# SOLUTION OF ERDÖS-MOSER EQUATION
## $1 + 2^p + 3^p + ... + (k)^p = (k+1)^p$

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Abstract. I will provide the solution of Erdös-Moser equation based on the properties of Bernoulli polynomials and prove that there is only one solution satisfying the above-mentioned equation.

Keywords: Bernoulli polynomials; Summation; Diophantine equation

## 1. Notation

$1 + 2^p + 3^p + ... + (k)^p = (k+1)^p$ represents Erdös-Moser equation, where $k, p \in \mathbb{N}$.
Let $b_n$ denotes Bernoulli numbers.
Let

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} b_{n-k} x^k$$

denotes Bernoulli polynomials for $n \geq 0$.

## 2. Introduction

The Erdös-Moser equation (EM equation), named after Paul Erdös and Leo Moser has been studied by many number theorists through history since combines addition, powers and summation together. The open and very interesting conjecture of Erdös-Moser states that there is no other solution of EM equation than the trivial $1 + 2 = 3$. Investigation of the properties and identities of the EM equation and ultimately providing the proof of this conjecture is the main purpose of this article.

## 3. Solution

**Lemma 3.1.** *The EM equation is equivalent of*

$$\sum_{k=0}^{x} k^p \equiv \frac{B_{p+1}(x+1)}{p+1} = (x+1)^p \tag{3.1}$$

*for* $x \in \mathbb{N}$.

*Proof.* Sum of pth powers is defined as

$$\sum_{k=0}^{x} k^p = \frac{B_{p+1}(x+1) - B_{p+1}(0)}{p+1}$$

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Leo Moser proved that for another solution of EM equation two must divide $p$, see [1], what yields that $p+1$ must be odd and $B_{p+1}(0)$ with odd subscripts is equal zero. □

**Lemma 3.2.**

$$B_{p+1}(x+1) - B_{p+1}(x) = (p+1)x^p \tag{3.2}$$

$$B_{p+1}(x+2) - B_{p+1}(x+1) = (p+1)(x+1)^p \tag{3.3}$$

*Proof.* Relation of Bernoulli polynomials given by Whittaker and Watson, see [2], what in general form is defined as $B_n(x+1) - B_n(x) = nx^{n-1}$. □

**Lemma 3.3.** *Lemma (3.1) in combination with rearranged Eq. (3.2) gives a relation*

$$\frac{B_{p+1}(x+1)}{B_{p+1}(x)} = \frac{(x+1)^p}{(x+1)^p - x^p} \tag{3.4}$$

*Proof.* Let us express $p+1$ from Eq. (3.2) as

$$\frac{B_{p+1}(x+1)}{x^p} - \frac{B_{p+1}(x)}{x^p} = p+1 \tag{3.5}$$

then by putting LHS of Eq. (3.5) in Eq. (3.1) we get

$$B_{p+1}(x+1) = (x+1)^p \left( \frac{B_{p+1}(x+1)}{x^p} - \frac{B_{p+1}(x)}{x^p} \right)$$

and after elementary rearrangements we can rearrange Eq. (3.1) to the form as is defined in Lemma (3.3). □

**Theorem 3.4.** *The EM equation has other solution than trivial if and only if holds the relation in Eq. (3.6)*

$$\frac{B_{p+1}(x+2)}{B_{p+1}(x+1)} = 2 \tag{3.6}$$

*for* $x \in \mathbb{N}$.

*Proof.* Let us rearrange Eq. (3.1) as

$$B_{p+1}(x+1) = (p+1)(x+1)^p \tag{3.7}$$

the RHS of Eq. (3.3) and Eq. (3.7) are equal so we can define

$$B_{p+1}(x+2) - B_{p+1}(x+1) = B_{p+1}(x+1)$$
$$B_{p+1}(x+2) = 2B_{p+1}(x+1)$$
$$\frac{B_{p+1}(x+2)}{B_{p+1}(x+1)} = 2$$

□

**Theorem 3.5.** *The expression on the LHS of Eq. (3.6)* $\frac{B_{p+1}(x+2)}{B_{p+1}(x+1)}$ *can be always expressed by the expression on the LHS of Eq. (3.4)* $\frac{B_{p+1}(x+1)}{B_{p+1}(x)}$ *and therefore EM equation does not have any other solution than trivial.*

*Proof.* Let us define variable $X_2 = x_2 + 2$. Let us recall that by the definition of Bernoulli numbers $B_n(x)$

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} b_{n-k} x^k$$

$\frac{B_{p+1}(x_2+2)}{B_{p+1}(x_2+1)}$ is equal to the following

$$\frac{B_{p+1}(x_2+2) = \sum_{k=0}^{p+1} \binom{p+1}{k} b_{p+1-k}(x_2+2)^k}{B_{p+1}(x_2+1) = \sum_{k=0}^{p+1} \binom{p+1}{k} b_{p+1-k}(x_2+1)^k}$$

or in terms of $X_2$

$$\frac{B_{p+1}(X_2) = \sum_{k=0}^{p+1} \binom{p+1}{k} b_{p+1-k}(X_2)^k}{B_{p+1}(X_2-1) = \sum_{k=0}^{p+1} \binom{p+1}{k} b_{p+1-k}(X_2-1)^k}$$

Let us define variable $X_1 = x_1 + 1$.
$\frac{B_{p+1}(x_1+1)}{B_{p+1}(x_1)}$ is equal to the following

$$\frac{B_{p+1}(x_1+1) = \sum_{k=0}^{p+1} \binom{p+1}{k} b_{p+1-k}(x_1+1)^k}{B_{p+1}(x_1) = \sum_{k=0}^{p+1} \binom{p+1}{k} b_{p+1-k}(x_1)^k}$$

or in terms of $X_1$

$$\frac{B_{p+1}(X_1) = \sum_{k=0}^{p+1} \binom{p+1}{k} b_{p+1-k}(X_1)^k}{B_{p+1}(X_1-1) = \sum_{k=0}^{p+1} \binom{p+1}{k} b_{p+1-k}(X_1-1)^k}$$

