On Neutrosophic Soft Topological Space

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Abstract: In this paper, the concept of connectedness and compactness on neutrosophic soft topological space have been introduced along with the investigation of their several characteristics. Some related theorems have been established also. Then, the notion of neutrosophic soft continuous mapping on a neutrosophic soft topological space and it’s properties are developed here.

Keywords: Connectedness and compactness on neutrosophic soft topological space, Neutrosophic soft continuous mapping.

1 Introduction

Zadeh’s \cite{1} classical concept of fuzzy sets is a strong mathematical tool to deal with the complexity generally arising from uncertainty in the form of ambiguity in real life scenario. Researchers in economics, sociology, medical science and many other several fields deal daily with the vague, imprecise and occasionally insufficient information of modeling uncertain data. For different specialized purposes, there are suggestions for nonclassical and higher order fuzzy sets since from the initiation of fuzzy set theory. Among several higher order fuzzy sets, intuitionistic fuzzy sets introduced by Atanassov \cite{2} have been found to be very useful and applicable. But each of these theories has it’s different difficulties as pointed out by Molodtsov \cite{3}. The basic reason for these difficulties is inadequacy of parametrization tool of the theories.

Molodtsov \cite{3} presented soft set theory as a completely generic mathematical tool which is free from the parametrization inadequacy syndrome of different theory dealing with uncertainty. This makes the theory very convenient, efficient and easily applicable in practice. Molodtsov \cite{3} successfully applied several directions for the applications of soft set theory, such as smoothness of functions, game theory, operation reaserch, Riemann integration, Perron integration and probability etc. Now, soft set theory and it’s applications are progressing rapidly in different fields. Shabir and Naz \cite{4} presented soft topological spaces and defined some concepts of soft sets on this spaces and separation axioms. Moreover, topological structure on fuzzy, fuzzy soft, intuitionistic fuzzy and intuitionistic fuzzy soft set was defined by Coker \cite{5}, Li and Cui \cite{6}, Chang \cite{7}, Tanay and Kandemir \cite{8}, Osmanoglu and Tokat \cite{9}, Neog et al. \cite{10}, Varol and Aygun \cite{11}, Bayramov and Gunduz \cite{12,13}. Turanh and Es \cite{14} defined compactness in intuitionistic fuzzy soft topological spaces.

The concept of Neutrosophic Set (NS) was first introduced by Smarandache \cite{15,16} which is a generalisation of classical sets, fuzzy set, intuitionistic fuzzy set etc. Later, Maji \cite{17} has introduced a combined concept Neutrosophic soft set (NSS).

Using this concept, several mathematicians have produced their research works in different mathematical structures for instance Arockiarani et al.\cite{18,19}, Bera and Mahapatra \cite{20}, Deli \cite{21,22}, Deli and Broumi \cite{23}, Maji \cite{24}, Broumi and Smarandache \cite{25}, Salama and Albshowi \cite{26}, Saroja and Kalaichelvi \cite{27}, Broumi \cite{28}, Sahin et al.\cite{29}. Later, this concept has been modified by Deli and Broumi \cite{30}. Accordingly, Bera and Mahapatra \cite{31-36} have developed some algebraic structures over the neutrosophic soft set.

The present study introduces the notion of connectedness, compactness and neutrosophic soft continuous mapping on a neutrosophic soft topological space. Section 2 gives some preliminary necessary definitions which will be used in rest of this paper. The notion of connectedness and compactness on neutrosophic soft topological spaces along with investigation of related properties have been introduced in Section 3 and Section 4, respectively. The concept of neutrosophic soft continuous mapping has been developed in Section 5. Finally, the conclusion of the present work has been stated in Section 6.

2 Preliminaries

In this section, we recall some necessary definitions and theorems related to fuzzy set, soft set, neutrosophic set, neutrosophic soft set, neutrosophic soft topological space for the sake of completeness. Unless otherwise stated, \( E \) is treated as the parametric set through out this paper and \( e \in E \), an arbitrary parameter.

2.1 Definition \cite{31}

1. A binary operation \( * : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is continuous \( t \)-norm if \( * \) satisfies the following conditions:

(i) \( * \) is commutative and associative.

(ii) \( * \) is continuous.
Let $X$ be a space of points (objects), with a generic element in $X$ denoted by $x$. A neutrosophic set $A$ in $X$ is characterized by a truth-membership function $T_A$, an indeterminacy-membership function $I_A$ and a falsity-membership function $F_A$. $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $[0, 1]$. That is $T_A, I_A, F_A : X \rightarrow [0, 1]$. There is no restriction on the sum of $T_A(x), I_A(x), F_A(x)$ and so, $0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$. 

2.2 Definition [15] 
Let $X$ be a space of points (objects), with a generic element in $X$ denoted by $x$. A neutrosophic set $A$ in $X$ is characterized by a truth-membership function $T_A$, an indeterminacy-membership function $I_A$ and a falsity-membership function $F_A$. $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $[0, 1]$. That is $T_A, I_A, F_A : X \rightarrow [0, 1]$. There is no restriction on the sum of $T_A(x), I_A(x), F_A(x)$ and so, $0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$. 

2.3 Definition [3] 
Let $U$ be an initial universe set and $E$ be a set of parameters. Let $P(U)$ denote the power set of $U$. Then for $A \subseteq E$, a pair $(F, A)$ is called a soft set over $U$, where $F : A \rightarrow P(U)$ is a mapping. 

2.4 Definition [17] 
Let $U$ be an initial universe set and $E$ be a set of parameters. Let $NS(U)$ denote the set of all NSSs of $U$. Then for $A \subseteq E$, a pair $(F, A)$ is called an NSS over $U$, where $F : A \rightarrow NS(U)$ is a mapping. 

This concept has been modified by Deli and Broumi [30] as given below. 

2.5 Definition [30] 
Let $U$ be an initial universe set and $E$ be a set of parameters. Let $NS(U)$ denote the set of all NSSs of $U$. Then, a neutrosophic soft set $N$ over $U$ is a set defined by a set valued function $f_N$ representing a mapping $f_N : E \rightarrow NS(U)$ where $f_N$ is called approximating function of the neutrosophic soft set $N$. In other words, the neutrosophic soft set is a parameterized family of some elements of the set $NS(U)$ and therefore it can be written as a set of ordered pairs, 

$$N = \{ (e, < x, T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x) > : x \in U ) : e \in E \}$$ 

where $T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x) \in [0, 1]$, respectively called the truth-membership, indeterminacy-membership, falsity-membership function of $f_N(e)$. Since supremum of each $T, I, F$ is 1 so the inequality $0 \leq T_{f_N(e)}(x) + I_{f_N(e)}(x) + F_{f_N(e)}(x) \leq 3$ is obvious. 

2.5.1 Example 
Let $U = \{ h_1, h_2, h_3 \}$ be a set of houses and $E = \{ e_1(\text{beautiful}), e_2(\text{wooden}), e_3(\text{costly}) \}$ be a set of parameters with respect to which the nature of houses are described. Let, 

$$f_N(e_1) = < h_1, (0.8, 0.6, 0.3) >, < h_2, (0.4, 0.7, 0.6) >, < h_3, (0.6, 0.2, 0.3) >;$$

$$f_N(e_2) = < h_1, (0.6, 0.3, 0.5) >, < h_2, (0.7, 0.4, 0.3) >, < h_3, (0.8, 0.1, 0.2) >;$$

$$f_N(e_3) = < h_1, (0.7, 0.4, 0.3) >, < h_2, (0.6, 0.7, 0.2) >, < h_3, (0.7, 0.2, 0.5) >;$$

Then $N = \{ (e_1, f_N(e_1)), (e_2, f_N(e_2)), (e_3, f_N(e_3)) \}$ is an NSS over $(U, E)$. The tabular representation of the NSS $N$ is as : 

<table>
<thead>
<tr>
<th>$f_N(e_1)$</th>
<th>$f_N(e_2)$</th>
<th>$f_N(e_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0.5, 0.6, 0.3)$</td>
<td>$(0.6, 0.3, 0.5)$</td>
<td>$(0.7, 0.4, 0.3)$</td>
</tr>
<tr>
<td>$(0.4, 0.7, 0.6)$</td>
<td>$(0.7, 0.4, 0.3)$</td>
<td>$(0.6, 0.7, 0.2)$</td>
</tr>
<tr>
<td>$(0.6, 0.2, 0.3)$</td>
<td>$(0.8, 0.1, 0.2)$</td>
<td>$(0.7, 0.2, 0.5)$</td>
</tr>
</tbody>
</table>

2.6 Definition [30] 
1. The complement of a neutrosophic soft set $N$ is denoted by $N^c$ and is defined by 

$$N^c = \{ (e, < x, F_{f_N(e)}(x), 1 - I_{f_N(e)}(x), T_{f_N(e)}(x) > : x \in U ) : e \in E \}$$

2. Let $N_1$ and $N_2$ be two NSSs over the common universe $(U, E)$. Then $N_1$ is said to be the neutrosophic soft subset of $N_2$ if $\forall e \in E \in U$, 

$$T_{f_{N_1}(e)}(x) \leq T_{f_{N_2}(e)}(x), I_{f_{N_1}(e)}(x) \geq I_{f_{N_2}(e)}(x), F_{f_{N_1}(e)}(x) \geq F_{f_{N_2}(e)}(x).$$

We write $N_1 \subseteq N_2$ and then $N_2$ is the neutrosophic soft superset of $N_1$. 

