Elementary Proof of the Goldbach Conjecture

Stephen Michael Marshall

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Abstract

Christian Goldbach (March 18, 1690 – November 20, 1764) was a German mathematician. He is remembered today for Goldbach's conjecture.

Goldbach's conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states: Every even integer greater than 2 can be expressed as the sum of two primes.

On 7 June 1742, the German mathematician Christian Goldbach wrote a letter to Leonhard Euler (letter XLIII) in which he proposed the following conjecture:
Every even integer which can be written as the sum of two primes (the strong conjecture) He then proposed a second conjecture in the margin of his letter:

Every odd integer greater than 5 can be written as the sum of three primes (the weak conjecture).

In number theory, Goldbach's weak conjecture, also known as the ternary Goldbach problem, states that every odd number greater than 5 can be expressed as the sum of three primes. (A prime may be used more than once in the same sum). In 2013, Harald Helfgott finally proved Goldbach's weak conjecture, a huge contribution to mathematics and number theory.

The "strong" conjecture has been shown to hold up through 4×10^{18}, but remains unproven for almost 300 years despite considerable effort by many mathematicians throughout history.

The author would like to give many thanks to Harald Helfgott for his proof of the weak conjecture, because this elementary proof of the strong conjecture is completely dependent on Helfgott's proof. Without Helfgott's proof, this elementary proof would not be possible.

Proof

Goldbach's conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states: Every even integer greater than 2 can be expressed as the sum of two primes.

The Goldbach Conjecture states that for every even integer N, and N > 2, then $N = P_1 + P_2$, where P_1, and P_2, are prime numbers.

For example, when N = 4, then 4 = 2 + 2, and since 2 is prime then the Goldbach Conjecture is satisfied. When N = 6, then 6 = 3 + 3, and since 3 is prime then the Goldbach Conjecture is satisfied again.

A proof of the strong Goldbach conjecture implies the ternary Goldbach conjecture, that is, all odd numbers greater than 5 are the sum of three primes. For example, in order to express an odd number $n \geq 5$ as the sum of three primes, subtract 3 and obtain an even number $n - 3 \geq 2$. If the strong conjecture is true, we can express $n – 3$ as a sum of two primes p_1, p_2; thus, $n = (n - 3) + 3$ is the sum of the primes p_1, p_2 and 3, which is the sum of three prime numbers. Thus, proving the ternary Goldbach conjecture, if the strong conjecture is true.

While the weak Goldbach conjecture was finally proved, by Helfgott [1][2] in 2013, however the strong conjecture has remained unsolved. In this paper we shall use Helfgott's proof of the ternary Goldbach conjecture to prove the strong conjecture of even numbers.

Helfgott's proof of the ternary Goldbach conjecture does establish that every even number can be written as the sum of at most 4 primes. For example, subtract any odd prime number, p_4, from every even number, m, that is greater than the prime number being subtracted results with another odd number. That is, $m - p_4$ = an odd number. Now, to Helfgott's credit we can write the odd number, $m - p_4$, as the sum of three primes. This can be written as:

$$m - p_4 = p_1 + p_2 + p_3$$

$$m = p_1 + p_2 + p_3 + p_4$$

Thus, proving every even number can be written as the sum of at most 4 primes. However, to prove that the strong Goldbach conjecture we must reduce this improvement of sum of four primes down to the sum of two primes.

Let n be any odd number, then it is the sum of three primes p_1, p_2, p_3, then the ternary Goldbach conjecture can be written as follows:

$$n = p_1 + p_2 + p_3$$

Subtract p_3 from both sides and the following even number is generated:

$$n - p_3 = p_1 + p_2$$

$$\text{and, } n - p_3 \geq 4$$

This proves that this even number is the sum of two primes, but it does not guarantee that every even number is the sum of two primes. To prove the strong Goldbach conjecture we will state the following conjecture, and then prove this conjecture.

Conjecture: Let p_3 represent any prime number from the infinite set of primes. For every prime number, p_3, $n - p_3 = p_1 + p_2$ represents all even numbers. In other words, let p_3 be any prime number and $n - p_3$ includes all even numbers. Therefore, we are not stating that $n - p_3$ includes all even numbers, but something much more comprehensive, that is, for every prime number p_3, then $n - p_3$ generates every even number.

Since Helfgott has proven the weak Goldbach Conjecture, if we can prove the above conjecture, then we will prove the strong Goldbach Conjecture as a result of Helfgott's proof. The proof is as follows:

Again, $n - p_3 = p_1 + p_2$

For every prime p_3 and an odd number n, such that, $n - p_3 \geq 4$, then let $n = p_3 + 4$, then $(p_3 + 4) - p_3 = 4$. We can continue to increase n by increments of two to generate all even numbers ≥ 4. For example, let $n = p_3 + 6$, then $(p_3 + 6) - p_3 = 6$. Again, let $n = p_3 + 8$, then $(p_3 + 8) - p_3 = 8 = p_1 + p_2$. This can be repeated an infinite number of times to generate all even numbers ≥ 4.

Mathematically this can be stated as follows:

$$\text{Let } k = 2, 3, 4, 5, 6, \ldots$$

$$\text{Or, } k = \text{an integer} > 1$$

Then, $\sum_{k=2}^{\infty}(2k)$ = all even numbers ≥ 4

Then, let $n_k = p_3 + \sum_{k=2}^{\infty}(2k)$

Therefore, for all p_3, $n - p_3$ implies $n_k - p_3 = (p_3 + \sum_{k=2}^{\infty}(2k)) - p_3 = \sum_{k=2}^{\infty}(2k)$

Then, since $n - p_3 = p_1 + p_2$

$n_k - p_3 = p_1 + p_2 = \sum_{k=2}^{\infty}(2k)$ = all even numbers ≥ 4

Thus, any prime number p_3 can generate all even numbers ≥ 4

Thus, $n - p_3 = p_1 + p_2$ = all even numbers ≥ 4, for any prime p_3

Thus, this proves our conjecture, and proves the strong Goldbach Conjecture by this elementary proof, since Helfgott has proven the weak Goldbach Conjecture. We have simply shown that Helfgott's proof of the weak Goldbach Conjecture does imply that the strong Goldbach Conjecture is true.

The tables below are provided to show a few examples of this proof, of course each table can be continued infinitely to show all even numbers.

Table 1.

Odd Numbers (Prime + 2k, where k>1	Prime	Difference
15	11	4
17	11	6
19	11	8
21	11	10
23	11	12
25	11	14
27	11	16
29	11	18
31	11	20
33	11	22
35	11	24
37	11	26
39	11	28
41	11	30
43	11	32
45	11	34

47	11	36
49	11	38
51	11	40

For prime number 1,371,117 the following example is provided in Table 2 below.

Table 2.

Odd Numbers (Prime + 2k, where k>1	Prime	Difference
1371117	1371113	4
1371119	1371113	6
1371121	1371113	8
1371123	1371113	10
1371125	1371113	12
1371127	1371113	14
1371129	1371113	16
1371131	1371113	18
1371133	1371113	20
1371135	1371113	22
1371137	1371113	24
1371139	1371113	26
1371141	1371113	28
1371143	1371113	30
1371145	1371113	32
1371147	1371113	34
1371149	1371113	36
1371151	1371113	38
1371153	1371113	40

References:

1) Helfgott, H.A. (2013). "Major arcs for Goldbach's theorem". arXiv:1305.2897 Freely accessible [math.NT].
2) Helfgott, H.A. (2012). "Minor arcs for Goldbach's problem". arXiv:1205.5252 Freely accessible [math.NT].