q-ANALOGUES FOR RAMANUJAN-TYPE SERIES

BING HE AND HONGCUN ZHAI

ABSTRACT. From a very-well-poised ${}_6\phi_5$ series formula we deduce a general series expansion formula involving the q-gamma function. With this formula we can give q-analogues of many Ramanujan-type series.

1. INTRODUCTION

In [10] Ramanujan listed 17 series expansions for $1/\pi$ without proof and the proof of the first three was sketched in [9]. The Borwein brothers found the first complete proof of all the 17 formulas in [2]. D.V. Chudnovsky and G.V. Chudnovsky [3] proved several series representations of the Ramanujan's independently and established certain new series as well. Please see [1] for the history of the Ramanujan-type series for $1/\pi$. Recently, Liu [7, 8] established many series expansions for $1/\pi$ by using properties of the general rising shifted factorial and the gamma function. In the recent paper [6], Guo and Liu supplied q-analogues of two Ramanujan-type series for $1/\pi$ by using q-WZ pairs and some basic hypergeometric identities. Motivated by the work of Liu [7, 8] and Guo and Liu [6] we shall establish q-analogues for Ramanujan-type series in this work. Our method is different from that of Guo and Liu.

Throughout this paper we assume $|q| < 1$. Gosper [5] introduced q-analogues of $\sin x$ and π :

$$\sin_q(\pi x) := q^{(x-1/2)^2} \frac{(q^{2-2x};q^2)_\infty (q^{2x};q^2)_\infty}{(q;q^2)_\infty^2}$$

and

$$\pi_q := (1-q^2)q^{1/4} \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2},$$

where $(z;q)_\infty$ is given by

$$(z;q)_\infty = \prod_{n=0}^{\infty}(1-zq^n).$$

They satisfy the following relations:

$$\lim_{q\to 1} \sin_q x = \sin x, \quad \lim_{q\to 1} \pi_q = \pi$$

and

$$\Gamma_{q^2}(x)\Gamma_{q^2}(1-x) = \frac{\pi_q}{\sin_q(\pi x)} q^{x(x-1)}, \tag{1.1}$$

where $\Gamma_q(x)$ is the q-gamma function defined by

2000 *Mathematics Subject Classification.* 33D05, 33D15, 65B10.

Key words and phrases. q-Analogue; Ramanujan-type series; a very-well-poised ${}_6\phi_5$ series formula.

The first author is the corresponding author.

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}. \tag{1.2}$$

From the definition of the q-gamma function we can see that

$$\frac{(q^x;q)_n}{(1-q)^n} = \frac{\Gamma_q(x+n)}{\Gamma_q(x)}, \tag{1.3}$$

where n is a non-negative integer and $(z;q)_n$ is the q-shifted factorial defined as

$$(z;q)_0 = 1,\ (z;q)_n = \prod_{k=0}^{n-1}(1-zq^k) \text{ for } n \geq 1.$$

We now extend the definition of $(q^x;q)_n$ to any complex α :

$$(q^x;q)_\alpha = \frac{\Gamma_q(x+\alpha)}{\Gamma_q(x)}(1-q)^\alpha \tag{1.4}$$

and denote $\frac{(q^x;q)_\alpha}{(1-q)^\alpha}$ by $(x|q)_\alpha$. Then, for any non-negative integer n, we have

$$(x|q)_n = \prod_{k=0}^{n-1}[x+k]_q$$

and

$$(x|q)_{-n} = \frac{\Gamma_q(x-n)}{\Gamma_q(x)} = \frac{(1-q)^n}{(q^{x-n};q)_n},$$

where $[z]_q$ is the q-integer defined by

$$[z]_q = \frac{1-q^z}{1-q}.$$

Our main aim of the present work is to establish the following general series expansion.