2.7 Definition [30] 
1. Let $N_1$ and $N_2$ be two NSSs over the common universe $(U, E)$. Then their union is denoted by $N_1 \cup N_2 = N_3$ and is defined as : 

$$N_3 = \{ (e, < x, T_{f_{N_1}(e)}(x), I_{f_{N_1}(e)}(x), F_{f_{N_1}(e)}(x) > : x \in U ) : e \in E \}$$

where $T_{f_{N_1}(e)}(x) = T_{f_{N_1}(e)}(x) \circ T_{f_{N_2}(e)}(x), I_{f_{N_1}(e)}(x) = I_{f_{N_1}(e)}(x) \ast I_{f_{N_2}(e)}(x), F_{f_{N_1}(e)}(x) = F_{f_{N_1}(e)}(x) \ast F_{f_{N_2}(e)}(x).$ 

2. Their intersection is denoted by $N_1 \cap N_2 = N_4$ and is defined as :
\( N_4 = \{(e, \{< x, T_{fN_4}(e)(x), I_{fN_4}(e)(x), F_{fN_4}(e)(x) > : x \in U\}) : e \in E\} \)

where \( T_{fN_4}(e)(x) = T_{fN_1}(e)(x) \ast T_{fN_2}(e)(x), I_{fN_4}(e)(x) = I_{fN_1}(e)(x) \circ I_{fN_2}(e)(x), F_{fN_4}(e)(x) = F_{fN_1}(e)(x) \circ F_{fN_2}(e)(x) \).

### 2.8 Definition [33]

1. Let \( M, N \) be two NSSs over \((U, E)\). Then \( M - N \) may be defined as, \( \forall x \in U, e \in E, \)

\[
M - N = \{< x, T_{fM}(e)(x) \ast F_{fM}(e)(x), I_{fM}(e)(x) \circ (1 - I_{fM}(e)(x)), F_{fM}(e)(x) \circ T_{fM}(e)(x) >\}
\]

2. A neutrosophic soft set \( N \) over \((U, E)\) is said to be null neutrosophic soft set if \( T_{fN}(e)(x) = 0, I_{fN}(e)(x) = 1, F_{fN}(e)(x) = 1, \forall e \in E, \forall x \in U \). It is denoted by \( \phi_u \).

A neutrosophic soft set \( N \) over \((U, E)\) is said to be absolute neutrosophic soft set if \( T_{fN}(e)(x) = 1, I_{fN}(e)(x) = 0, F_{fN}(e)(x) = 0, \forall x \in U \). It is denoted by \( 1_u \).

Clearly, \( \phi_u = 1_u \).

### 2.9 Definition [33]

Let \( NSS(U, E) \) be the family of all neutrosophic soft sets over \( U \) via parameters in \( E \) and \( \tau_u \subset NSS(U, E) \). Then \( \tau_u \) is called neutrosophic soft topology on \((U, E)\) if the following conditions are satisfied:

(i) \( \phi_u, 1_u \in \tau_u \)

(ii) The intersection of any finite number of members of \( \tau_u \) also belongs to \( \tau_u \).

(iii) The union of any collection of members of \( \tau_u \) belongs to \( \tau_u \).

Then the triplet \((U, E, \tau_u)\) is called a neutrosophic soft topological space. Every member of \( \tau_u \) is called \( \tau_u \)-open neutrosophic soft set. An NSS is called \( \tau_u \)-closed iff it’s complement is \( \tau_u \)-open. There may be a number of topologies on \((U, E)\). If \( \tau_{u_1} \) and \( \tau_{u_2} \) are two topologies on \((U, E)\) such that \( \tau_{u_1} \subset \tau_{u_2} \), then \( \tau_{u_1} \) is called neutrosophic soft strictly weaker (coarser) than \( \tau_{u_2} \) and in that case \( \tau_{u_2} \) is neutrosophic soft strict finer than \( \tau_{u_1} \). Moreover \( NSS(U, E) \) is a neutrosophic soft topology on \((U, E)\).

### 2.9.1 Example

1. Let \( U = \{h_1, h_2\}, E = \{e_1, e_2\} \) and \( \tau_u = \{\phi_u, 1_u, N_1, N_2, N_3, N_4\} \) where \( N_1, N_2, N_3, N_4 \) being NSSs are defined as following:

\[
\begin{align*}
\text{f}_{N_1}(e_1) &= \{< h_1, (1, 0, 1) >, < h_2, (0, 0, 1) >\}, \\
\text{f}_{N_1}(e_2) &= \{< h_1, (0, 1, 0) >, < h_2, (1, 0, 0) >\}; \\
\text{f}_{N_2}(e_1) &= \{< h_1, (0, 1, 0) >, < h_2, (1, 1, 0) >\}, \\
\text{f}_{N_2}(e_2) &= \{< h_1, (1, 0, 1) >, < h_2, (0, 1, 1) >\}; \\
\end{align*}
\]

Here \( N_1 \cap N_1 = N_1, N_1 \cap N_2 = \phi_u, N_1 \cap N_3 = N_3, N_1 \cap N_4 = N_4 \).

2. Let \( U = \{x_1, x_2, x_3\}, E = \{e_1, e_2\} \) and \( \tau_u = \{\phi_u, 1_u, N_1, N_2, N_3\} \) where \( N_1, N_2, N_3 \) being NSSs over \((U, E)\) are defined as follow:

\[
\begin{align*}
\text{f}_{N_1}(e_1) &= \{< x_1, (0, 0, 0.5, 0.4) >, < x_2, (0.6, 0.6, 0.6) >, < x_3, (0.5, 0.6, 0.4) >\}, \\
\text{f}_{N_1}(e_2) &= \{< x_1, (0.8, 0.4, 0.5) >, < x_2, (0.7, 0.7, 0.3) >, < x_3, (0.7, 0.5, 0.6) >\}; \\
\text{f}_{N_2}(e_1) &= \{< x_1, (0.8, 0.5, 0.6) >, < x_2, (0.5, 0.7, 0.6) >, < x_3, (0.4, 0.7, 0.5) >\}, \\
\text{f}_{N_2}(e_2) &= \{< x_1, (0.7, 0.6, 0.5) >, < x_2, (0.6, 0.8, 0.4) >, < x_3, (0.5, 0.8, 0.6) >\}; \\
\text{f}_{N_3}(e_1) &= \{< x_1, (0.6, 0.6, 0.7) >, < x_2, (0.4, 0.8, 0.8) >, < x_3, (0.3, 0.8, 0.6) >\}; \\
\text{f}_{N_3}(e_2) &= \{< x_1, (0.5, 0.8, 0.6) >, < x_2, (0.5, 0.9, 0.5) >, < x_3, (0.2, 0.9, 0.7) >\}; \\
\end{align*}
\]

The \( t \)-norm and \( s \)-norm are defined as \( a \ast b = \max\{a + b - 1, 0\} \) and \( a \circ b = \min\{a + b, 1\} \). Here \( N_1 \cap N_1 = N_1, N_1 \cap N_2 = N_2, N_1 \cap N_3 = N_3, N_1 \cap N_4 = N_4 \).

