ON SURFACE MEASURES ON CONVEX BODIES AND
GENERALIZATIONS OF KNOWN TANGENTIAL IDENTITIES

J. ASPEGREN

Abstract. One theme of this paper is to extend known results from polygons
and balls to the general convex bodies in $n$-dimensions. Another theme
stems from approximating a convex surface with a polytope surface. Our
result gives a sufficient and necessary condition for an natural approximation
method to succeed (in principle) in the case of surfaces of convex bodies. Thus,
Schwarz's paradox does not affect our method. This allows us to define certain
surface measures on surfaces of convex bodies in a novel and simple way.

1. Introduction
Archimedes discovered the 2-dimensional formula for a disk $D(x, r)$

$$\frac{|A|}{|\partial A|} = \frac{1}{2} R.$$  

In the early millennia Apostol and Mnatsakanian extended above equality to the
class of polygons that they called "tangential" [1]. We will prove that if we extend
the definition on tangentiality we obtain in $n$-dimensions for that class of polytopes

$$\frac{|A|}{|\partial A|} = \frac{1}{n} R,$$

where again $R$ is the radius of the inscribed maximal ball (John's ball [2]).

2. Some definitions and known results
A convex body is a compact and convex set. We expect always that $0 \in K$. For
any measurable set $K \subset \mathbb{R}^n$ we use $|K|$ as sign of the Lebesgue measure of that set.
For sets $A, B \subset \mathbb{R}^n$ we define the Minkowski addition as

$$A + B = \{a + b : a \in A \quad b \in B\}.$$  

We define the set dilation as

$$\lambda A = \{\lambda a : a \in A\}.$$  

We always use $B^n_2$ an unit ball in $n$ dimensions. The surface area of an convex
body is defined as

$$|\partial(K)| := \lim_{\epsilon \to 0} \frac{|K + \epsilon B^n_2| - |K|}{\epsilon}.$$  

We use $H_\theta$ for the $n-1$ dimensional hyperplane going through the origo and
orthogonal to $\theta \in S^n$. In our sense a polyhedra is an intersection of finite number

1991 Mathematics Subject Classification. 52A20, 52B11.

Key words and phrases. Convex Geometry, Polytope Approximation, Surface Measures.
of closed half-spaces containing origo. A polytype is an polyhedra with a finite Lebesque measure. See for example [4] and references therein. For convex bodies in John’s position, the inradius $R$ is defined as the radius of the John’s ball [2].

3. On Polytypes and Normal Points

A convex body can be defined as a compact set

$$ K = \bigcap_{i \in I} H^i, $$

where each $H^i$ is the half space containing origo that is bounded by $H$ [4]. It can be proved that in each $n - 1$-dimensional hyperplane $H$ there exists an unique point $x$ with the vector $x$ orthogonal to the hyperplane [4]. These points $x_i$ are the unique solutions to the minimization problems

$$ \min_{y \in H} ||y|| = ||x_i||, $$

where for each hyperplane $y \in H$ and $H$ is the boundary of a half space used to define $K$. We call these points normal points. They depend on the position of $K$. They exists as long as the origo belongs to the interior of $K$. We prove that the convex body inherits the normal points from the half spaces used to define it.

**Theorem 1.** Let $K$ be a convex set containing origo. If the halfspace $H$ is used in the definition of $K$, then the normal point of $H$ belongs $K \cap H$.

**Proof.** If $x_i$ is defined as the solution to

$$ <y, x_i> = 0, $$

then a closed origo centered ball of radius $||x||_2$ contains only that point from the hyperplane $H$, and because the ball has that common supporting hyperplane with $K$, we have the claim. \( \square \)

In light of the theorem the next corollary is clear.

**Corollary 1.** Let $P$ be a polytope. Then each facet of $P$ contains an unique normal point in it’s interior.

We define minimal normal points as follows. They are the normal points of minimal value $||.||_2$ norm. We will use the following definition.

**Definition 1.** A convex body is tangential iff every normal point is minimal.

With respect to any normal point $x_i$ we denote $F_i$ as the corresponding facet. For each $K$ we define the Borel space $(K, \mathcal{A})$. We consider the finite intersection of half spaces used to define $K$. For each intersection $\bigcap_{i=1}^m H'$ of half spaces forming a polytope and used to define $K$ we define a probability measure

$$ \mu_j := \frac{\sum_{i=1}^m |F_i| \delta_{x_i}(x)}{\sum_{i=1}^m |F_i|}, $$

where every $\delta_{x_i}(x)$ is a Dirac measure.

