General relativity and representation of solutions

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ABSTRACT

In the general relativity theory, we find the representation of the gravity field equation and solutions. We treats the representation of Schwarzschild solution, Reissner-Nodstrom solution, Kerr-Newman solution, Robertson -Walker solution. Specially, Robertson -Walker solution is an uniqueness.

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1. Introduction

In the general relativity theory, our article's aim is that we find the representation of the gravity field equation and solutions.

First, the gravity potential $g_{\mu\upsilon}$ is

$$ds^2 = g_{\mu\upsilon} dx^\mu dx^\upsilon \tag{1}$$

In gravity potential $g_{\mu\upsilon}$, we introduce tensor $f_{\mu\upsilon}$ and scalar K.

$$f_{\mu\upsilon} = K g_{\mu\upsilon} \ , \quad \frac{\partial K}{\partial x^\lambda} = 0$$

$$ds'^2 = f_{\mu\upsilon} dx^\mu dx^\upsilon = \bar{g}_{\mu\upsilon} d\bar{x}^\mu d\bar{x}^\upsilon = \bar{g}_{\mu\upsilon} \frac{\partial \bar{x}^\mu}{\partial \bar{x}'^\alpha} \frac{\partial \bar{x}^\upsilon}{\partial \bar{x}'^\beta} d\bar{x}'^\alpha \, d\bar{x}'^\beta$$

$$= \bar{g}'_{\alpha\beta} \, d\bar{x}'^\alpha \, d\bar{x}'^\beta = f'_{\alpha\beta} \, dx'^\alpha \, dx'^\beta$$

$$\bar{g}'_{\alpha\beta} = \bar{g}_{\mu\upsilon} \frac{\partial \bar{x}^\mu}{\partial \bar{x}'^\alpha} \frac{\partial \bar{x}^\upsilon}{\partial \bar{x}'^\beta} \ , \quad f'_{\alpha\beta} = f_{\mu\upsilon} \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\upsilon}{\partial x'^\beta} \tag{2}$$

In inverse gravity potential $g^{\mu\upsilon}$,

$$f^{\mu\upsilon} f_{\mu\upsilon} = \delta_\mu{}^\upsilon = (\frac{1}{K} g^{\mu\upsilon})(K g_{\mu\upsilon}), \ f^{\mu\upsilon} = \frac{1}{K} g^{\mu\upsilon} \tag{3}$$

In Christoffel symbol $\Gamma^\rho{}_{\mu\upsilon}$,

$$\Gamma'^\rho{}_{\mu\upsilon} = \frac{1}{2} f^{\rho\lambda} (\frac{\partial f_{\mu\lambda}}{\partial x^\upsilon} + \frac{\partial f_{\upsilon\lambda}}{\partial x^\mu} - \frac{\partial f_{\mu\upsilon}}{\partial x^\lambda})$$

$$= \frac{1}{2} (\frac{1}{K} g^{\rho\lambda})(K \frac{\partial g_{\mu\lambda}}{\partial x^\upsilon} + K \frac{\partial g_{\upsilon\lambda}}{\partial x^\mu} - K \frac{\partial g_{\mu\upsilon}}{\partial x^\lambda}) = \Gamma^\rho{}_{\mu\upsilon}$$

$$\bar{\Gamma}^\rho{}_{\mu\upsilon} = \frac{1}{2} \bar{g}^{\rho\lambda} (\frac{\partial \bar{g}_{\mu\lambda}}{\partial \bar{x}^\upsilon} + \frac{\partial \bar{g}_{\upsilon\lambda}}{\partial \bar{x}^\mu} - \frac{\partial \bar{g}_{\mu\upsilon}}{\partial \bar{x}^\lambda}) = \frac{1}{\sqrt{K}} \Gamma^\rho{}_{\mu\upsilon} \tag{4}$$