Theorem 1.1. *For any complex number α and $\mathrm{Re}(1+a+b+c+d) > 0$ we have*

$$\sum_{n=0}^{\infty} \frac{(1-q^{4n+2a+2\alpha})(\alpha|q^2)_{a+n}(\beta|q^2)_{n-b}(\gamma|q^2)_{n-c}(\delta|q^2)_{n-d}}{(1-q^2)[n]_{q^2}!(1+\alpha-\beta|q^2)_{a+b+n}(1+\alpha-\gamma|q^2)_{a+c+n}(1+\alpha-\delta|q^2)_{a+d+n}} q^{An}$$
$$= \frac{\Gamma_{q^2}(1+\alpha-\beta)\Gamma_{q^2}(1+\alpha-\gamma)\Gamma_{q^2}(1+\alpha-\delta)\Gamma_{q^2}(2+\alpha-\beta-\gamma-\delta)}{\Gamma_{q^2}(\alpha)\Gamma_{q^2}(1+\alpha-\beta-\gamma)\Gamma_{q^2}(1+\alpha-\beta-\delta)\Gamma_{q^2}(1+\alpha-\gamma-\delta)}$$
$$\times \frac{(\beta|q^2)_{-b}(\gamma|q^2)_{-c}(\delta|q^2)_{-d}(2+\alpha-\beta-\gamma-\delta|q^2)_{a+b+c+d-1}}{(1+\alpha-\beta-\gamma|q^2)_{a+b+c}(1+\alpha-\beta-\delta|q^2)_{a+b+d}(1+\alpha-\gamma-\delta|q^2)_{a+c+d}},$$

where $A = 2(a+b+c+d+1+\alpha-\beta-\gamma-\delta)$ and $[n]_q!$ is given by

$$[0]_q! = 1, \quad [n]_q! = \prod_{k=1}^{n}[k]_q \text{ for } n \geq 1.$$

The next section is devoted to our proof of Theorem 1.1. In Section 3 we deduce q-analogues of certain Ramanujan type series for $1/\pi$. In the last section several q-analogues of series expansions for π^2 are also obtained.

2. Proof of Theorem 1.1

Recall the following summation formula for the basic hypergeometric series [4, (2.7.1)]:

$$ {}_6\phi_5\left(\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, b, c, d \\ a^{\frac12}, -a^{\frac12}, aq/b, aq/c, aq/d \end{matrix}; q, \frac{aq}{bcd}\right) = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd; q)_\infty}, \tag{2.1} $$

where $\left|\dfrac{aq}{bcd}\right| < 1$ and ${}_6\phi_5$ is the basic hypergeometric series given by

$$ {}_6\phi_5\left(\begin{matrix} a_1, a_2, a_3, a_4, a_5, a_6 \\ b_1, b_2, b_3, b_4, b_5 \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, a_3, a_4, a_5, a_6; q)_n}{(q, b_1, b_2, b_3, b_4, b_5; q)_n} z^n. $$

Replacing (q, a, b, c, d) by $(q^2, q^{2a}, q^{2b}, q^{2c}, q^{2d})$ in (2.1) and employing (1.2) and (1.3) we have

(2.2)

$$ \sum_{n=0}^{\infty} \frac{(1-q^{4n+2a})\Gamma_{q^2}(a+n)\Gamma_{q^2}(b+n)\Gamma_{q^2}(c+n)\Gamma_{q^2}(d+n)}{(1-q^2)[n]_{q^2}!\Gamma_{q^2}(1+a-b+n)\Gamma_{q^2}(1+a-c+n)\Gamma_{q^2}(1+a-d+n)} q^{2n(1+a-b-c-d)} $$
$$ = \frac{\Gamma_{q^2}(b)\Gamma_{q^2}(c)\Gamma_{q^2}(d)\Gamma_{q^2}(1+a-b-c-d)}{\Gamma_{q^2}(1+a-b-c)\Gamma_{q^2}(1+a-b-d)\Gamma_{q^2}(1+a-c-d)}. $$