If $X_2 = x_2 + 2$ and $X_1 = x_1 + 1$ we can define relation between $X_2$ and $X_1$ as follows

$$X_2 - 1 = X_1 \tag{3.8}$$

in other words every natural number representing variable $X_2$ can be expressed by the natural number representing variable $X_1$ in this relation, in other words the expression on the LHS of Eq. (3.6) $\frac{B_{p+1}(x_2+2)}{B_{p+1}(x_2+1)} = \frac{B_{p+1}(X_2)}{B_{p+1}(X_2-1)}$ is equal to the expression on the LHS of Eq. (3.4) $\frac{B_{p+1}(x_1+1)}{B_{p+1}(x_1)} = \frac{B_{p+1}(X_1)}{B_{p+1}(X_1-1)}$ if and only if

$$X_2 = X_1 \tag{3.9}$$

and Eq. (3.9) holds always if

$$X_2 = X_1 \tag{3.10}$$

$$x_2 + 2 = x_1 + 1 \tag{3.11}$$

$$x_2 = x_1 - 1 \tag{3.12}$$

By mathematical induction can be easily proved that this relations holds $\forall X_2 > 2$ (considering the EM equation we are only focusing on positive integers).

**Example 3.1.** *Let*

$$\frac{B_{p+1}(x_2+2)}{B_{p+1}(x_2+1)} = \frac{B_{p+1}(X_2)}{B_{p+1}(X_2-1)} = \frac{B_{p+1}(5+2)}{B_{p+1}(5+1)}$$

*then* $x_2 = 5$ *and* $X_2 = 5 + 2$*. In order to satisfy Eq. (3.9)* $x_1$ *will be equal to six (based on Eq. (3.12) and therefore the expression will be*

$$\frac{B_{p+1}(x_1+1)}{B_{p+1}(x_1)} = \frac{B_{p+1}(X_1)}{B_{p+1}(X_1-1)} = \frac{B_{p+1}(6+1)}{B_{p+1}(6)}$$

*what proves that both expressions are equal. It does not matter how the expressions in the parentheses are expressed so* $(5+2)$ *is equal to* $(6+1)$ *and*

$$B_{p+1}(5+2) = \sum_{k=0}^{p+1} \binom{p+1}{k} b_{p+1-k}(5+2)^k = B_{p+1}(6+1) = \sum_{k=0}^{p+1} \binom{p+1}{k} b_{p+1-k}(6+1)^k$$

.

Since we have proved that thanks to the Eq. (3.12) we are always able to express (define) the expression on the LHS of Eq. (3.6) $\frac{B_{p+1}(x+2)}{B_{p+1}(x+1)}$ by the expression on the LHS of Eq. (3.4) $\frac{B_{p+1}(x+1)}{B_{p+1}(x)}$ in other words if $x_2 = x_1 - 1$ then

$$\text{(3.13)} \qquad LHS\ of\ Eq.\ (3.6)\ \frac{B_{p+1}(x_2+2)}{B_{p+1}(x_2+1)} = \frac{B_{p+1}(x_1+1)}{B_{p+1}(x_1)}\ LHS\ of\ Eq.\ (3.4)$$

Now is enough to prove (mentioned below) that there is no $x_1$ in the expression on LHS of Eq. (3.4) $\frac{B_{p+1}(x_1+1)}{B_{p+1}(x_1)} = \frac{B_{p+1}(X_1)}{B_{p+1}(X_1-1)}$ for which Eq. (3.4) has an integral solution equal to two for $p > 1$ (in order to eliminate the trivial solution) since it will be in contradiction with Theorem (3.4) and it will unconditionally prove Theorem (3.5). Let us recall that Eq. (3.4) is defined as

$$\frac{B_{p+1}(x+1)}{B_{p+1}(x)} = \frac{B_{p+1}(x_1+1)}{B_{p+1}(x_1)} = \frac{(x+1)^p}{(x+1)^p - x^p}$$

and as was mentioned above it is enough to prove that Eq. (3.4) does not have an integral solution equal to two for $p > 1$. It is trivial to see that the expression $\frac{(x+1)^p}{(x+1)^p - x^p}$ has integral solutions for $x > 1$ if and only if $0 < p < 2$ (considering the EM equation, for this moment is important the exponent $p$ not the variable $x$) since

$$\frac{(x+1)^p}{(x+1)^p - x^p} = \frac{x^p + px^{p-1} + ... + 1}{px^{p-1} + ... + 1} = \frac{x^p}{px^{p-1} + ... + 1} + 1$$

On the basis of this facts we can state that if there is an other solution, than the trivial, of the EM equation it is possible if and only if the exponent $p = 1$ what is impossible since there is only one solution - trivial when $p = 1$ as it follows from the basic formula of summation

$$\sum_{k=0}^{x} k^1 \equiv \frac{x*(x+1)}{2} = x + 1 \Rightarrow \frac{x}{2} = 1$$

where $x$ must be equal to two. All of the above-mentioned facts unconditionally prove the Theorem (3.5) and at the same time the Erdös-Moser conjecture. □

## References

[1] L.Moser, On the Diophantine Equation $1^k + 2^k + ...(m-1)^k = m^k$, *Scripta Math. 19,* (1953), 84-88.
[2] E. T. Whittaker, G. N. Watson, A course of MODERN ANALYSIS, *Cambridge University Press 3rd edition,* (1920), 127.