Therefore, in the curvature tensor $R^\rho{}_{\mu\upsilon\lambda}$,

$$R'^\rho{}_{\mu\upsilon\lambda} = \frac{\partial \Gamma'^\rho{}_{\mu\upsilon}}{\partial x^\lambda} - \frac{\partial \Gamma'^\rho{}_{\mu\lambda}}{\partial x^\upsilon} + \Gamma'^\sigma{}_{\mu\upsilon} \Gamma'^\rho{}_{\sigma\lambda} - \Gamma'^\sigma{}_{\mu\lambda} \Gamma'^\rho{}_{\sigma\upsilon}$$

$$= \frac{\partial \Gamma^\rho{}_{\mu\upsilon}}{\partial x^\lambda} - \frac{\partial \Gamma^\rho{}_{\mu\lambda}}{\partial x^\upsilon} + \Gamma^\sigma{}_{\mu\upsilon} \Gamma^\rho{}_{\sigma\lambda} - \Gamma^\sigma{}_{\mu\lambda} \Gamma^\rho{}_{\sigma\upsilon} = R^\rho{}_{\mu\upsilon\lambda}$$

$$\overline{R}^{\rho}{}_{\mu\upsilon\lambda} = \frac{\partial\overline{\Gamma}^{\rho}{}_{\mu\upsilon}}{\partial\overline{x}^{\lambda}} - \frac{\partial\overline{\Gamma}^{\rho}{}_{\mu\lambda}}{\partial\overline{x}^{\upsilon}} + \overline{\Gamma}^{\sigma}{}_{\mu\upsilon}\overline{\Gamma}^{\rho}{}_{\sigma\lambda} - \overline{\Gamma}^{\sigma}{}_{\mu\lambda}\overline{\Gamma}^{\rho}{}_{\sigma\upsilon}$$

$$= \frac{1}{K}\left(\frac{\partial\Gamma^{\rho}{}_{\mu\upsilon}}{\partial x^{\lambda}} - \frac{\partial\Gamma^{\rho}{}_{\mu\lambda}}{\partial x^{\upsilon}} + \Gamma^{\sigma}{}_{\mu\upsilon}\Gamma^{\rho}{}_{\sigma\lambda} - \Gamma^{\sigma}{}_{\mu\lambda}\Gamma^{\rho}{}_{\sigma\upsilon}\right) = \frac{1}{K}R^{\rho}{}_{\mu\upsilon\lambda} \quad (5)$$

In Ricci tensor $R_{\mu\upsilon}$,

$$R'_{\mu\upsilon} = R'^{\rho}{}_{\mu\rho\upsilon} = R^{\rho}{}_{\mu\rho\upsilon} = R_{\mu\upsilon} \quad , \overline{R}_{\mu\upsilon} = \overline{R}^{\rho}{}_{\mu\rho\upsilon} = \frac{1}{K}R^{\rho}{}_{\mu\rho\upsilon} = \frac{1}{K}R_{\mu\upsilon}$$

(6)

In curvature scalar R

$$R' = f^{\mu\upsilon}R'_{\mu\upsilon} = \frac{1}{K}g^{\mu\upsilon}R_{\mu\upsilon} = \frac{1}{K}R$$

$$\overline{R} = \overline{g}^{\mu\upsilon}\overline{R}_{\mu\upsilon} = \frac{1}{K}g^{\mu\upsilon}R_{\mu\upsilon} = \frac{1}{K}R \quad (7)$$

Hence, in the gravity field equation of Einstein,

$$R'_{\mu\upsilon} - \frac{1}{2}f_{\mu\upsilon}R' = R_{\mu\upsilon} - \frac{1}{2}Kg_{\mu\upsilon}\left(\frac{1}{K}R\right)$$

$$= R_{\mu\upsilon} - \frac{1}{2}g_{\mu\upsilon}R = -\frac{8\pi G}{c^4}T_{\mu\upsilon}$$

$$\overline{R}_{\mu\upsilon} - \frac{1}{2}\overline{g}_{\mu\upsilon}\overline{R} = \frac{1}{K}\left(R_{\mu\upsilon} - \frac{1}{2}g_{\mu\upsilon}R\right)$$

$$= -\frac{8\pi G}{c^4}\frac{1}{K}T_{\mu\upsilon} = -\frac{8\pi G}{c^4}\overline{T}_{\mu\upsilon} \quad (8)$$