It follows from (1.4) that

$$ \Gamma_{q^2}(a+n+\alpha) = (\alpha|q^2)_{a+n}\Gamma_{q^2}(\alpha), \Gamma_{q^2}(n-b+\beta) = (\beta|q^2)_{n-b}\Gamma_{q^2}(\beta), $$
$$ \Gamma_{q^2}(n-c+\gamma) = (\gamma|q^2)_{n-c}\Gamma_{q^2}(\gamma), \Gamma_{q^2}(n-d+\delta) = (\delta|q^2)_{n-d}\Gamma_{q^2}(\delta), $$
$$ \Gamma_{q^2}(\beta-b) = (\beta|q^2)_{-b}\Gamma_{q^2}(\beta), \Gamma_{q^2}(\gamma-c) = (\gamma|q^2)_{-c}\Gamma_{q^2}(\gamma), \Gamma_{q^2}(\delta-d) = (\delta|q^2)_{-d}\Gamma_{q^2}(\delta), $$
$$ \Gamma_{q^2}(a+b+n+1+\alpha-\beta) = (1+\alpha-\beta|q^2)_{a+b+n}\Gamma_{q^2}(1+\alpha-\beta), $$
$$ \Gamma_{q^2}(a+c+n+1+\alpha-\gamma) = (1+\alpha-\gamma|q^2)_{a+c+n}\Gamma_{q^2}(1+\alpha-\gamma), $$
$$ \Gamma_{q^2}(a+d+n+1+\alpha-\delta) = (1+\alpha-\delta|q^2)_{a+d+n}\Gamma_{q^2}(1+\alpha-\delta), $$
$$ \Gamma_{q^2}(a+b+c+1+\alpha-\beta-\gamma) = (1+\alpha-\beta-\gamma|q^2)_{a+b+c}\Gamma_{q^2}(1+\alpha-\beta-\gamma), $$
$$ \Gamma_{q^2}(a+b+d+1+\alpha-\beta-\delta) = (1+\alpha-\beta-\delta|q^2)_{a+b+d}\Gamma_{q^2}(1+\alpha-\beta-\delta), $$
$$ \Gamma_{q^2}(a+c+d+1+\alpha-\gamma-\delta) = (1+\alpha-\gamma-\delta|q^2)_{a+c+d}\Gamma_{q^2}(1+\alpha-\gamma-\delta) $$

and

$$ \Gamma_{q^2}(a+b+c+d+1+\alpha-\beta-\gamma-\delta) $$
$$ = (2+\alpha-\beta-\gamma-\delta|q^2)_{a+b+c+d-1}\Gamma_{q^2}(2+\alpha-\beta-\gamma-\delta). $$

Making the sunbstitutions: $a \to a+\alpha$, $b \to \beta - b$, $c \to \gamma - c$, $d \to \delta - d$ in (2.2) and then substituting the above identities into the resulting equation we can easily deduce the result. This finishes the proof of Theorem 1.1. □

3. q-Analogues of Ramanujan type series for $1/\pi$

In this section we employ Theorem 1.1 to deduce certain q-analogues of Ramanujan type series for $1/\pi$.

Theorem 3.1. *For* $\mathrm{Re}(a+b+c+d)>0$ *we have*

$$\sum_{n=0}^{\infty}\frac{(1-q^{4n+2a+1})(1/2|q^2)_{a+n}(1/2|q^2)_{n-b}(1/3|q^2)_{n-c}(2/3|q^2)_{n-d}}{(1-q^2)[n]_{q^2}!(1|q^2)_{a+b+n}(7/6|q^2)_{a+c+n}(5/6|q^2)_{a+d+n}}q^{2(a+b+c+d)n}$$
$$=\frac{(1/2|q^2)_{-b}(1/3|q^2)_{-c}(2/3|q^2)_{-d}(1|q^2)_{a+b+c+d-1}}{(1/3|q^2)_{a+b+d}(2/3|q^2)_{a+b+c}(1/2|q^2)_{a+c+d}}\cdot\frac{[1/6]_{q^2}(q^{4/3},q^{2/3};q^2)_\infty q^{1/4}}{(q^{1/3},q^{5/3};q^2)_\infty\pi_q}.$$