**Theorem 2.** Let $x_1, ..., x_m$ be the normal points of a polytope $P$, then

$$ \frac{1}{n} \mathbb{E}(||x_i||) = \frac{|A|}{|\partial A|}. $$
ON SURFACE MEASURES ON CONVEX BODIES AND GENERALIZATIONS OF KNOWN TANGENTIAL IDENTITIES

Proof. We know that a volume of an $n$-dimensional cone $A_i$ is given by

$$|A_i| = \frac{|\text{base}| \times \text{height}}{n}.$$

The theorem 1 implies that the polytope $P$'s volume is

$$|P| = \frac{1}{n} \sum_{i=1}^{m} ||x_i||F_i|,$$

where $F_i$ is the facet corresponding to the normal point $x_i$. Thus,

$$\frac{|P|}{|\partial P|} = \frac{1}{n} \sum_{i=1}^{m} ||x_i||F_i| = \frac{1}{n} E(||x_i||).$$

□

Corollary 2. Let a polytope $P$ be tangential, then

$$\frac{1}{n} R = \frac{|A|}{|\partial A|},$$

where $R$ is the inradius of $K$.

Naturally we want to extend the theorem and it's corollary to the general convex bodies. We will first do that for the corollary. For that we need a lemma.

Lemma 1. For any point $x \in \partial K$, there exists a normal point $x'$ s.t $x \in H_{x'}$.

Proof. Now, $x$ belongs to some supporting hyperplane. To this supporting hyperplane there corresponds at least one normal point and via theorem 1 the normal point belongs to $\partial K$. □

Next we proof that the space of normal points $N$ is separable as a metric compact space.

Theorem 3. The space of normal points is closed in the subspace topology and the space is separable.

Proof. Suppose that there exists a Cauchy sequence of normal points $(x_j)$ s.t

$$\lim_{j=1} x_j = x.$$

It holds that $x \in \partial K$ because the surface is closed. Thus, via lemma 1 it holds that $x \in H_{x'}$, where $x'$ is a normal point. Now, if $y \in H_{x'}$ it holds that

$$\lim_{j \to \infty} <x_j, x' - y> = <x, x' - y> = 0$$

because of the continuity of the inner product. Thus, $H_{x'}$ is orthogonal to $x$ and $x' = x$. In addition it's well known that every compact metric space is separable [9]. □

For any sequence of normal points $(x_j)$ we denote the corresponding sequence of nested polyhedras to be the sequence $\cap_{j=1}^{m} H_{x_j}$. The following theorem is a key theorem.
Theorem 4. Let \((x_j)\) be any sequence of normal points. Then for the corresponding nested sequence of polyhedras \((P_m)\), that have facets containing the normal points \(x_i \in P_m\), it holds that

\[
\bigcap_{m=1}^{\infty} P_m = K,
\]

iff the set of the corresponding normal points is dense in the space of normal points of \(K\).

Proof. Let the set of normal points corresponding to the sequence \((x_j)\) be dense in \(N\). Because \(\bigcap_{m=1}^{\infty} P_m\) is some convex body, the set of it's normal points is closed. Thus, it must be that \(K = \bigcap_{m=1}^{\infty} P_m\). On the other hand, if the sequence of normal points is not dense in \(N\), then there exist a normal point of \(K\) and a halfspace used in the definition of \(K\) that has a boundary that is not a supporting hyperplane to \(\bigcap_{m=1}^{\infty} P_m\). Thus,

\[
\bigcap_{i=1}^{\infty} P_m \neq K.
\]

□

Corollary 3. It holds that

\[
\lim_{m \to \infty} |P_m| = |K|
\]

iff the set of normal points corresponding to the polyhedras is dense in \(N\).

Proof. We need only to prove the following. If the set of normal points is not dense then

\[
|K| < \bigcap_{i=1}^{\infty} P_m|.
\]

Via the theorem 4 the sets are not the same. So there exists \(x \in \bigcap_{i=1}^{\infty} P_m \cap K^c\).

Now,

\[
\min_{y_i \in K} ||x - y_i|| = ||x - y|| > 0,
\]

because there exists a hyperplane separating \(K\) and \(x\). It follows that

\[
|K| < |\text{conv}(x, K)| \leq |\bigcap_{i=1}^{\infty} P_m|,
\]

because there exists a small ball \(B(x, \epsilon) \subset \bigcap_{i=1}^{\infty} P_m \cap K^c\). □

4. On normal measures on convex bodies

In this section we will define a surface measure on convex surfaces in \(n\) dimensions. In general the measure will be a mixed measure, because it's discrete in the segments of an convex body (for example on the ends of an cylinder). First we will prove the existence of the measure. For the sake of completeness we prove a lemma.

Lemma 2. For nested polytopes it holds that

\[
|\partial(P_{m+1})| \leq |\partial(P_m)|.
\]
Proof. Now,
\[ \partial P_m = \partial (P_{m+1} \cup P_m / P_{m+1}). \]
Thus,
\[ |\partial P_{m+1}| \leq |\partial P_m|, \]
via additivity of the surface volume.
\[ \square \]

We use the sequences of measures defined in the last section.