In Newtonian approximation, Energy-momentum tensor $T'_{\mu\upsilon}$ is

$$\nabla^2 f_{00} = \nabla^2 K g_{00} \approx -\frac{8\pi G}{c^4}KT_{00} = -\frac{8\pi G}{c^4}T'_{00} \quad (9)$$

$$\rho c^2 = T_{00}, \quad K\rho c^2 = T'_{00}$$

$$\overline{T}_{\mu\upsilon} = \frac{1}{K}T_{\mu\upsilon} \quad (10)$$

$$\overline{\nabla}^2\overline{g}_{00} = \frac{1}{K}\nabla^2 g_{00} \approx -\frac{8\pi G}{c^4}\frac{1}{K}T_{00} = -\frac{8\pi G}{c^4}\overline{T}_{00} \quad (11)$$

$$\rho c^2 = T_{00}, \quad \frac{1}{K}\rho c^2 = \overline{T}_{00}$$

$$T'_{\mu\upsilon} = KT_{\mu\upsilon}, \quad \frac{1}{K}T_{\mu\upsilon} = \overline{T}_{\mu\upsilon} \tag{12}$$

Einstein's gravity field equation is

$$R'_{\mu\upsilon} - \frac{1}{2}f_{\mu\upsilon}R' = R_{\mu\upsilon} - \frac{1}{2}g_{\mu\upsilon}R = -\frac{8\pi G}{c^4}T_{\mu\upsilon} = -\frac{8\pi G}{c^4}\frac{T'_{\mu\upsilon}}{K}$$

$$\overline{R}_{\mu\upsilon} - \frac{1}{2}\overline{g}_{\mu\upsilon}\overline{R} = \frac{1}{K}(R_{\mu\upsilon} - \frac{1}{2}g_{\mu\upsilon}R) = -\frac{1}{K}\frac{8\pi G}{c^4}T_{\mu\upsilon} = -\frac{8\pi G}{c^4}\overline{T}_{\mu\upsilon} \tag{13}$$

Therefore, tensor $f_{\mu\upsilon}$ satisfy new gravity field equation of Einstein.

$$f^{\mu\upsilon}[R'_{\mu\upsilon} - \frac{1}{2}f_{\mu\upsilon}R'] = \frac{1}{K}g^{\mu\upsilon}[R_{\mu\upsilon} - \frac{1}{2}g_{\mu\upsilon}R] = -\frac{8\pi G}{c^4}\frac{1}{K}g^{\mu\upsilon}T_{\mu\upsilon} = -\frac{8\pi G}{c^4}\frac{1}{K}T^{\lambda}{}_{\lambda}$$

$$\overline{g}^{\mu\upsilon}[\overline{R}_{\mu\upsilon} - \frac{1}{2}\overline{g}_{\mu\upsilon}\overline{R}] = \frac{1}{K}g^{\mu\upsilon}[R_{\mu\upsilon} - \frac{1}{2}g_{\mu\upsilon}R] = -\frac{8\pi G}{c^4}\frac{1}{K}g^{\mu\upsilon}T_{\mu\upsilon} = -\frac{8\pi G}{c^4}\frac{1}{K}T^{\lambda}{}_{\lambda}$$

$$= -\frac{8\pi G}{c^4}\overline{g}^{\mu\upsilon}\overline{T}_{\mu\upsilon} = -\frac{8\pi G}{c^4}\overline{T}^{\lambda}{}_{\lambda} \qquad \rightarrow$$

$$-\overline{R} = -\frac{1}{K}R = -\frac{8\pi G}{c^4}\frac{1}{K}T^{\lambda}{}_{\lambda} = -\frac{8\pi G}{c^4}\overline{T}^{\lambda}{}_{\lambda}$$

$$= -\frac{8\pi G}{c^4}f^{\mu\upsilon}\frac{T'_{\mu\upsilon}}{K} = -\frac{8\pi G}{c^4}\frac{1}{K}T'^{\lambda}{}_{\lambda}$$

$$\rightarrow \quad -R' = -\frac{1}{K}R = -\frac{8\pi G}{c^4}\frac{1}{K}T^{\lambda}{}_{\lambda} = -\frac{8\pi G}{c^4}\frac{1}{K}T'^{\lambda}{}_{\lambda} \quad ,$$

$$T'^{\lambda}{}_{\lambda} = T^{\lambda}{}_{\lambda}, \frac{1}{K}T^{\lambda}{}_{\lambda} = \overline{T}^{\lambda}{}_{\lambda} \tag{14}$$