Proof. It follows from (1.1) that

$$\Gamma^2_{q^2}(1/2)=\pi_q q^{-1/4}, \tag{3.1}$$
$$\Gamma_{q^2}(1/3)\Gamma_{q^2}(2/3)=\frac{\pi_q}{\sin_q(\pi/3)}q^{-2/9},$$
$$\Gamma_{q^2}(7/6)\Gamma_{q^2}(5/6)=[1/6]_{q^2}\Gamma_{q^2}(1/6)\Gamma_{q^2}(5/6)$$
$$=\frac{\pi_q}{\sin_q(\pi/6)}[1/6]_{q^2}q^{-5/36}.$$

Then, by the definition of $\sin_q$,

$$\frac{\Gamma_{q^2}(7/6)\Gamma_{q^2}(5/6)}{\Gamma_{q^2}(1/3)\Gamma_{q^2}(2/3)}=\frac{\sin_q(\pi/3)}{\sin_q(\pi/6)}[1/6]_{q^2}q^{1/12}=\frac{(q^{4/3},q^{2/3};q^2)_\infty[1/6]_{q^2}}{(q^{1/3},q^{5/3};q^2)_\infty}. \tag{3.2}$$

Therefore, the result follows easily by setting $(\alpha,\beta,\gamma,\delta)=(1/2,1/2,1/3,2/3)$ in Theorem 1.1 and applying the identities $\Gamma_q(1)=1$, (3.1) and (3.2). □

Taking $(a,b,c,d)=(1,0,0,0)$ in Theorem 3.1 we can get

Example 3.1. We have

$$\sum_{n=0}^{\infty}\frac{(1-q^{4n+3})(1-q^{2n+1})(1/2|q^2)_n^2(1/3|q^2)_n(2/3|q^2)_n}{(1-q^2)(1-q^{2n+2})([n]_{q^2}!)^2(7/6|q^2)_{1+n}(5/6|q^2)_{1+n}}q^{2n}$$
$$=\frac{[1/6]_{q^2}(q^{4/3},q^{2/3};q^2)_\infty q^{1/4}}{[1/3]_{q^2}[2/3]_{q^2}[1/2]_{q^2}(q^{1/3},q^{5/3};q^2)_\infty\pi_q}.$$

This series expansion for $1/\pi_q$ can be regarded as a q-analogue of the series for $1/\pi$:

$$\sum_{n=0}^{\infty}\frac{(4n+3)(2n+1)(1/2)_n^2(1/3)_n(2/3)_n}{(n+1)(6n+1)(6n+5)(6n+7)(n!)^2(1/6)_n(5/6)_n}=\frac{\sqrt{3}}{6\pi}.$$

Putting $(a,b,c,d)=(0,0,0,1)$ in Theorem 3.1 we can deduce that

Example 3.2. We have

$$\frac{q^{2/3}}{(1+q)[1/3]_{q^2}[5/6]_{q^2}}-\sum_{n=1}^{\infty}\frac{(1-q^{4n+1})(1/2|q^2)_n^2(1/3|q^2)_n(2/3|q^2)_{n-1}}{(1-q^2)([n]_{q^2}!)^2(7/6|q^2)_n(5/6|q^2)_{1+n}}q^{2n}$$
$$=\frac{[1/6]_{q^2}}{[1/3]^2_{q^2}[1/2]_{q^2}}\cdot\frac{(q^{4/3},q^{2/3};q^2)_\infty q^{11/12}}{(q^{1/3},q^{5/3};q^2)_\infty\pi_q}.$$

This series expansion for $1/\pi_q$ can be considered as a q-analogue of the series for $1/\pi$:

$$1-\frac{5}{18}\sum_{n=1}^{\infty}\frac{(4n+1)(1/2)_n^2(1/3)_n(2/3)_{n-1}}{(n!)^2(7/6)_n(5/6)_{1+n}}=\frac{5}{\sqrt{3}\pi}.$$

4. q-Analogues of series expansions for π^2

In this section we use Theorem 1.1 to give q-analogues of some series expansions for π^2.