### 3. Let \( NSS(U, E) \) be the family of all neutrosophic soft sets over \((U, E)\). Then \( \phi_u, 1_u \) and \( NSS(U, E) \) are two examples of the neutrosophic soft topology over \((U, E)\). They are called, respectively, indiscrete (trivial) and discrete neutrosophic soft topology. Clearly, they are the smallest and largest neutrosophic soft topology on \((U, E)\), respectively.

### 2.10 Definition [33]

Let \((U, E, \tau_u)\) be a neutrosophic soft topological space over \((U, E)\) and \( M \in NSS(U, E) \) be arbitrary. Then the interior of \( M \) is denoted by \( M^o \) and is defined as:

\[
M^o = \bigcup\{N_1 : N_1 \text{is neutrosophic soft open and } N_1 \subset M\}
\]

i.e., it is the union of all open neutrosophic soft subsets of \( M \).
2.10.1 Theorem [33]

Let \((U, E, \tau_u)\) be a neutrosophic soft topological space over \((U, E)\) and \(M, P \in \text{NSS}(U, E)\). Then,
(i) \(M^0 \subset M\) and \(M^0\) is the largest open set.
(ii) \(M \subset P \Rightarrow M^0 \subset P^0\).
(iii) \(M^0\) is an open neutrosophic soft set i.e., \(M^0 \in \tau_u\).
(iv) \(M\) is neutrosophic soft open set iff \(M^0 = M\).
(v) \((M^0)^o = M^o\).
(vi) \((\phi_u)^o = \phi_u\) and \(1_u^o = 1_u\).
(vii) \((M \cap P)^o = M^o \cap P^o\).
(viii) \(M^0 \cap P^0 \subset (M \cup P)^o\).

2.11 Definition [33]

Let \((U, E, \tau_u)\) be a neutrosophic soft topological space over \((U, E)\) and \(M \in \text{NSS}(U, E)\) be arbitrary. Then the closure of \(M\) is denoted by \(\overline{M}\) and is defined as :
\[
\overline{M} = \cap \{N_1 : N_1 \text{ is neutrosophic soft closed and } N_1 \supset M\}
\]
i.e., it is the intersection of all closed neutrosophic soft supersets of \(M\).

2.11.1 Theorem [33]

Let \((U, E, \tau_u)\) be a neutrosophic soft topological space over \((U, E)\) and \(M, P \in \text{NSS}(U, E)\). Then,
(i) \(M \subset \overline{M}\) and \(\overline{M}\) is the smallest closed set.
(ii) \(M \subset P \Rightarrow \overline{M} \subset \overline{P}\).
(iii) \(\overline{M}\) is closed neutrosophic soft set i.e., \(\overline{M} \in \tau_u\).
(iv) \(M\) is neutrosophic soft closed set iff \(\overline{M} = M\).
(v) \(\overline{\overline{M}} = \overline{M}\).
(vi) \(\overline{\phi_u} = \phi_u\) and \(1_u = 1_u\).
(vii) \(\overline{M \cup P} \subset \overline{M} \cup \overline{P}\).
(viii) \(\overline{M \cap P} \subset \overline{M} \cap \overline{P}\).

2.11.2 Theorem [33]

Let \((U, E, \tau_u)\) be a neutrosophic soft topological space over \((U, E)\) and \(M \in \text{NSS}(U, E)\). Then,
(i) \((\overline{M})^o = (M^o)^c\)
(ii) \((M^o)^c = (M^c)^o\)

2.12 Definition [33]

1. A neutrosophic soft point in an NSS \(N\) is defined as an element \((e, f_N(e))\) of \(N\), for \(e \in E\) and is denoted by \(e_N\), if \(f_N(e) \notin \phi_u\) and \(f_N(e') \in \phi_u, \forall e' \in E - \{e\}\).
2. The complement of a neutrosophic soft point \(e_N\) is another neutrosophic soft point \(e_N^c\) such that \(f_N^c(e) = (f_N(e))^c\).
3. A neutrosophic soft point \(e_N \in M, M\) being an NSS if for the element \(e \in E\), \(f_N(e) \leq f_M(e)\).

2.12.1 Example

Let \(U = \{x_1, x_2, x_3\}\) and \(E = \{e_1, e_2\}\). Then,
\[
e_{1N} = \{< x_1, (0.6, 0.4, 0.8) >, < x_2, (0.8, 0.3, 0.5) >, < x_3, (0.3, 0.7, 0.6) >\}
\]
is a neutrosophic soft point whose complement is
\[
e_{1N}^c = \{< x_1, (0.8, 0.6, 0.6) >, < x_2, (0.5, 0.7, 0.8) >, < x_3, (0.6, 0.3, 0.3) >\}.
\]
For another NSS \(M\) defined on same \((U, E)\), let,
\[
f_M(e_1) = \{< x_1, (0.7, 0.4, 0.7) >, < x_2, (0.8, 0.2, 0.4) >, < x_3, (0.5, 0.6, 0.5) >\}.
\]
Then, \(f_N(e_1) \leq f_M(e_1)\) i.e., \(e_{1N} \in M\).

2.13 Definition [33]

Hausdorff space : Let \((U, E, \tau_u)\) be a neutrosophic soft topological space over \((U, E)\). For two distinct neutrosophic soft points \(e_K, e_S\), if there exists disjoint neutrosophic soft open sets \(M, P\) such that \(e_K \in M\) and \(e_S \in P\) then \((U, E, \tau_u)\) is called T\(2\) space or Hausdorff space.

2.13.1 Example

Let \(U = \{h_1, h_2\}, E = \{e\}\) and \(\tau_u = \{\phi_u, 1_u, M, P\}\) where \(M, P\) being neutrosophic soft subsets of \(N\) are defined as following :
\[
f_M(e) = \{< h_1, (1, 0, 1) >, < h_2, (0, 0, 1) >\};
\]
\[
f_P(e) = \{< h_1, (0, 1, 0) >, < h_2, (1, 1, 0) >\};
\]
Then \(\tau_u\) is a neutrosophic soft topology on \((U, E)\) with respect to the t-norm and s-norm defined as \(a \ast b = \max\{a + b - 1, 0\}\) and \(a \circ b = \min\{a + b, 1\}\). Here \(e_M \in M\) and \(e_P \in P\) with \(e_M \neq e_P\) and \(M \cap P = \phi_u\).

2.14 Definition [33]

Let \((U, E, \tau_u)\) be a neutrosophic soft topological space over \((U, E)\) where \(\tau_u\) is a topology on \((U, E)\) and \(M \in \text{NSS}(U, E)\) an arbitrary NSS. Suppose \(\tau_M = \{M \cap N_i : N_i \in \tau_u\}\). Then \(\tau_M\) forms also a topology on \((U, E)\). Thus \((U, E, \tau_M)\) is a neutrosophic soft topological subspace of \((U, E, \tau_u)\).

2.14.1 Example

Let us consider the example (2) in [2.9.1]. We define \(M \in \text{NSS}(U, E)\) as following :
\[
f_M(e_1) = \{< x_1, (0.4, 0.6, 0.8) >, < x_2, (0.7, 0.3, 0.2) >, < x_3, (0.5, 0.5, 0.7) >\};
\]
\[
f_M(e_2) = \{< x_1, (0.6, 0.3, 0.5) >, < x_2, (0.4, 0.7, 0.6) >, < x_3, (0.8, 0.3, 0.5) >\};
\]
We denote \(M \cap \phi_u = \phi_M, M \cap 1_u = 1_M, M \cap N_1 = M_1, M \cap N_2 = M_2, M \cap N_3 = M_3\); Then \(M_1, M_2, M_3\) are defined as following :

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3.1 Definition

A neutrosophic soft topological space \((U, E, \tau_u)\) is said to be neutrosophic soft disconnected if there does not exist a neutrosophic soft separation of \((U, E, \tau_u)\). Otherwise, \((U, E, \tau_u)\) is called neutrosophic soft connected.

The topological space in the Example (2) of [2.9.1] is connected but (1) of [2.9.1] is disconnected.

3.4 Theorem

A neutrosophic soft topological space \((U, E, \tau_u)\) is said to be neutrosophic soft disconnected iff there exists a nonempty proper neutrosophic soft subset of \(1_u\) which is both neutrosophic soft open and neutrosophic soft closed.