Ricci tensor is

$$R'_{\mu\upsilon} = R_{\mu\upsilon} = -\frac{8\pi G}{c^4}(T_{\mu\upsilon} - \frac{1}{2}g_{\mu\upsilon}T^{\lambda}{}_{\lambda}) = -\frac{8\pi G}{c^4}(\frac{T'_{\mu\upsilon}}{K} - \frac{1}{2}\frac{f_{\mu\upsilon}}{K}T'^{\lambda}{}_{\lambda})$$

$$\overline{R}_{\mu\upsilon} = \frac{1}{K}R_{\mu\upsilon} = -\frac{8\pi G}{c^4}\frac{1}{K}(T_{\mu\upsilon} - \frac{1}{2}g_{\mu\upsilon}T^{\lambda}{}_{\lambda}) = -\frac{8\pi G}{c^4}(\overline{T}_{\mu\upsilon} - \frac{1}{2}\overline{g}_{\mu\upsilon}\overline{T}^{\lambda}{}_{\lambda})$$

$$f_{\mu\upsilon} = Kg_{\mu\upsilon} \quad , \quad \frac{\partial K}{\partial x^{\lambda}} = 0$$

$$ds^2 = g_{\mu\upsilon}dx^{\mu}dx^{\upsilon}, \; ds'^2 = f_{\mu\upsilon}dx^{\mu}dx^{\upsilon} = \overline{g}_{\mu\upsilon}d\overline{x}^{\mu}d\overline{x}^{\upsilon} \tag{15}$$

2. Weak gravity field approximation.

Weak gravity field approximation is

$$g_{\mu\upsilon} = \eta_{\mu\upsilon} + h_{\mu\upsilon} \ , \ \bar{g}_{\mu\upsilon} = \bar{\eta}_{\mu\upsilon} + \bar{h}_{\mu\upsilon}$$

$$R_{\mu\upsilon} = R'_{\mu\upsilon} = -\frac{8\pi G}{c^4}(T_{\mu\upsilon} - \frac{1}{2}g_{\mu\upsilon}T^{\lambda}{}_{\lambda})$$

$$\frac{1}{K}R_{\mu\upsilon} = \bar{R}_{\mu\upsilon} = -\frac{8\pi G}{c^4}(\bar{T}_{\mu\upsilon} - \frac{1}{2}\bar{g}_{\mu\upsilon}\bar{T}^{\lambda}{}_{\lambda})$$

$$R_{\mu\upsilon} = -\frac{8\pi G}{c^4}S_{\mu\upsilon}, \bar{R}_{\mu\upsilon} = -\frac{8\pi G}{c^4}\bar{S}_{\mu\upsilon}$$

$$S_{\mu\upsilon} = T_{\mu\upsilon} - \frac{1}{2}g_{\mu\upsilon}T^{\lambda}{}_{\lambda} \qquad ,\bar{S}_{\mu\upsilon} = \bar{T}_{\mu\upsilon} - \frac{1}{2}\bar{g}_{\mu\upsilon}\bar{T}^{\lambda}{}_{\lambda} \qquad (16)$$

The solution is

$$h_{\mu\upsilon}(t,\vec{x}) = \frac{4G}{c^2}\int d^4x' \frac{S_{\mu\upsilon}(t - |\vec{x} - \vec{x}'|, \vec{x}')}{|\vec{x} - \vec{x}'|} \qquad , \qquad \int d^3x T_{00} = M$$

$$\bar{h}_{\mu\upsilon}(t,\vec{\bar{x}}) = \frac{4G}{c^2}\int d^4\bar{x}' \frac{\bar{S}_{\mu\upsilon}(\bar{t} - |\vec{\bar{x}} - \vec{\bar{x}}'|, \vec{\bar{x}}')}{|\vec{\bar{x}} - \vec{\bar{x}}'|} \qquad ,$$

$$\int d^3\bar{x}\bar{T}_{00} = \int K\sqrt{K}d^3x\frac{1}{K}T_{00} = \sqrt{K}M = \bar{M}$$

$$\bar{h}_{00}(\vec{\bar{x}}) \approx \frac{4G}{\bar{r}c^2}\int d^3\bar{x}'[\bar{T}_{00} - \frac{1}{2}\bar{T}_{00}] = \frac{2\sqrt{K}GM}{\bar{r}c^2},$$

$$\int d^3\bar{x}\bar{T}_{00} = \int d^3xK\sqrt{K}\bar{T}_{00} = \int d^3x\sqrt{K}T_{00} = \bar{M}$$

$$\bar{h}_{ij}(\vec{\bar{x}}) \approx \frac{4G}{\bar{r}c^2}\int d^3\bar{x}'[\frac{1}{2}\delta_{ij}\bar{T}_{00}] = \frac{2\sqrt{K}GM}{\bar{r}c^2}\delta_{ij}, \qquad T_{\mu\upsilon} = K\bar{T}_{\mu\upsilon} \qquad (17)$$