Theorem 4.1. *For* $\mathrm{Re}(a+b+c+d-1/2)>0$ *we have*

$$\sum_{n=0}^{\infty}\frac{(1-q^{4n+2a})(1|q^2)_{a+n-1}(1/2|q^2)_{n-b}(1/2|q^2)_{n-c}(1/2|q^2)_{n-d}}{(1-q^2)[n]_{q^2}!(1/2|q^2)_{a+b+n}(1/2|q^2)_{a+c+n}(1/2|q^2)_{a+d+n}}q^{2(a+b+c+d)n-n}$$
$$=\frac{\pi_q^2(1/2|q^2)_{-b}(1/2|q^2)_{-c}(1/2|q^2)_{-d}(1/2|q^2)_{a+b+c+d-1}}{(1|q^2)_{a+b+c-1}(1|q^2)_{a+b+d-1}(1|q^2)_{a+c+d-1}q^{1/2}}$$

Proof. It can be dedeuced from $\Gamma_q(x+1)=[x]_q\Gamma_q(x)$ and Theorem 1.1 that

$$\sum_{n=0}^{\infty}\frac{(1-q^{4n+2a+2\alpha})(\alpha+1|q^2)_{a+n-1}(\beta|q^2)_{n-b}(\gamma|q^2)_{n-c}(\delta|q^2)_{n-d}}{(1-q^2)[n]_{q^2}!(1+\alpha-\beta|q^2)_{a+b+n}(1+\alpha-\gamma|q^2)_{a+c+n}(1+\alpha-\delta|q^2)_{a+d+n}}q^{An}$$
$$=\frac{\Gamma_{q^2}(1+\alpha-\beta)\Gamma_{q^2}(1+\alpha-\gamma)\Gamma_{q^2}(1+\alpha-\delta)\Gamma_{q^2}(2+\alpha-\beta-\gamma-\delta)}{\Gamma_{q^2}(\alpha+1)\Gamma_{q^2}(2+\alpha-\beta-\gamma)\Gamma_{q^2}(2+\alpha-\beta-\delta)\Gamma_{q^2}(2+\alpha-\gamma-\delta)}$$
$$\times\frac{(\beta|q^2)_{-b}(\gamma|q^2)_{-c}(\delta|q^2)_{-d}(2+\alpha-\beta-\gamma-\delta|q^2)_{a+b+c+d-1}}{(2+\alpha-\beta-\gamma|q^2)_{a+b+c-1}(2+\alpha-\beta-\delta|q^2)_{a+b+d-1}(2+\alpha-\gamma-\delta|q^2)_{a+c+d-1}}.$$

Then the result follows readily from by setting $(\alpha,\beta,\gamma,\delta)=(0,1/2,1/2,1/2)$ in the above identity and applying the identities $\Gamma_q(1)=1$ and (3.1). ☐

Taking $(a,b,c,d)=(1,0,0,0)$ in Theorem 4.1 we can obtain

Example 4.1. We have

$$\sum_{n=0}^{\infty}\frac{(1+q^{2n+1})q^n}{(1-q^{2n+1})^2}=\frac{\pi_q^2}{(1-q^2)^2q^{1/2}}.$$

This series expansion for π_q^2 can be regarded as a q-analogue of the series for π^2 :

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}.$$

Setting $(a,b,c,d)=(1,1,1,0)$ in Theorem 4.1 we can derive

Example 4.2. We have

$$\sum_{n=0}^{\infty}\frac{(1+q^{2n+1})q^{5n}}{(1-q^{2n-1})^2(1-q^{2n+1})^2(1-q^{2n+3})^2}=\frac{\pi_q^2(1+q+q^2)q^{3/2}}{(1+q^2)(1-q^2)^6}.$$

