An elementary approach for Fermat's Last Theorem of Case II

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Abstract

We discuss an elementary approach to prove the second case of Fermat's last theorem (FLT). This is a continuation of an earlier paper that deals with the first case of FLT.

Introduction

In an earlier paper *(1)* (hereafter Paper I), we examine the first case of FLT using an elementary approach. The essence of the proof is to notice that $a+b+c$ is of order N^α if $a^N+b^N+c^N=0$. To prove FLT, one shows that α can not be 2; and then show that α can not be 3, etc. This is a standard practice of the method of induction, but we refer to it as the "infinite ascent" technique, in contrast to Fermat's original "infinite descent" technique.

Here we focus on Case II of FLT. We assume $N \mid c$. The Barlow-Abel relations for case II will be used and is repeated here (see e.g. *(2)*): **Barlow-Abel Relation case II:** If $N \mid c$ and

$N \nmid (ab)$, then we have,

$$\begin{aligned} a+b &= N^{\alpha_0 N-1}t^N, \frac{a^N+b^N}{a+b} = Nt_1^N, c = -N^{\alpha_0}tt_1, & (1) \\ b+c &= r^N, \frac{b^N+c^N}{b+c} = r_1^N, a = -rr_1 & (2) \\ c+a &= s^N, \frac{c^N+a^N}{c+a} = s_1^N, b = -ss_1 & (3) \end{aligned}$$

where $\gamma = N\alpha_0 - 1$; t and t_1 are co-prime and t_1 is odd; r and r_1 are co-prime and r_1 is odd; s and s_1 are co-prime and s_1 is odd.

the Proof

Consider $a+b+c$. One can readily show that[1],

$$a+b+c = rstN^{\alpha_0}\tilde{\theta} \quad (4)$$

where $\tilde{\theta} \neq 1$ and is co-prime with t_1, r_1 and s_1. We can write $\tilde{\theta} = \theta_1\theta_2...\theta_l$ where θ_i are prime factors of $\tilde{\theta}$. Let θ be any θ_i, by multiplying q, one can write a, b and c as,

$$c = N + c_1 N\theta, \quad a = k + a_1 N\theta, \quad b = -(k+N) + b_1 N\theta. \quad (5)$$

Note that since $a+b$ is of order N^γ, so $a_1 + b_1 \neq 0 \bmod \mathrm{N}$.

We now have,

$$a^N + b^N + c^N = (N^N + k^N - (N+k)^N) + (N^{N-1}c_1 + k^{N-1}a_1 + (k+N)^{N-1}b_1)N^2\theta + O(N^3\theta^2) \quad (6)$$

Because $a_1 + b_1 \neq 0 \bmod \mathrm{N}$, so the second term in equation (6) is not a multiple of $N^3\theta$, i.e., $(N^{N-1}c_1 + k^{N-1}a_1 + (k+N)^{N-1}b_1) \neq 0 \bmod N^3\theta$. Requiring $a^N + b^N + c^N = 0$ leads to,

$$N^N + k^N - (N+k)^N = \alpha k(k+N)N^2\theta \quad \alpha \neq 0 \bmod \mathrm{N} \quad (7)$$

[1] Showing $\tilde{\theta} \neq 1$ is straightforward but non-trivial.

Multiplying $(1+\epsilon\theta)$ to equation (5), a, b and c become,

$$c \to c' = N + c_1' N\theta, \quad a \to a' = k' + a_1' N\theta, \quad b \to b' = -(k'+N) + b_1' N\theta \tag{8}$$

where $k' = k(1+\epsilon\theta)$, $a_1' = a_1(1+\epsilon\theta)$, $b_1' = b_1(1+\epsilon\theta) - \epsilon$, and $c_1' = \epsilon + c_1 + \epsilon c_1 \theta$. Similar to equation (6), we now have,

$$\begin{aligned} a'^N + b'^N + c'^N &= (N^N + k'^N - (N+k')^N) \\ &+ (N^{N-1}c_1' + (k')^{N-1}a_1' + (k'+N)^{N-1}b_1')N^2\theta + O(N^3\theta^2) \end{aligned} \tag{9}$$

Again $a_1' + b_1' \neq 0 \bmod \mathrm{N}$ and the second term in equation (9) is not a multiple of $N^3\theta$.

Lemma 1: Assuming that $\delta \geq 1$ is an integer, $N > 3$ and θ are two prime numbers, α is co-prime to N, and $(N^\delta)^N + k^N - (N^\delta + k)^N = k(N^\delta + k)N^{\delta+1}\alpha\theta$, then

$$(N^\delta)^N + (k+\Delta)^N - (N^\delta + k + \Delta)^N = (k+\Delta)(N^\delta + k + \Delta)N^{\delta+1}(\alpha\theta + t\Delta) \tag{10}$$

where t and Δ are two integers and $t \neq 0 \bmod \mathrm{N}$.

proof: the form of equation (10) is straightforward. By setting $k + \Delta = N^\delta$ one can see immediately that $t \neq 0 \bmod \mathrm{N}$.

Now let $\delta = 1$ and $\Delta = k\epsilon\theta$, we then find that the first term in the right hand side of equation (9) becomes,

$$N^N + (k')^N - (N+k')^N = N^2 k'(N+k')(\alpha + tk\epsilon)\theta \tag{11}$$

We can now choose ϵ such that $(\alpha + tk\epsilon) = 0 \bmod \mathrm{N}$, then

$$N^N + k'^N - (N+k')^N = \xi N^3 \theta \tag{12}$$

Now we see that at the order of $N^3\theta$ equation (9) can not be balanced. This proves the case II of FLT.

Two illustration examples:

We use two examples to illustrate the idea presented here. For $N = 11$ and $\theta = 17$, one finds $k = 6$ is the only number smaller than θ satisfying,

$$N^N + k^N - (N+k)^N = 0 \bmod \mathrm{N}^2\theta \tag{13}$$

It is easy to verify that for $k' = k(1+9\theta) = 924$, we have

$$N^N + k'^N - (N+k')^N = 0 \bmod \mathrm{N}^3\theta \tag{14}$$

As another example, take $N = 13$ and $\theta = 19$: one finds that $k = 6$, $k = 10$, and $k = 15$ are the numbers smaller than θ satisfying (13). For $k = 10$ we find $k' = k(1+2\theta) = 390$; for $k = 6$ we find $k' = k(1+2\theta) = 234$; for $k = 15$ we find $k' = k(1+2\theta) = 585$; for all these k's, equation (14) holds.

Discussion

Mr. Fermat is arguably the best amateur mathematician in history. Less known is that he was also a very insightful physicist. He discovered that between two points light travels along a path which yields the least travel time. This stimulated the later development of the least action principle in theoretical physics. Mr. Fermat's impact to Physics is no less than his contribution to Mathematics.

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References and Notes

1. The original version of Paper I can be found at http://vixra.org/abs/1706.0288.

2. Ribenboim, Paulo, *13 lectures on Fermat's last theorem*, Verlag: Springer. (1979).