The proper distance is

$$-ds^2 = c^2d\tau^2 = -g_{\mu\upsilon}dx^{\mu}dx^{\upsilon} \approx (1 - \frac{2GM}{rc^2})c^2dt^2 - (1 + \frac{2GM}{rc^2})\delta_{ij}dx^idx^j$$

$$-ds'^2 = -Kds^2 = Kc^2d\tau^2 = -Kg_{\mu\upsilon}dx^{\mu}dx^{\upsilon}$$

$$\approx K(1 - \frac{2GM}{rc^2})c^2dt^2 - K(1 + \frac{2GM}{rc^2})\delta_{ij}dx^idx^j$$

$$= (1 - \frac{2\sqrt{K}GM}{\bar{r}c^2})c^2d\bar{t}^2 - (1 + \frac{2\sqrt{K}GM}{\bar{r}c^2})\delta_{ij}d\bar{x}^id\bar{x}^j$$

$$= (1 - \frac{2G\bar{M}}{\bar{r}c^2})c^2d\bar{t}^2 - (1 + \frac{2G\bar{M}}{\bar{r}c^2})\delta_{ij}d\bar{x}^id\bar{x}^j$$

$$= \bar{g}_{\mu\upsilon} d\bar{x}^{\mu} d\bar{x}^{\upsilon}$$

$$\sqrt{K}t = \bar{t}\ , \sqrt{K}x^{i} = \bar{x}^{i},\ \sqrt{K}x^{j} = \bar{x}^{j}\ ,\ \sqrt{K}r = \bar{r}\ , \sqrt{K}M = \overline{M} \tag{18}$$

3. The other representation in Schwarzschild solution, Reissner-Nodstrom solution, Kerr-Newman solution and Robertson-Walker solution

Schwarzschild solution (vaccum solution) is

$$R_{\mu\upsilon} = R'_{\mu\upsilon} = 0$$

$$ds^2 = -c^2(1 - \frac{2GM}{rc^2})dt^2 + \frac{dr^2}{1 - \frac{2GM}{rc^2}} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{19}$$

The other representation of Schwarzschild solution is

$$ds'^2 = f_{\mu\upsilon} dx^{\mu} dx^{\upsilon} = K g_{\mu\upsilon} dx^{\mu} dx^{\upsilon} = K \cdot ds^2$$

$$= -c^2 K(1 - \frac{2GM}{rc^2})dt^2 + \frac{K dr^2}{1 - \frac{2GM}{rc^2}} + K r^2 d\theta^2 + K r^2 \sin^2\theta d\phi^2$$

$$= -c^2(1 - \frac{2\sqrt{K}GM}{\bar{r}c^2})d\bar{t}^2 + \frac{d\bar{r}^2}{1 - \frac{2\sqrt{K}GM}{\bar{r}c^2}} + \bar{r}^2 d\bar{\theta}^2 + \bar{r}^2 \sin^2\bar{\theta} d\bar{\phi}^2$$

$$= -c^2(1 - \frac{2G\overline{M}}{\bar{r}c^2})d\bar{t}^2 + \frac{d\bar{r}^2}{1 - \frac{2G\overline{M}}{\bar{r}c^2}} + \bar{r}^2 d\bar{\theta}^2 + \bar{r}^2 \sin^2\bar{\theta} d\bar{\phi}^2 = \bar{g}_{\mu\upsilon} d\bar{x}^{\mu} d\bar{x}^{\upsilon}$$

$$\sqrt{K}t = \bar{t}\ ,\ \sqrt{K}r = \bar{r}\ , \theta = \bar{\theta}, \phi = \bar{\phi}\ , \sqrt{K}M = \overline{M} \tag{20}$$