**Proof.** Let \(M \subset 1_u, M \neq \phi_u\) and \(M\) is both neutrosophic soft open and closed. Then \(M^c \subset 1_u, M^c \neq \phi_u\) and \(M^c\) is both neutrosophic soft open and closed, also. Let \(P = M^c\). Then \(M = M^c = \overline{P} = P\). Thus \(1_u\) can be expressed as the union of two separated neutrosophic soft sets \(M, P\) and so, is neutrosophic soft disconnected.

Conversely, let \(1_u\) be neutrosophic soft disconnected. Then there exists nonempty neutrosophic soft open sets \(N_1, N_2\) such that \(1_u = N_1 \cup N_2\) and \(N_1 \cap N_2 = \phi_u\). Then \(N_1 = N_2^c\) i.e., \(N_1\) is closed, also. Similarly, \(N_2 = N_1^c\) and so, \(N_2\) is closed.

3.5 Theorem

A neutrosophic soft topological space \((U, E, \tau_u)\) is said to be neutrosophic soft connected iff there exists neutrosophic soft sets in \(NSS(U, E)\) which are both neutrosophic soft open and neutrosophic soft closed, are \(\phi_u\) and \(1_u\).

**Proof.** Let \((U, E, \tau_u)\) be a connected neutrosophic soft topological space. For contrary, suppose that \(M\) is both neutrosophic soft open and closed different from \(\phi_u, 1_u\). Then \(M^c\) is also both neutrosophic soft open and closed different from \(\phi_u, 1_u\). Also \(M \cap M^c = \phi_u\) and \(M \cup M^c = 1_u\). Therefore \(M, M^c\) is a neutrosophic soft separation of \(1_u\). This is a contradiction. So, only neutrosophic soft connected and open sets in \(NSS(U, E)\) are \(\phi_u\) and \(1_u\).

Conversely, let \(M, P\) be a neutrosophic soft separation of \((U, E, \tau_u)\). Then \(M \neq N\) i.e., \(M = P^c\), otherwise \(M = 1_u\) implies \(P = \phi_u\); a contradiction. This shows that \(M\) is both neutrosophic soft open and neutrosophic soft closed different from \(\phi_u, 1_u\). This is a contradiction. Hence, \((U, E, \tau_u)\) is connected.

3.6 Theorem

If the neutrosophic soft sets \(N_1, N_2\) form a neutrosophic soft separation of \((U, E, \tau_u)\) and if \((U, E, \tau_M)\) is a neutrosophic soft connected subspace of \((U, E, \tau_u)\), then \(M \subset N_1\) or \(M \subset N_2\).
3.7 Theorem

Let \((U, E, \tau_M)\) be a neutrosophic soft topological subspace of \((U, E, \tau_u)\). A separation of \((U, E, \tau_M)\) is a pair of disjoint nonempty neutrosophic soft sets \(M_1, M_2\) whose union is \(M\) such that \(M_1 \cap M_2 = \emptyset\) and \(M_2 \cap M_1 = \emptyset\).

**Proof.** Suppose \(M_1, M_2\) forms a separation of \((U, E, \tau_M)\). Then \(M_k\) is both neutrosophic soft open and closed subset of \(M\) by Theorem [3.7]. The neutrosophic soft closure of \(M_1\) in \(M\) is \(\overline{M_1} \cap M\) by Theorem [2.19]. Since \(M_1\) is neutrosophic soft closed in \(M\) then \(M_1 = \overline{M_1} \cap M\). It implies \(\overline{M_1} \cap M_2 = (\overline{M_1} \cap M) \cap M_2 = M_1 \cap M_2 = \emptyset\). Similarly, \(\overline{M_2} \cap M_1 = \emptyset\).

Conversely, let \(M = M_1 \cup M_2\) with \(M_1 \cap M_2 = \emptyset\) such that \(\overline{M_1} \cap M_2 = \emptyset\) and \(\overline{M_2} \cap M_1 = \emptyset\). Then \(M \cap \overline{M_1} = \emptyset\) and \(M \cap \overline{M_2} = \emptyset\) implies both are neutrosophic soft open in \(M\).

3.8 Theorem

Let \((U, E, \tau_M)\) be a connected neutrosophic soft subspace of \((U, E, \tau_u)\). If \((U, E, \tau_P)\) be any neutrosophic soft subspace of \((U, E, \tau_u)\) such that \(M \subset P < M\), then \((U, E, \tau_P)\) is also neutrosophic soft connected.

**Proof.** Let the neutrosophic soft set \(P\) satisfy the hypothesis. If possible, let \(P_1, P_2\) form a neutrosophic soft separation of \((U, E, \tau_P)\). Then \(M \subset P_1\) or \(M \subset P_2\). Let \(M \cap P_1 = \emptyset\). So \(M \subset P_1\) and \(P_1\) is closed NSS. It implies \(M \subset P < M\) and \(P_1 < P \Rightarrow P \cap P_1 = \emptyset\). This is a contradiction to the fact that \(P_1 \cup P_2 = P\). Hence, \((U, E, \tau_P)\) is neutrosophic soft connected.

3.9 Theorem

Arbitrary union of connected neutrosophic soft subspaces of \((U, E, \tau_u)\) having nonempty intersection is also neutrosophic soft connected.

**Proof.** Let \(\{(U, E, \tau_{N_i}) : i \in \Gamma\}\) be a class of connected neutrosophic soft subspaces of \((U, E, \tau_u)\) with nonempty intersection. Let \(\tau_M = \bigcup_{i \in \Gamma} (\tau_{N_i})\). If possible, we take a neutrosophic soft separation \(P, Q\) of \((U, E, \tau_M)\). For each \(i\), \(P \cap N_i\) and \(Q \cap N_i\) are disjoint neutrosophic soft open sets in the subspace such that their union is \(N_i\). Since each \((U, E, \tau_{N_i})\) is connected, any of \(P \cap N_i\) and \(Q \cap N_i\) must be empty. Let \(P \cap N_i = \emptyset\Rightarrow Q \cap N_i = N_i \Rightarrow N_i \subset Q, \forall i \in \Gamma \Rightarrow \cup_i N_i \subset Q \Rightarrow M \subset Q \Rightarrow P \cup Q \subset Q \Rightarrow P\) is empty, a contradiction. So, \((U, E, \tau_M)\) is neutrosophic soft connected.

3.10 Theorem

Arbitrary union of a family of connected neutrosophic soft subspaces of \((U, E, \tau_u)\) such that one of the members of the family has nonempty intersection with every member of the family, is neutrosophic soft connected.

**Proof.** Let \(\{(U, E, \tau_{N_i}) : i \in \Gamma\}\) be a class of connected neutrosophic soft subspaces of \((U, E, \tau_u)\) and \(N_k\) be a fixed member such that \(N_k \cap N_i \neq \emptyset\) for each \(i \in \Gamma\). Let \(M_i = N_k \cup N_i\). Then by Theorem [3.9], \((U, E, \tau_M)\) is a neutrosophic soft connected for each \(i \in \Gamma\). Now, \(\cup_i M_i = \cup_i (N_k \cup N_i) = (N_k \cup N_1) \cup (N_k \cup N_2) \cup \cdots = N_k \cup (N_k \cup N_2) \cup \cdots = N_k \cup (N_1 \cup N_2) \cup \cdots \neq \emptyset\).

This completes the theorem.

4 Compactness

Here, the notion of compactness on neutrosophic soft topological space is developed with some basic theorems.

4.1 Definition

Let \((U, E, \tau_u)\) be a neutrosophic soft topological space and \(M \in \tau_u\). A family \(\Omega = \{Q_i : i \in \Gamma\}\) of neutrosophic soft sets is said to be a cover of \(M\) if \(M \subset \bigcup Q_i\).

If every member of that family which covers \(M\) is neutrosophic soft open then it is called open cover of \(M\). A subfamily of \(\Omega\) which also covers \(M\) is called a subcover of \(M\).

4.1.1 Definition

Let \((U, E, \tau_u)\) be a neutrosophic soft topological space and \(M \in \tau_u\). Suppose \(\Omega\) be an open cover of \(M\). If \(\Omega\) has a finite subcover which also covers \(M\) then \(M\) is called neutrosophic soft compact.

4.1.2 Example

In the Example (1) of [2.9], \(1_u = \bigcup_{i=1}^4 N_i\). So \(\{N_1, N_2, N_3, N_4\}\) is an open cover of \((U, E, \tau_u)\). Also, \(1_u = N_1 \cup N_2\) or \(1_u = N_1 \cup N_4\). So \((U, E, \tau_u)\) is neutrosophic soft compact topological space.