This series expansion for π_q^2 can also be considered as a q-analogue of the series for π^2 :

$$\sum_{n=0}^{\infty}\frac{1}{(2n-1)^2(2n+1)^2(2n+3)^2}=\frac{3\pi^2}{256}.$$

Putting $(a,b,c,d)=(1,1,1,1)$ in Theorem 4.1 we can deduce

Example 4.3. We have

$$\frac{(1+q)q^3}{(1-q)^5(1-q^3)^3}-\sum_{n=1}^{\infty}\frac{(1+q^{2n+1})q^{7n}}{(1-q^{2n-1})^3(1-q^{2n+1})^2(1-q^{2n+3})^3}$$
$$=\frac{\pi_q^2(1+q+q^2)(1+q+q^2+q^3+q^4)q^{5/2}}{(1+q^2)^3(1-q^2)^8}.$$

This series expansion for π_q^2 is also a q-analogue of the series for π^2 :

$$\frac{1}{27}-\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3(2n+1)^2(2n+3)^3}=\frac{15\pi^2}{4096}.$$

Remark. Besides those formulas displayed in Theorems 3.1 and 4.1 and their consequences, we can give a general series expansions for $1/\pi_q^2$ by taking $(\alpha,\beta,\gamma,\delta)=(1/2,1/2,1/2,1/2)$ in Theorem 1.1, from which many series expansions for $1/\pi_q^2$ can be deduced. We shall not display them out one by one in this work.

Acknowledgements

The first author was supported by the Natural Science Basic Research Plan in Shaanxi Province of China (No. 2017JQ1001), the Initial Foundation for Scientific Research of Northwest A&F University (No. 2452015321) and the Fundamental Research Funds for the Central Universities (No. 2452017170). The second author was supported by the National Natural Science Foundation of China (Grant No. 11371184) and the Natural Science Foundation of Henan Province (Grant No. 162300410086, 2016B259, 172102410069).

References

[1] N.D. Baruah, B.C. Berndt and H.H. Chan, Ramanujan's series for $1/\pi$: A survey, Amer. Math. Monthly 116 (2009), 567–587.
[2] J.M. Borwein and P.B. Borwein, Pi and the AGM. Wiley, New York, 1987.
[3] D.V. Chudnovsky, G.V. Chudnovsky, Approximation and complex multiplication according to Ramanujan, in: G.E. Andrews, R.A. Askey, B.C. Berndt, K.G. Ramanathan, R.A. Rankin (Eds.), Ramanujan Revisited, Academic Press, Boston, 1988, pp. 375–472.
[4] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge University Press, Cambridge, 1990.
[5] R.W. Gosper, Experiments and discoveries in q-trigonometry, in: F.G. Garvan, M.E.H. Ismail (Eds.), Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics, Kluwer, Dordrecht, Netherlands, 2001, pp.79–105.
[6] V.J.W. Guo and J.-C. Liu, q-analogues of two Ramanujan-type formulas for $1/\pi$, arXiv: 1802.01944.
[7] Z.-G. Liu, A summation formula and Ramanujan type series, J. Math. Anal. Appl. 389(2) (2012), 1059–1065.
[8] Z.-G. Liu, Gauss summation and Ramanujan-type series for $1/\pi$. Int. J. Number Theory, 8(2) (2012), 289–297.
[9] S. Ramanujan, Collected Papers, Cambridge University Press, Cambridge, 1927, reprinted by Chelsea, New York, 1962, reprinted by the American Mathematical Society, Providence, RI, 2000.
[10] S. Ramanujan, Modular equations and approximations to π, Quart. J. Pure Appl. Math. 45 (1914), 350–372.

College of Science, Northwest A&F University, Yangling 712100, Shaanxi, People's Republic of China
E-mail address: yuhe001@foxmail.com; yuhelingyun@foxmail.com

Department of Mathematics, Luoyang Normal University, Luoyang 471934, People's Republic of China
E-mail address: zhai_hc@163.com