Reissner-Nodstrom solution is

$$ds^2 = g_{\mu\upsilon} dx^{\mu} dx^{\upsilon}$$

$$= -c^2(1 - \frac{2GM}{rc^2} + \frac{kGQ^2}{r^2c^4})dt^2 + \frac{dr^2}{1 - \frac{2GM}{rc^2} + \frac{kGQ^2}{r^2c^4}} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{21}$$

The other representation of Reissner-Nodstrom solution is

$$ds'^2 = f_{\mu\upsilon} dx^{\mu} dx^{\upsilon} = K g_{\mu\upsilon} dx^{\mu} dx^{\upsilon} = K \cdot ds^2$$

$$= -Kc^2(1-\frac{2GM}{rc^2}+\frac{kGQ^2}{r^2c^4})dt^2+\frac{Kdr^2}{1-\frac{2GM}{rc^2}+\frac{kGQ^2}{r^2c^4}}+Kr^2d\theta^2+Kr^2\sin^2\theta d\phi^2$$

$$= -c^2(1-\frac{2\sqrt{K}GM}{\bar{r}c^2}+\frac{KkGQ^2}{\bar{r}^2c^4})d\bar{t}^2+\frac{d\bar{r}^2}{1-\frac{2\sqrt{K}GM}{\bar{r}c^2}+\frac{KkGQ^2}{\bar{r}^2c^4}}+\bar{r}^2d\bar{\theta}^2+\bar{r}^2\sin^2\bar{\theta}d\bar{\phi}^2$$

$$= -c^2(1-\frac{2G\overline{M}}{\bar{r}c^2}+\frac{kG\overline{Q}^2}{\bar{r}^2c^4})d\bar{t}^2+\frac{d\bar{r}^2}{1-\frac{2G\overline{M}}{\bar{r}c^2}+\frac{kG\overline{Q}^2}{\bar{r}^2c^4}}+\bar{r}^2d\bar{\theta}^2+\bar{r}^2\sin^2\bar{\theta}d\bar{\phi}^2$$

$$= \bar{g}_{\mu\upsilon}d\bar{x}^{\mu}d\bar{x}^{\upsilon}$$

$$\sqrt{K}t=\bar{t}\ ,\ \sqrt{K}r=\bar{r}\ \ ,\theta=\bar{\theta},\phi=\bar{\phi}\,,\sqrt{K}M=\overline{M}\ ,KQ^2=\overline{Q}^2 \qquad (22)$$

Kerr-Newman solution is

$$ds^2=g_{\mu\upsilon}dx^{\mu}dx^{\upsilon}$$

$$= -c^2(1-\frac{2c^2GMr-kGQ^2}{c^4\Sigma})dt^2+2(2c^2MGr-kGQ^2)\frac{a\sin^2\theta}{c^4\Sigma}cdtd\phi$$
$$-\frac{c^4\Sigma}{r^2-c^22GMr+a^2+kGQ^2}dr^2-\Sigma d\theta^2$$
$$-\sin^2\theta[r^2+a^2+(2c^2GMr-kGQ^2)\frac{a^2\sin^2\theta}{c^4\Sigma}]d\phi^2$$

$$\Sigma=r^2+a^2\cos^2\theta \qquad (23)$$

The other representation of Kerr-Newman solution is

$$ds'^2=f_{\mu\upsilon}dx^{\mu}dx^{\upsilon}=Kg_{\mu\upsilon}dx^{\mu}dx^{\upsilon}=K\cdot ds^2$$

$$= -Kc^2(1-\frac{2c^2GMr-kGQ^2}{c^4\Sigma})dt^2+2K(2c^2MGr-kGQ^2)\frac{a\sin^2\theta}{c^4\Sigma}cdtd\phi$$
$$-\frac{K\Sigma c^4}{r^2-2c^2GMr+a^2+kGQ^2}dr^2-K\Sigma d\theta^2$$
$$-K\sin^2\theta[r^2+a^2+(2c^2GMr-kGQ^2)\frac{a^2\sin^2\theta}{c^4\Sigma}]d\phi^2$$

$$= -c^2(1-\frac{2c^2G\sqrt{K}M\sqrt{K}r-kGKQ^2}{K\Sigma c^4})d\bar{t}^2+2(2c^2\sqrt{K}MG\sqrt{K}r-kGKQ^2)\frac{\sqrt{K}a\sin^2\theta}{K\Sigma c^4}cd\sqrt{K}td\phi$$