4.2 Theorem

Let \((U, E, \tau_u)\) be a neutrosophic soft compact topological space and \(M\) be a neutrosophic soft closed set of that space. Then \(M\) is also compact.

**Proof.** Let \(\Omega = \{Q_i : i \in \Gamma\}\) be an open cover of \(M\).
Then \( \{Q_i\} \cup M^c \) is an open cover of \((U, E, \tau_u)\), obviously. Since \((U, E, \tau_u)\) is compact so there exists a finite subcover of \( \{Q_i\} \cup M^c \) such that

\[
1_u = Q_1 \cup Q_2 \cup \cdots \cup Q_n \cup M^c \\
\Rightarrow M \subset 1_u = Q_1 \cup Q_2 \cup \cdots \cup Q_n \cup M^c \\
\Rightarrow M \subset Q_1 \cup Q_2 \cup \cdots \cup Q_n \text{ as } M \cap M^c = \phi_u.
\]
Hence, \( M \) has a finite subcover and so is compact.

### 4.3 Theorem

Let \((U, E, \tau_u)\) be a neutrosophic soft Hausdorff topological space and \( M \) be a neutrosophic soft compact set belonging to that space. Then \( M \) is a closed NSS.

**Proof.** Let \( e \in K \subset M \) be a neutrosophic soft point. Then for each \( e \in M \), we have \( e \in K \neq e \). So by definition of Hausdorff space, there are disjoint neutrosophic soft open sets \( N_K, N_S \) so that \( e \in K \neq e \) and \( e \in S \). Let \( \{N_s : e \in M\} \) be a neutrosophic soft open cover of \( M \). Since \( M \) is neutrosophic soft compact so it has a finite subcover, say, \( \{N_{s_1}, N_{s_2}, \ldots, N_{s_n}\} \) i.e., \( M \subset N_{s_1} \cup N_{s_2} \cup \cdots \cup N_{s_n} = P, \) say. Then \( P \) is neutrosophic soft open.

Let \( Q = N_{K_1} \cap N_{K_2} \cap \cdots \cap N_{K_n} \) where each \( N_{K_i} \) is open NSS corresponding to \( e_{K_i} \subset M^c \). Now, \( N_{s_1} \cap N_{K_i} = \phi_u \Rightarrow N_{s_i} \cap Q = \phi_u \) for each \( i \). Then \( P \cap Q = (N_{s_1} \cup N_{s_2} \cup \cdots \cup N_{s_n}) \cap Q = (N_{s_1} \cap Q) \cup (N_{s_2} \cap Q) \cup \cdots \cup (N_{s_n} \cap Q) = \phi_u. \)
Since \( M \subset P \) and \( P \cap Q = \phi_u \), so \( M \cap Q = \phi_u \Rightarrow Q \subset M^c \) and \( Q \) is open NSS. This implies \( M^c \) is open NSS i.e., \( M \) is closed.

### 4.4 Theorem

A neutrosophic soft topological space is compact iff each family of neutrosophic soft closed sets with the finite intersection property has a nonempty intersection.

**Proof.** Let \((U, E, \tau_u)\) be a compact neutrosophic soft topological space. Consider \( \Omega = \{Q_i : i \in \Gamma\} \) be a family of closed NSSs such that \( \cap \Omega_i \neq \phi_u \). We show \( \Omega \) cannot have finite intersection property. Let \( \Delta = \{Q_i : \Omega_i \neq \cap \Omega_i\} \). Then \( \Delta \) is an open cover of \((U, E, \tau_u)\) such that there exists a finite subcover \( \{Q_1, Q_2, \ldots, Q_n\}. \) Now \( \cap \frac{n}{i=1} Q_i = 1_u = \bigcup \frac{n}{i=1} Q_i \neq \phi_u \) by Definition [2.8]. Hence, the ‘if part’ holds.

Next assume that \((U, E, \tau_u)\) is not compact. Then, a neutrosophic soft open cover \( \{Q_i : i \in \Gamma\}, \) say, of \((U, E, \tau_u)\) has no finite subcover i.e., \( Q_1 \cup Q_2 \cup \cdots \cup Q_n \neq 1_u \). This implies \( Q_1 \cap Q_2 \cap \cdots \cap Q_n \neq 1_u \) by Definition [2.8] and Proposition [2.16]. Thus \( \{Q_i : i \in \Gamma\} \) has finite intersection property. Then by hypothesis, \( \cap \frac{n}{i=1} Q_i \neq 1_u \) which is a contradiction. Hence, \((U, E, \tau_u)\) is compact.

### 5 Neutrosophic soft continuous mappings

In this section, first we define neutrosophic soft mapping, then define image and pre-image of an NSS under a neutrosophic soft mapping. In continuation, we introduce the notion of neutrosophic soft continuous mapping in a neutrosophic soft topological space along with some of its properties.

In rest of the paper, if \( M \) be an NSS over \( U \) via parameter set \( E \), we write \((M, E)\), an NSS over \( U \) i.e., \((M, E) = \{e, f_M(e) : e \in E\}\).

#### 5.1 Definition

Let \( \varphi : U \to V \) and \( \psi : E \to E \) be two functions where \( E \) is the parameter set for each of the crisp sets \( U \) and \( V \). Then the pair \((\varphi, \psi)\) is called an NSS function from \((U, E)\) to \((V, E)\).

**5.1.1 Definition**

Let \((M, E)\) and \((N, E)\) be two NSSs defined over \( U \) and \( V \), respectively and \((\varphi, \psi)\) be an NSS function from \((U, E)\) to \((V, E)\). Then,

1. The image of \((M, E)\) under \((\varphi, \psi)\), denoted by \((\varphi, \psi)(M, E)\), is an NSS over \( V \) and is defined as :
   \[
   (\varphi, \psi)(M, E) = \{\psi(a), f_M(\psi)(a) : a \in E\} \text{ where } \forall b \in \psi(E), \forall y \in V.
   \]
   \[
   T_{\varphi(\psi)}(b)(y) = \begin{cases} \max_{\varphi(x)=y} \max_{\psi(a)=b} [T_{f_M}(a)(x)], & \text{if } x \in \varphi^{-1}(y) \\ 0, & \text{otherwise} \end{cases}
   \]
   \[
   I_{\varphi(\psi)}(b)(y) = \begin{cases} \min_{\varphi(x)=y} \min_{\psi(a)=b} [I_{f_M}(a)(x)], & \text{if } x \in \varphi^{-1}(y) \\ 1, & \text{otherwise} \end{cases}
   \]
   \[
   F_{\varphi(\psi)}(b)(y) = \begin{cases} \min_{\varphi(x)=y} \min_{\psi(a)=b} [F_{f_M}(a)(x)], & \text{if } x \in \varphi^{-1}(y) \\ 1, & \text{otherwise} \end{cases}
   \]

2. The pre-image of \((N, E)\) under \((\varphi, \psi)\), denoted by \((\varphi, \psi)^{-1}(N, E)\), is an NSS over \( U \) and is defined by :
   \[
   (\varphi, \psi)^{-1}(N, E) = \{(\varphi^{-1}(N), \psi^{-1}(E)) : \forall a \in \psi^{-1}(E), \forall x \in U\}
   \]
   \[
   T_{\varphi^{-1}}(I_{\psi^{-1}})(a)(x) = T_{f_N}(\psi(a))(\varphi(x))
   \]
   \[
   I_{\varphi^{-1}}(I_{\psi^{-1}})(a)(x) = I_{f_N}(\psi(a))(\varphi(x))
   \]
   \[
   F_{\varphi^{-1}}(I_{\psi^{-1}})(a)(x) = F_{f_N}(\psi(a))(\varphi(x))
   \]

If \( \varphi \) and \( \psi \) are injective (surjective), then \((\varphi, \psi)\) is injective (surjective).

#### 5.1.2 Proposition

Let, \((\varphi, \psi) : (U, E) \to (V, E)\) be a neutrosophic soft mapping and \((M_1, E)\) and \((M_2, E)\) be two NSSs defined over \( U \). Then the followings hold.

