = (e^{S_f(t)+(\Delta t)S'_f(t)})$

$U_f(t + \Delta t) = (e^{(\Delta t)S'_f(t)})U_f(t)$

If we take the above function and plug it into $R_f(t)$ we'll get $R_f(t + \Delta t)$, but it won't be in a form useful to us. We use a result from [1], p.27 for this purpose. In order to state this result within the context of this paper, we first need the functional forms of the B-matrix, C-matrix, F-matrix (these were defined in section 2):

$$B_f(t) = U_f(t)B_0U_f^*(t) \tag{4.3}$$

$$C_f(t) = A_0 + B_f(t) \tag{4.4}$$

$$F_f(t) = C_f^{-1}(t)A_0 - A_0C_f^{-1}(t) \tag{4.5}$$

Note, $F_f(t)$ is only defined if $R_f(t) \neq 0$. Also $F_f(t) = 0$ only when $U_f(t)$ is a permutation matrix.

Now we can state the result from [1]:

When $F_f(t) \neq 0$,

$$R_f(t + \Delta t) = R_f(t) + (\Delta t)\det(C_f(t))tr(S'_f(t)F_f(t)) + O((\Delta t)^2) \tag{4.6}$$

$$R'_f(t) = det(C_f(t))tr(S'_f(t)F_f(t)) \tag{4.7}$$

Now we have the tools needed to prove Theorem 3.1.

Proof. Given any non-permutation unitary matrix U with $R_m(U) \neq 0$. let C be the C-matrix of U. let F be the F-matrix of U. Given Z is some arbitrary skew-hermitian matrix. We can find a skew-hermitian matrix S such that $U = e^S$.

We choose:

$$S_f(t) = S + tZ \tag{4.8}$$

Note that $S_f(t)$ is a smooth function of skew-hermitian matrices. We use it with (4.1),(4.2),(4.4),(4.5) and (4.7) to get $R_f(t), U_f(t), C_f(t), F_f(t)$ and $R'_f(t)$. Note that $U_f(0) = U$, the unitary matrix we're originally given. The choice of $t = 0$ is merely for simplicity and has no special significance. We could time-shift $S_f(t)$ to the right by t_1 to make $U_f(t_1) = U$ instead.

Note that $C_f(0) = C$

Note that $F_f(0) = F$

Note that $R_f(0) = R_m(U)$. See (1.3) and (4.1).

$R'_f(t) = det(C_f(t))tr(ZF_f(t))$

$R'_f(0) = det(C_f(0))tr(ZF_f(0))$

$R'_f(0) = det(C)tr(ZF)$

therefore

$$R'_f(0) = R_m(U)tr(ZF) \tag{4.9}$$

This proves Theorem 3.1. ∎

5. Proof of Theorem 3.2. We will prove the contrapositive. ie: We'll start with a multidirectional matrix U, and prove that it is not a boundary matrix.

Proof. Given we have a multidirectional matrix U. Let F be its F-matrix and C-matrix C. We know $R_m(U) = det(C) \neq 0$ and we know F is non-zero. See the discussion on multidirectional matrices in the second last paragraph of section 2.

There exist two skew-hermitian matrices Z_1 and Z_2 such that

$$T_1 = tr(Z_1F) \tag{5.1}$$

$$T_2 = tr(Z_2F) \tag{5.2}$$

are both non-zero and non-collinear.

By Theorem 3.1, there exist two functions $R_1(t)$ and $R_2(t)$ such that $R_1(0) = R_2(0) = R_m(U)$ and such that

$R'_1(0) = R_m(U)tr(Z_1F)$

$R'_2(0) = R_m(U)tr(Z_2F)$

substitute in (5.1) and (5.2),

$R'_1(0) = R_m(U)T_1$

$R'_2(0) = R_m(U)T_2$

Since we know T_1 and T_2 are non-collinear, $R'_1(0)$ and $R'_2(0)$ are non-collinear. They are also non-zero. Therefore they form a linear basis for all the complex numbers over the real numbers. Let Q be an arbitrary complex number.

$Q = aR'_1(0) + bR'_2(0)$ where a and b are real.

$Q = a(R_m(U))T_1 + b(R_m(U))T_2$

$Q = R_m(U)(aT_1 + bT_2)$

substitute in (5.1) and (5.2),

$Q = R_m(U)(tr(aZ_1F) + tr(bZ_2F))$

$Q = R_m(U)tr((aZ_1 + bZ_2)F)$

let $Z_3 = aZ_1 + bZ_2$

$Q = R_m(U)tr(Z_3F)$

Note that Z_3 is also a skew-hermitian matrix.