$$-\frac{K\Sigma c^4}{Kr^2 - 2c^2G\sqrt{K}M\sqrt{K}r + Ka^2 + kGKQ^2}d(\sqrt{K}r)^2 - K\Sigma d\theta^2$$

$$-\sin^2\theta[Kr^2 + Ka^2 + (2c^2G\sqrt{K}M\sqrt{K}r - kGKQ^2)\frac{Ka^2\sin^2\theta}{K\Sigma c^4}]d\phi^2$$

$$= -c^2(1 - \frac{2c^2G\overline{M}\bar{r} - kG\overline{Q}^2}{\overline{\Sigma}c^4})d\bar{t}^2 + 2(2c^2\overline{M}G\bar{r} - kG\overline{Q}^2)\frac{\bar{a}\sin^2\bar{\theta}}{\overline{\Sigma}c^4}cd\bar{t}d\bar{\phi}$$

$$-\frac{\overline{\Sigma}c^4}{\bar{r}^2 - 2c^2G\overline{M}\bar{r} + \bar{a}^2 + kG\overline{Q}^2}d\bar{r}^2 - \overline{\Sigma}d\bar{\theta}^2$$

$$-\sin^2\bar{\theta}[\bar{r}^2 + \bar{a}^2 + (2c^2G\overline{M}\bar{r} - kG\overline{Q}^2)\frac{\bar{a}^2\sin^2\bar{\theta}}{c^4\overline{\Sigma}}]d\bar{\phi}^2$$

$$= \bar{g}_{\mu\upsilon}d\bar{x}^{\mu}d\bar{x}^{\upsilon}$$

$$\overline{\Sigma} = K\Sigma = Kr^2 + Ka^2\cos^2\theta = \bar{r}^2 + \bar{a}^2\cos^2\bar{\theta}$$

$$\sqrt{K}t = \bar{t}\,,\ \sqrt{K}r = \bar{r}\ ,\theta = \bar{\theta}, \phi = \bar{\phi}, \sqrt{K}M = \overline{M}\ , KQ^2 = \overline{Q}^2, \sqrt{K}a = \bar{a}$$

(24)

Robertson-Warker solution is

$$ds^2 = g_{\mu\upsilon}dx^{\mu}dx^{\upsilon}$$

$$= -c^2dt^2 + \Omega^2(t)[\frac{dr^2}{1-Kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2] \qquad (25)$$

The other representation of Robertson-Warker solution is by the other scalar K',

$$ds'^2 = f_{\mu\upsilon}dx^{\mu}dx^{\upsilon} = K'g_{\mu\upsilon}dx^{\mu}dx^{\upsilon} = K'\cdot ds^2$$

$$= -K'c^2dt^2 + \Omega^2(t)[\frac{K'dr^2}{1-k\frac{K'r^2}{K'}} + K'r^2d\theta^2 + K'r^2\sin^2\theta d\phi^2]$$

$$= -c^2d\bar{t}^2 + \overline{\Omega}^2(\bar{t})[\frac{d\bar{r}^2}{1-\frac{k\bar{r}^2}{K'}} + \bar{r}^2d\bar{\theta}^2 + \bar{r}^2\sin^2\bar{\theta}d\bar{\phi}^2]$$

$$= -c^2d\bar{t}^2 + \overline{\Omega}^2(\bar{t})[\frac{d\bar{r}^2}{1-k'\bar{r}^2} + \bar{r}^2d\bar{\theta}^2 + \bar{r}^2\sin^2\bar{\theta}d\bar{\phi}^2] = \bar{g}_{\mu\upsilon}d\bar{x}^{\mu}d\bar{x}^{\upsilon}$$

$$\sqrt{K'}t = \bar{t}\ , \Omega(t) = \overline{\Omega}(\bar{t}),$$

$$\sqrt{K'}r = \bar{r}\ , \theta = \bar{\theta}, \phi = \bar{\phi}$$

$$k=(0,1,-1),\ \ k'=\frac{k}{K'}=(0,\frac{1}{K'},-\frac{1}{K'}) \tag{26}$$

Hence, $K'=1$, In this time, ds^2 is an uniqueness.

4. Conclusion

We find the other representation of solutions in the General relativity theory. In this time, Robertson-Warker solution is an uniqueness.

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