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(1) \((M_1, E) \subseteq (\varphi, \psi)^{-1}[(\varphi, \psi)(M_1, E)]\)

(2) \([(\varphi, \psi)(M_1, E)]^c \subseteq (\varphi, \psi)(M_1, E)^c\), if \(\varphi\) is surjective.

(3) \((\varphi, \psi)[(M_1, E) \cup (M_2, E)] = (\varphi, \psi)(M_1, E) \cup (\varphi, \psi)(M_2, E)\)

(4) \((\varphi, \psi)[(M_1, E) \cap (M_2, E)] = (\varphi, \psi)(M_1, E) \cap (\varphi, \psi)(M_2, E)\)

Proof.

(1) \((\varphi, \psi)^{-1}[(\varphi, \psi)(M_1, E)] = (\varphi, \psi)^{-1}[(\varphi(M_1), \psi(E))] = [\varphi^{-1}(\varphi(M_1)), \psi(E)]\). Then for \(a \in \psi^{-1}(\psi(E))\) and \(x \in U\), we have, \(T_{f_a^{-1}(\varphi(M_1))(a)}(x) = T_{f_a^{-1}(\varphi(M_1))(\varphi(x))}\) = \(\max_{a(x)}\max_{a(a)}[T_{f_a}(a)(x)]\). Now, \(T_{f_a}(a)(x) \leq \max_{a(x)}\max_{a(a)}[T_{f_a}(a)(x)] = T_{f_a^{-1}(\varphi(M_1))(a)}(x)\).

Similarly, \(I_{f_a^{-1}(\varphi(M_1))(a)}(x) = I_{f_a^{-1}(\varphi(M_1))(\varphi(x))}\) = \(\max_{a(x)}\max_{a(a)}[I_{f_a}(a)(x)]\). Thus, \(I_{f_a^{-1}(\varphi(M_1))(a)}(x) \leq T_{f_a^{-1}(\varphi(M_1))(a)}(x)\).

Hence, \((M_1, E) \subseteq (\varphi, \psi)^{-1}[(\varphi, \psi)(M_1, E)]\).

(2) Suppose, \(\varphi\) is surjective mapping. Here, \([(\varphi, \psi)(M_1, E)]^c = [(\varphi(M_1))^c, \psi(E)]\) and \((\varphi, \psi)(M_1, E)^c = [\varphi(M_1)^c, \psi(E)]\). For \(b \in \psi(E)\) and \(y \in V\), we have, \(T_{f_b^{-1}(\varphi(M_1))(b)}(y) = T_{f_b^{-1}(\varphi(M_1))(\varphi(y))}\) = \(\min_{b(x)}\min_{b(a)}[T_{f_b}(a)(x)]\). But, \(T_{f_b^{-1}(\varphi(M_1))(b)}(y) = \max_{b(x)}\max_{b(a)}[T_{f_b}(a)(x)]\). Thus, \(T_{f_b^{-1}(\varphi(M_1))(b)}(y) \leq T_{f_b^{-1}(\varphi(M_1))(b)}(y)\).

Similarly, \(I_{f_b^{-1}(\varphi(M_1))(b)}(y) \geq I_{f_b^{-1}(\varphi(M_1))(\varphi(y))}\) = \(\max_{b(x)}\max_{b(a)}[I_{f_b}(a)(x)]\). Thus, \(I_{f_b^{-1}(\varphi(M_1))(b)}(y) \leq I_{f_b^{-1}(\varphi(M_1))(b)}(y)\).

This completes the 2nd part.

(3) Let, \((M_1, E) \cup (M_2, E) = (M, E)\).

Then, \((\varphi, \psi)[(M_1, E) \cup (M_2, E)] = (\varphi, \psi)(M, E) = [\varphi(M), \psi(E)]\). So, for \(b \in \psi(E)\) and \(y \in V\), we have, \(T_{f_b}(b)(y) = \max_{b(x)}\max_{b(a)}[T_{f_b}(a)(x)]\).

Next, \((\varphi, \psi)(M_1, E) \cup (\varphi, \psi)(M_2, E) = [\varphi(M_1) \cup \varphi(M_2), \psi(E)] = \{P, \psi(E)\}\), say. Then, \(T_{f_b}(b)(y) = \max_{b(x)}\max_{b(a)}[T_{f_b}(a)(x) \circ T_{f_b}(a)(x)]\).

Thus, \(T_{f_b}(b)(y) = T_{f_b}(b)(y)\). Similar results also hold for \(I, F\).

This completes the proof of part (3).

(4) Let, \((M_1, E) \cap (M_2, E) = (M, E)\).

Then, \((\varphi, \psi)(M_1, E) \cap (M_2, E) = [\varphi(M), \psi(E)]\). So, for \(b \in \psi(E)\) and \(y \in V\), we have, \(T_{f_b}(b)(y) = \max_{b(x)}\max_{b(a)}[T_{f_b}(a)(x)]\).

Next, \((\varphi, \psi)(M_1, E) \cap (\varphi, \psi)(M_2, E) = [\varphi(M_1) \cap \varphi(M_2), \psi(E)] = [Q, \psi(E)]\), say. Then, \(T_{f_b}(b)(y) = T_{f_b}(b)(y) \ast T_{f_b}(b)(y)\).

Thus, \(T_{f_b}(b)(y) = T_{f_b}(b)(y)\). Similar results also hold for \(I, F\).

This ends the last part.

5.1.3 Proposition

Let, \((\varphi, \psi) : (U, E) \rightarrow (V, E)\) be a neutrosophic soft mapping and \((N_1, E)\) and \((N_2, E)\) be two NSSS defined over \(V\). Then the followings hold.

(1) \([(\varphi, \psi)^{-1}(N_1, E)] = (N_1, E), \text{if } (\varphi, \psi) \text{ is surjective.}\)

(2) \([(\varphi, \psi)^{-1}(N_1, E)]^c = [(\varphi^{-1}(N_1))^c, \psi(E)]\).

(3) \((\varphi, \psi)^{-1}(N_1, E) \cup (N_2, E) = (\varphi, \psi)^{-1}(N_1, E) \cup (\varphi, \psi)^{-1}(N_2, E)\).

(4) \((\varphi, \psi)^{-1}(N_1, E) \cap (\varphi, \psi)^{-1}(N_2, E) = (\varphi, \psi)^{-1}(N_1, E) \cap (\varphi, \psi)^{-1}(N_2, E)\).

Proof. We shall prove (2) and (3) only. The others can be proved similarly.

(2) Here, \([(\varphi, \psi)^{-1}(N_1, E)]^c = [(\varphi^{-1}(N))^c, \psi^{-1}(E)]\). Then, for \(a \in \psi^{-1}(E), x \in U,\)

\(T_{f_b^{-1}(\varphi^{-1}(N))^c(a)}(x) = T_{f_b^{-1}(\varphi^{-1}(N)^c)(a)}(x) = F_{f_b}(\psi(a))(\varphi(x)),\)

\(I_{f_b^{-1}(\varphi^{-1}(N))^c(a)}(x) = I_{f_b^{-1}(\varphi^{-1}(N)^c)(a)}(x) = I_{f_b}(\psi(a))(\varphi(x)),\)

\(F_{f_b^{-1}(\varphi^{-1}(N))^c(a)}(x) = T_{f_b^{-1}(\varphi^{-1}(N))^c}(x) = T_{f_b}(\psi(a))(\varphi(x)).\)

Thus, \(T_{f_b^{-1}(\varphi^{-1}(N))^c(a)}(x) = T_{f_b}(\psi(a))(\varphi(x))\).

Next, \((\varphi, \psi)^{-1}(N_1, E) \cap (\varphi, \psi)^{-1}(N_2, E) = (\varphi, \psi)^{-1}(N_1, E) \cap (\varphi, \psi)^{-1}(N_2, E)\).

Thus, \(T_{f_b^{-1}(\varphi^{-1}(N))^c(a)}(x) = T_{f_b}(\psi(a))(\varphi(x))\).

Hence, the result is proved.

(3) Let, \((N_1, E) \cup (N_2, E) = (N, E)\).

Then, \((\varphi, \psi)^{-1}(N_1, E) \cup (N_2, E) = [\varphi^{-1}(N), \psi^{-1}(E)]\). So, for \(a \in \psi^{-1}(E)\) and \(x \in U,\)

\(T_{f_b^{-1}(\varphi^{-1}(N))^c(a)}(x) = T_{f_b}(\psi(a))(\varphi(x))\).