**Theorem 5.** For any Borel set \( A \subset K \) the sequence \( (\mu_j(A)) \) is monotonic.

*Proof.* We can estimate \( \mu_j(A) \leq \mu_j(K) = 1 \). We need to prove that \( \mu_j(A) \leq \mu_{j+1}(A) \).

Now,
\[ \mu_j(A) = \sum_{i=1}^m |F_i| \delta_{x_i}(x) \sum_{i=1}^m |F_i| = |\partial P'_j|/|\partial P_j|, \]
where \( \partial P'_j \) is the union of those facets that contain the normal points of \( A \cap P_j \).

Now, \( \partial P'_{j+1} \) contains those and at least one more normal point, and in addition \( \partial P'_{j+1} \subset \partial P'_j \). So we have
\[ \mu_j(A) \leq |\partial P'_{j+1}|/|\partial P_{j+1}| = \mu_{j+1}(A), \]
via previous lemma 2.
\[ \square \]

The total variation distance of probability measures is defined as
\[ ||\mu - v||_{TV} := 2 \sup_A |\mu(A) - v(A)|. \]

**Theorem 6.** The set function defined by \( \lim_{j \to \infty} u_j = \mu \) is a Radon measure on \((K, \mathcal{X})\), where the metric is understood in the total variation sense.

*Proof.* By lemma 5 and Riesz representation theorem \( K \lim_{j \to \infty} u_j = \mu \) uniformly. Thus \( u_j \) is a Radon measure. See for example [8] and [6].
\[ \square \]

We need the following lemma.

**Lemma 3.** Suppose that every normal point of \( K' \) is a normal point of \( K \). Then for normal measures \( \mu \) and \( v \) of \( K \subset K' \) and \( K' \), respectively, it holds that
\[ \int_{K'} |f| \mu \leq \int_K |f| v. \]

*Proof.* It’s enough to prove the theorem for characteristic functions and use a standard limit process. Now,
\[ \mu(P_m \cap A) \leq v(P_m \cap A), \]
for any polytopes by definition (2) of the last section and the normal point condition. Thus,
\[ \mu(A) = \lim_{m \to \infty} \mu(P_m \cap A) \leq \lim_{m \to \infty} v_j(P_m \cap A) = v(A), \]
by continuity of measures for nested sequence of polyhedras.
\[ \square \]
We need to prove that the measure does not depend on the sequence of polytopes, if the sequence of normal points is dense.

**Theorem 7.** If $A$ is Borel set it holds that

\begin{equation}
\lim_{j \to \infty} \int_{K} 1_{A}\mu_j = \int_{K} 1_{A}\mu = \lim_{l \to \infty} \int_{K} 1_{A}v_l = \int_{K} 1_{A}v.
\end{equation}

**Proof.** If $v(A) < \mu(A)$, then there exists $P_l$ s.t

$v_l(A) < \mu(A),
\]

thus via previous lemma 3 $K$ does not contain the normal points of $P_l$ which is a contradiction. □

For each convex body $K$ we call the measure just defined the normal measure of $K$.

Next we generalize the result 2 from polytopes to general convex bodies.

**Theorem 8.** For any convex body it holds that

\begin{equation}
\frac{1}{n} \mathbb{E}[||x||] = \frac{|K|}{|\partial K|},
\end{equation}

where $x \in X$ are the normal points of $K$ and the expected value is understood with respect to the measure given by the theorem 6.

**Proof.** Now, via theorem 3

\[
\int 1_{A}\mu_j \leq \int 1_{A}\mu_{j+1},
\]

for all closed sets. It follows that

\begin{equation}
\mathbb{E}(||f||)\mu_j \to \mathbb{E}(||f||)\mu
\end{equation}

for all bounded continuous functions. □

The tangential case is an easy corollary.

**Corollary 4.** For tangential convex body $K$, it holds that

\[
\frac{R}{n} = \frac{|K|}{|\partial K|},
\]

where $R$ is the inradius of $K$.

The next corollary is an analogue of the corollary 3 from the last section.

**Corollary 5.**

\begin{equation}
\lim_{m \to \infty} |\partial P_m| = |\partial K|
\end{equation}

iff the set of normal points corresponding to the polyhedra is dense in $X$.

**Proof.** Via the theorem 4 we need only to prove that if

\[
|\partial K| = |\partial \bigcap_{i=1}^{\infty} P_m|,
\]
then the set of normal points is dense. Now, if it holds that 
\[
|K| < \left| \bigcap_{i=1}^{\infty} P_m \right|.
\]

It follows that 
\[
|\partial\left(\bigcap_{i=1}^{\infty} P_m \right)| - |\partial K| = \lim_{\epsilon \to 0} \frac{\left| \bigcap_{i=1}^{\infty} P_m + \epsilon B_2^n \right| - \left| \bigcap_{i=1}^{\infty} P_m \right| - |K + \epsilon B_2^n| + |K|}{\epsilon} < 0
\]
which is a contradiction. 

REFERENCES


E-mail address: jaspengren@outlook.com