Again by Theorem 3.1, there exists a function $R_3(t)$ such that

$R_3(0) = R_m(U)$

and

$R_3'(0) = R_m(U)tr(Z_3F) = Q$

Therefore $R_3(t)$ goes through $R_m(U)$ in a direction parallel to Q. Q was chosen arbitrarily. So through $R_m(U)$ there exists curves $R_3(t) \subseteq \Delta$ going in all directions. Therefore $R_m(U)$ is an internal point of Δ. So it's not a boundary point. Therefore U is not a boundary matrix. That gives us Theorem 3.2. ∎

6. Proof of Theorem 3.3. For n = 3, we define the following 12 skew-hermitian matrices with zero diagonal:

$$Z_{12} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad Z_{13} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad Z_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Z_{21} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad Z_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad Z_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$Z_{12,i} = Z_{21,i} = \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad Z_{13,i} = Z_{31,i} = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \qquad Z_{23,i} = Z_{32,i} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$$

Note that the commas do not indicate tensors. They're just used here as a label to distinguish imaginary and real matrices.

We define Z_{ab} and $Z_{ab,i}$ similarly for all $n > 3$, where $a \neq b$. For a given n we have $n(n-1)$ real matrices and $n(n-1)$ imaginary matrices.

Proof. Given a boundary matrix U with $R_m(U) \neq 0$. Let F be its F-matrix. We know that F is zero-diagonal by (4.5).

Suppose $F_{ab} = F_{ab,r} + iF_{ab,i}$ where $F_{ab,r}$ and $F_{ab,i}$ are real numbers.

$tr(Z_{ab}F) = F_{ab} - F_{ba}$

$tr(Z_{ab,i}F) = (F_{ab} + F_{ba})i$

Substitute in for F_{ab} and F_{ba}

$$tr(Z_{ab}F) = (F_{ab,r} - F_{ba,r}) + i(F_{ab,i} - F_{ba,i}) \tag{6.1}$$

$$tr(Z_{ab,i}F) = (-F_{ab,i} - F_{ba,i}) + i(F_{ab,r} + F_{ba,r}) \tag{6.2}$$

By Theorem 3.2, we know that U is not multidirectional.

Therefore

$(F_{ab,i} - F_{ba,i})(-F_{ab,i} - F_{ba,i}) = (F_{ab,r} + F_{ba,r})(F_{ab,r} - F_{ba,r})$

We can simplify this to get:

$F^2_{ab,r} + F^2_{ab,i} = F^2_{ba,r} + F^2_{ba,i}$

$|F_{ab}| = |F_{ba}|$

We can write:

$F_{ab} = |F_{ab}|\angle\theta_{ab}$

$F_{ba} = |F_{ab}|\angle\theta_{ba}$

There are multiple cases we need to deal with.

Case 1: F-matrix is 0

F=0 is hermitian so we're finished.

Case 2: $|F_{ab}|$ is non-zero for only one pair (a,b) where $a \neq b$

In this case,

$H = e^{-(\theta_{ab}+\theta_{ba})/2}F$ is a hermitian matrix, and we're finished.

Case 3: $|F_{ab}|$ is non-zero for multiple pairs (a,b) where $a \neq b$. For an arbitrary skew-hermitian Z, when tr(ZF) is non-zero, it is imaginary.

If $|F_{ab}| \neq 0$, then by (6.1) and (6.2), $\theta_{ab} = -\theta_{ba}$. So our F-matrix is already hermitian, and we're done.

Case 4: $|F_{ab}|$ is non-zero for multiple pairs (a,b) where $a \neq b$. For an arbitrary skew-hermitian Z, when tr(ZF) is non-zero, it is real.

If $|F_{ab}| \neq 0$, then by (6.1) and (6.2), $\theta_{ab} = \pi - \theta_{ba}$.

$H = e^{-(\frac{\pi}{2})}F$ is hermitian and we're done.

Case 5: $|F_{ab}|$ is non-zero for multiple pairs (a,b) where $a \neq b$. For an arbitrary skew-hermitian Z, when tr(ZF) is non-zero, it isn't real or imaginary.

Suppose $|F_{ab}| \neq 0$ and $|F_{cd}| \neq 0$

if $tr(Z_{ab}F) \neq 0$, then

slope of $tr(Z_{ab}F) = \frac{\sin(\theta_{ab})-\sin(\theta_{ba})}{\cos(\theta_{ab})-\cos(\theta_{ba})} = -\cot(\frac{\theta_{ab}+\theta_{ba}}{2})$

if $tr(Z_{ab,i}F) \neq 0$:

slope of $tr(Z_{ab,i}F) = \frac{\cos(\theta_{ab})+\cos(\theta_{ba})}{-\sin(\theta_{ab})-\sin(\theta_{ba})} = -\cot(\frac{\theta_{ab}+\theta_{ba}}{2})$

similarly,

slope of $tr(Z_{cd}F) = -\cot(\frac{\theta_{cd}+\theta_{dc}}{2})$

or

slope of $tr(Z_{cd,i}F) = -\cot(\frac{\theta_{cd}+\theta_{dc}}{2})$

$\cot(\frac{\theta_{cd}+\theta_{dc}}{2}) = \cot(\frac{\theta_{ab}+\theta_{ba}}{2})$

therefore either:

$\frac{\theta_{cd}+\theta_{dc}}{2} = \frac{\theta_{ab}+\theta_{ba}}{2}$

or,

$\frac{\theta_{cd}+\theta_{dc}}{2} = \frac{\theta_{ab}+\theta_{ba}}{2} + \pi$

For some specific x, y where $x \neq y$ and $|F_{xy}| \neq 0$

let $\beta = \frac{\theta_{xy}+\theta_{yx}}{2}$

let $H = e^{-i\beta}F$

For any $a \neq b$,

$H_{ab} = |H_{ab}|\angle\alpha_{ab}$

$\frac{\alpha_{ab}+\alpha_{ba}}{2} = 0$ or π

Therefore H is zero-diagonal, with transpositional elements of equal magnitude and opposite arguments. Therefore H is hermitian.

So in all 5 cases we can write $F = e^{i\beta}H$ for some hermitian matrix H and some real β.

This completes our proof of Theorem 3.3. ∎

7. Proof of Theorem 3.4. Given a regular boundary matrix U. Let F be the F-matrix of U.

Proof. Therefore by Theorem 3.3 we know that

$$F = e^{i\theta}H \tag{7.1}$$

for some real θ and some zero-diagonal hermitian matrix H.

We can substitute (7.1) into (6.1) and (6.2) and simplify to get:

$$tr(Z_{ab}F) = 2H_{ab,i}e^{i(\theta+\pi/2)} \tag{7.2}$$

$$tr(Z_{ab,i}F) = 2H_{ab,r}e^{i(\theta+\pi/2)} \tag{7.3}$$

As expected the vectors are collinear.

Since U is a regular boundary matrix, $\partial\Delta$ is smooth at $R_m(U)$ ie: the tangent to the curve exists at $R_m(U)$.

So using Theorem 3.1, we see that the tangent line forms an angle $arg(R_m(U)) + \theta + \pi/2$ with the positive real axis. This completes our proof of Theorem 3.4. ∎

8. Conjectures. Before we state our conjectures we define a region Δ_S which is a restriction of Δ. See (1.1).

$$\Delta_S = \left\{det(A_0 + OB_0O^*) : O \in O(n)\right\} \tag{8.1}$$

where $O(n)$ is the set of $n \times n$ real orthogonal matrices.

As proven in [3], p.207, theorem 4.4.7, a matrix is normal and symmetric if and only if it is diagonalizable by a real orthogonal matrix.

Therefore Δ_S is the set of determinants of sums of normal, symmetric matrices with prescribed eigenvalues. We know Δ_S contains all the permutation points.

CONJECTURE 8.1 (Restricted Marcus-de Oliveira Conjecture).

$$\Delta_S \subseteq co\left\{\prod(a_i + b_{\sigma(i)})\right\}$$

CONJECTURE 8.2 (Boundary Conjecture).

$$\partial\Delta \subseteq \partial\Delta_S$$

THEOREM 8.3. *If the boundary conjecture is true, the restricted Marcus-de Oliveira conjecture imples the full Marcus-de Oliveira conjecture.*

Proof. Suppose we know Conjecture 8.1 is true. Then Δ_S along with its boundary is within the convex-hull. Suppose we also know that Conjecture 8.2 is true. Then we know that $\partial\Delta$ is inside the convex-hull. Can we have a unitary matrix U such that $R_m(U)$ is outside the convex-hull? No, because that would mean we have points of Δ on both the inside and outside of $\partial\Delta$. This is impossible since Δ is a closed set (See the second last paragraph of section 1). So Δ is within the convex hull proving Conjecture 1.1. ∎

9. Conclusion. We hope that further analysis on boundary matrices of Δ, either by expanding on the results in this paper, or novel research, leads to a proof of the Boundary Conjecture. Then proving the full Marcus-de Oliveira conjecture would amount to proving the restricted conjecture. Whether the restricted conjecture is any easier to prove is unknown, but it's an avenue worth exploring.

REFERENCES

[1] N. Bebiano and J. Querió, *The determinant of the sum of two normal matrices with prescribed eigenvalues*, Linear Algebra and its Applications, 71 (1985), pp. 23–28.
[2] G. N. de Oliveira, *Research problem: Normal matrices*, Linear and Multilinear Algebra, 12 (1982), pp. 153–154.
[3] R. Horn and C. Johnson, *Matrix Analysis*, Cambridge University Press, 1990.
[4] M. Marcus, *Derivations, plücker relations and the numerical range*, Indiana University Math Journal, 22 (1973), pp. 1137–1149.