Thus, \(T_{f_b^{-1}(\varphi^{-1}(N))^c(a)}(x) = T_{f_b}(\psi(a))(\varphi(x))\).

This completes the proof of (3).
Next, \((\varphi, \psi)^{-1}(N_1, E) \cup (\varphi, \psi)^{-1}(N_2, E) = [\varphi^{-1}(N_1) \cup \varphi^{-1}(N_2), \psi^{-1}(E)] = [R, \psi^{-1}(E)],\) say. Then,

\[
T_{fR(a)}(x) = T_{f_{\varphi^{-1}(N_1)}(a)}(x) \cap T_{f_{\varphi^{-1}(N_2)}(a)}(x)
= T_{f_{\varphi^{-1}(N_1)}(\varphi(a))}(x) \cap T_{f_{\varphi^{-1}(N_2)}(\varphi(a))}(\varphi(x))
\]

Thus, \(T_{f_{\varphi^{-1}(N_1)}(a)}(x) = T_{fR(a)}(x).\) Similar results also hold for \(I, F.\)

This completes the proof of part (3).

### 5.2 Definition

Let \((\varphi, \psi) : (U, E, \tau_u) \to (V, E, \tau_v)\) be a mapping where \((U, E, \tau_u)\) and \((V, E, \tau_v)\) be two neutrosophic soft topological spaces.

1. For each neutrosophic soft open set \((M, E) \in (U, E, \tau_u),\) if the image \((\varphi, \psi)(M, E)\) is open in \((V, E, \tau_v)\) then \((\varphi, \psi)\) is said to be neutrosophic soft open mapping.

2. For each neutrosophic soft closed set \((Q, E) \in (U, E, \tau_u),\) if the image \((\varphi, \psi)(Q, E)\) is closed in \((V, E, \tau_v)\) then \((\varphi, \psi)\) is said to be neutrosophic soft closed mapping.

### 5.3 Theorem

Let, \((U, E, \tau_u)\) and \((V, E, \tau_v)\) be two neutrosophic soft topological spaces and \((\varphi, \psi) : (U, E, \tau_u) \to (V, E, \tau_v)\) be a mapping. Then,

1. \((\varphi, \psi)\) is a neutrosophic soft open mapping if for each neutrosophic soft set \((M, E) \in (U, E, \tau_u),\) there hold \([(\varphi, \psi)(M, E)^o) \subset [(\varphi, \psi)(M, E)]^o].\)

2. \((\varphi, \psi)\) is a neutrosophic soft closed mapping if for each neutrosophic soft set \((Q, E) \in (U, E, \tau_u),\) there hold \([(\varphi, \psi)(Q, E) \subset (\varphi, \psi)[Q, E]).\]

**Proof.**

1. Let \((\varphi, \psi)\) is a neutrosophic soft open mapping and \((M, E) \in (U, E, \tau_u).\) Then \((M, E)^o\) is a neutrosophic soft open set and \((M, E)^o \subset (M, E).\) Since \((\varphi, \psi)\) is a neutrosophic soft open mapping, \((\varphi, \psi)(M, E)^o) \subset (\varphi, \psi)(M, E).\)

2. Let \((\varphi, \psi)\) is a neutrosophic soft closed mapping and \((Q, E) \in (U, E, \tau_u).\) Then \((\varphi, \psi)[Q, E]) \subset (\varphi, \psi)(Q, E).\)

Conversely, suppose \((M, E)\) be an open NSS in \((U, E, \tau_u)\) such that the given condition holds. Then \((M, E) = (M, E)^o\) and so \((\varphi, \psi)(M, E) = (\varphi, \psi)(M, E)^o \subset [(\varphi, \psi)(M, E)]^o \subset (\varphi, \psi)(M, E).\)

### 5.4 Definition

Let, \((U, E, \tau_u)\) and \((V, E, \tau_v)\) be two neutrosophic soft topological spaces. Then \((\varphi, \psi) : (U, E, \tau_u) \to (V, E, \tau_v)\) is said to be a neutrosophic soft continuous mapping if for each \((N, E) \in \tau_v,\)

Conversely, suppose \((Q, E)\) be a closed NSS in \((U, E, \tau_u)\) such that the given condition holds. Then \((Q, E) = (Q, E)^c\) and so \((\varphi, \psi)(Q, E) \subset [(\varphi, \psi)(Q, E)]^c \subset (\varphi, \psi)(Q, E).\)

This completes the proof.

### 5.4.1 Example

For two neutrosophic soft topological spaces \((U, E, \tau_u)\) and \((V, E, \tau_v)\), let \((\varphi, \psi) : (U, E, \tau_u) \to (V, E, \tau_v)\) be a mapping.

1. If \(\tau_u\) is the neutrosophic soft indiscrete topology on \(U,\) then \((\varphi, \psi)\) is a neutrosophic soft continuous mapping.

2. If \(\tau_u\) is the neutrosophic soft discrete topology on \(U,\) then \((\varphi, \psi)\) is a neutrosophic soft continuous mapping.
as: \( \varphi(u_1) = v_1, \varphi(u_2) = v_3, \varphi(u_3) = v_2 \) and \( \psi(e_1) = e_2, \psi(e_2) = e_1 \). Then \( (\varphi, \psi)^{-1}(N_1, E) \) is as follows:

\[
\begin{align*}
T_{I_{\varphi^{-1}(N_1)}(e_1)}(u_1) &= T_{I_{\varphi}(e_1)}(\varphi(u_1)) = T_{I_{\varphi}(e_1)}(v_1) = 0.7, \\
I_{\varphi^{-1}(N_1)}(e_1)(u_1) &= I_{\varphi}(e_1)(\varphi(u_1)) = I_{\varphi}(e_1)(v_1) = 0.6, \\
F_{I_{\varphi^{-1}(N_1)}(e_1)}(u_1) &= F_{I_{\varphi}(e_1)}(\varphi(u_1)) = F_{I_{\varphi}(e_1)}(v_1) = 0.5, \\
T_{I_{\varphi^{-1}(N_1)}(e_1)}(u_2) &= T_{I_{\varphi}(e_1)}(\varphi(u_2)) = T_{I_{\varphi}(e_1)}(v_2) = 0.5, \\
I_{\varphi^{-1}(N_1)}(e_1)(u_2) &= I_{\varphi}(e_1)(\varphi(u_2)) = I_{\varphi}(e_1)(v_2) = 0.8, \\
F_{I_{\varphi^{-1}(N_1)}(e_1)}(u_2) &= F_{I_{\varphi}(e_1)}(\varphi(u_2)) = F_{I_{\varphi}(e_1)}(v_2) = 0.6, \\
T_{I_{\varphi^{-1}(N_1)}(e_1)}(u_3) &= T_{I_{\varphi}(e_1)}(\varphi(u_3)) = T_{I_{\varphi}(e_1)}(v_3) = 0.6, \\
I_{\varphi^{-1}(N_1)}(e_1)(u_3) &= I_{\varphi}(e_1)(\varphi(u_3)) = I_{\varphi}(e_1)(v_3) = 0.8, \\
F_{I_{\varphi^{-1}(N_1)}(e_1)}(u_3) &= F_{I_{\varphi}(e_1)}(\varphi(u_3)) = F_{I_{\varphi}(e_1)}(v_3) = 0.4.
\end{align*}
\]

5.4.2 Proposition

Let \( (\varphi, \psi) : (U, E, \tau_u) \rightarrow (V, E, \tau_v) \) be a neutrosophic soft continuous mapping. Then for each \( e \in E \), \( (\varphi, \psi) : (U, \tau_u^e) \rightarrow (V, \tau_v^e) \) is a neutrosophic continuous mapping.

**Proof.** Let \( (N, E) \in \tau_u^e \). Since \( (\varphi, \psi) \) is a neutrosophic soft continuous mapping, so \( (\varphi, \psi)^{-1}(N, E) \in \tau_u^e \). It implies \( (\varphi, \psi)^{-1}(< e, f_N(e) > : e \in E) \in \tau_u^e \) i.e., \( (\varphi, \psi)^{-1}(< e, f_N(e) >) \in \tau_u^e \) for \( < e, f_N(e) > \in \tau_v^e \). This follows the theorem.

But the converse does not hold. The following example shows the fact.

Let, \( U = \{u_1, u_2, u_3\}, V = \{v_1, v_2, v_3\}, E = \{e_1, e_2\}, \tau_u = \{\phi_u, 1_u, (N_1, E), (N_2, E)\}, \tau_v = \{\phi_v, 1_v, (M_1, E), (M_2, E), (M_3, E)\} \).

where \( (N_1, E), (N_2, E) \) are as follows:

\[
\begin{align*}
f_{N_1}(e_1) &= \{< v_1, (0.8, 0.5, 0.6) >, < v_2, (0.5, 0.7, 0.6) >, < v_3, (0.4, 0.7, 0.5) >\}; \\
f_{N_1}(e_2) &= \{< v_1, (0.7, 0.6, 0.5) >, < v_2, (0.6, 0.8, 0.4) >, < v_3, (0.5, 0.8, 0.6) >\}; \\
f_{N_2}(e_1) &= \{< v_1, (1.0, 0.5, 0.4) >, < v_2, (0.6, 0.6, 0.6) >, < v_3, (0.5, 0.6, 0.4) >\}; \\
f_{N_2}(e_2) &= \{< v_1, (0.8, 0.4, 0.5) >, < v_2, (0.7, 0.7, 0.3) >, < v_3, (0.7, 0.5, 0.6) >\};
\end{align*}
\]

and \( (M_1, E), (M_2, E), (M_3, E) \) are given as follows:

\[
\begin{align*}
f_{M_1}(e_1) &= \{< u_1, (0.6, 0.6, 0.6) >, < u_2, (0.8, 0.6, 0.4) >, < u_3, (1.0, 0.5, 0.4) >\}; \\
f_{M_1}(e_2) &= \{< u_1, (0.7, 0.7, 0.3) >, < u_2, (0.7, 0.5, 0.6) >, < u_3, (0.8, 0.4, 0.5) >\}; \\
f_{M_2}(e_1) &= \{< u_1, (0.5, 0.7, 0.6) >, < u_2, (0.4, 0.7, 0.5) >, < u_3, (0.8, 0.5, 0.6) >\}; \]
\]

Then, \( \tau_u^e \) of \( (\varphi, \psi)^{-1}(N_1, E) \) is not neutrosophic soft continuous. Now, \( \tau_v^e \) of \( \tau_u^e \) is neutrosophic soft continuous mapping because \( (\varphi, \psi)^{-1}[f_{N_1}(e_1)] = f_{M_1}(e_1) \) and \( (\varphi, \psi)^{-1}[f_{N_1}(e_2)] = f_{M_1}(e_2) \). Similarly, \( \tau_v^e \) of \( \tau_u^e \) is neutrosophic soft continuous mapping as: \( (\varphi, \psi)^{-1}[f_{N_1}(e_2)] = f_{M_2}(e_1) \).
5.5 Theorem

For two neutrosophic soft topological spaces \((U, E, \tau_u)\) and \((V, E, \tau_v)\), let \((\varphi, \psi) : (U, E, \tau_u) \rightarrow (V, E, \tau_v)\) be a neutrosophic soft mapping. Then the following conditions are equivalent.

(1) \((\varphi, \psi)\) is neutrosophic soft continuous mapping.

(2) The inverse image of a closed NSS in \((V, E, \tau_v)\) is closed in \((U, E, \tau_u)\).

(3) For each \((M, E) \in NSS(U, E)\), \((\varphi, \psi)(M, E) \subset (\varphi, \psi)(M, E)\).

(4) For each \((N, E) \in NSS(V, E)\), \((\varphi, \psi)^{-1}(N, E) \subset (\varphi, \psi)^{-1}(N, E)\).

(5) For each \((N, E) \in NSS(V, E)\), \((\varphi, \psi)^{-1}(N, E)^c \subset [(\varphi, \psi)^{-1}(N, E)]^c\).

Proof. (1) \(\Rightarrow\) (2)

Let, \((Q, E)\) be a closed NSS in \((V, E, \tau_v)\). Then \((Q, E)^c \in \tau_v\) and so by (1), \((\varphi, \psi)^{-1}(Q, E)^c \in \tau_u\). But \((\varphi, \psi)^{-1}(Q, E)^c = \{((\varphi, \psi)^{-1}(Q, E))^c\}\). So \((\varphi, \psi)^{-1}(Q, E)\) is a closed NSS in \((U, E, \tau_u)\).

(2) \(\Rightarrow\) (3)

Let, \((M, E) \in NSS(U, E)\). Since \((\varphi, \psi)(M, E) \subset (\varphi, \psi)(M, E)\), we have \((M, E) \subset (\varphi, \psi)^{-1}((\varphi, \psi)(M, E)) \subset (\varphi, \psi)^{-1}((\varphi, \psi)^{-1}(M, E))\). Obviously, \((\varphi, \psi)^{-1}(M, E)\) is closed in \((V, E, \tau_v)\). Then by (2), \((\varphi, \psi)^{-1}(\varphi, \psi)(M, E)\) is closed in \((U, E, \tau_u)\). But, since \((M, E) \subset (\varphi, \psi)(M, E)\) and \((\varphi, \psi)(M, E)\) is the smallest closed NSS, so \((M, E) \subset (\varphi, \psi)^{-1}((\varphi, \psi)^{-1}(M, E))\). This implies \((\varphi, \psi)(M, E) \subset \varphi, \psi)^{-1}(\varphi, \psi)^{-1}((\varphi, \psi)(M, E))\) i.e., \((\varphi, \psi)(M, E) \subset (\varphi, \psi)(M, E)\) is obtained.

(3) \(\Rightarrow\) (4)

Let, \((N, E) \in NSS(V, E)\) and \((\varphi, \psi)^{-1}(N, E) = (M, E)\). Then \((\varphi, \psi)^{-1}(N, E) = (M, E)\). But by (3), we have \((M, E) \subset (\varphi, \psi)^{-1}((\varphi, \psi)(M, E))\) i.e., \((\varphi, \psi)^{-1}(N, E) \subset (\varphi, \psi)^{-1}((\varphi, \psi)(M, E))\). This shows \((\varphi, \psi)^{-1}(N, E) \subset (\varphi, \psi)^{-1}((\varphi, \psi)^{-1}(N, E))\) i.e., \((\varphi, \psi)^{-1}(N, E) \subset (\varphi, \psi)^{-1}(N, E)\).

(4) \(\Rightarrow\) (5)

Let, \((N, E) \in NSS(V, E)\). Replacing \((N, E)\) by \((N, E)^c\) and applying (4), we have \((\varphi, \psi)^{-1}(N, E)^c \subset (\varphi, \psi)^{-1}((\varphi, \psi)(N, E))^c\) i.e., \(((\varphi, \psi)^{-1}(N, E))^c \subset (\varphi, \psi)^{-1}(N, E)^c\). By Theorem (ii) of [2.15.2], since \((N, E)^c = [N, E]^c\), so \((\varphi, \psi)^{-1}(N, E)^c = (\varphi, \psi)^{-1}((\varphi, \psi)^{-1}(N, E))^c = (\varphi, \psi)^{-1}(N, E)^c = (\varphi, \psi)^{-1}(N, E)^c\).

(5) \(\Rightarrow\) (1)

Let, \((N, E)\) be an open NSS in \((V, E, \tau_v)\). Then \((N, E)^o = (N, E)\). Since \(((\varphi, \psi)^{-1}(N, E))^o \subset (\varphi, \psi)^{-1}(N, E) = (\varphi, \psi)^{-1}(N, E)^o \subset (\varphi, \psi)^{-1}(N, E)^o\), so \(((\varphi, \psi)^{-1}(N, E))^o = (\varphi, \psi)^{-1}(N, E)\) is obtained. Thus, \((\varphi, \psi)^{-1}(N, E)\) is an open NSS in \((U, E, \tau_u)\) and so \((\varphi, \psi)\) is neutrosophic soft continuous mapping.

6. Conclusion

Topology is a major sector in mathematics and it can give many relationships between other scientific area and mathematici models. The motivation of the present paper is to extend the concept of topological structure on neutrosophic soft set introduced in the paper [33]. Here, we have defined connectedness and compactness on neutrosophic soft topological space, neutrosophic soft continuous mappings. These are illustrated by suitable examples. Their several related properties and structural characteristics have been investigated. We expect, this paper will promote the future study on neutrosophic soft topological groups and many other general frameworks.

References


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