Idempotents in motivic quantum gravity

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Abstract

The quantization of rest mass is governed by strictly discrete data, in principle coming from the combinatorics of a universal category of motives. We outline the essential ingredients from a categorical perspective, giving a context for the Koide mass operators.

This paper started out in discussions with Michael Rios, and in a collaboration with Michael Rios and Raymond Aschheim from Quantum Gravity Research. There have also been valuable discussions with Fang Fang, Marcelo Amaral and Tony Smith.

1 Introduction

When spacetime and gravity emerge from the timeless realm of quantum gravity, we must be able to see how continuum geometries themselves emerge from combinatorial data. In the motivic philosophy, contrary to common sense, classical geometries are derived from the cohomological invariants supposedly derivative to them. A projective space, in particular, has first of all a good cover for sheaf cohomology. For $\mathbb{RP}^1$, this is the three point set representing three open sets. But even open sets are too zero dimensional, from an axiomatic viewpoint.

In a topos, a topological space is replaced by a lattice, or rather a Heyting algebra. Elements in a lattice are idempotents, as are basic operations in quantum mechanics. The rule $PP = P$ is also monadic, suggesting endofunctors on a category of motives, rather than the focus on functors from spatial categories into distinct algebraic ones. We wish to define this universal category of motives in a way that axiomatizes categories themselves, in arbitrary dimensions, since an ordinal is not a cardinality of a set in quantum mechanics. In particular, the associahedra describe directly the cohomology of real projective spaces. Dual to a triangulation of the associahedron is the permutohedron polytope for the group S_n.

All of this comes from considering the information theoretic generation of rest masses in quantum gravity, using the emergent Higgs mechanism. A topological surface, in analogy with topological insulators [1], defines an observational limit to our observable universe at around the Hubble radius [2], and this is related to a new infrared scale for the neutrino masses. Magnetic states associated to this horizon, and all (electric) Standard Model states, are responsible for the emergence of spacetime itself. No gravity without matter.

The central right handed neutrino mass in the Brannen-Koide scheme is $m_R = 0.00117$eV, exactly the present day CMB temperature [2][3]. This suggests a neutrino see-saw rule $m_H = \sqrt{m_R m_P}$, where m_H is the Higgs mass and m_P a Planck scale, in place of the usual Dirac and Majorana masses for neutrinos.

Particle states are represented initially as anyonic ribbon graphs [4], or by elements in related octonion algebras. Compactified Minkowski space $SU(2) \times U(1)$ emerges from the Fibonacci anyon braid group representations for B_3 and B_2, which underlie ribbon pictures. On the algebraic side, integral elements of composition algebras are also associated to lattice geometries. Every computation is ideally motivic, recalling that motivic methods underlie scattering amplitudes in the Standard Model [5][6][7].

2 A noncommutative Fourier transform

At the Streetfest in 2006, Kapranov [8] defined a discrete noncommutative Fourier transform, starting with noncommutative monomials (words) and complex coefficients. Each letter in the alphabet X, Y, $Z \cdots$ represents a path step in the given direction: one step for X, two for X^2 and so on. Thus integer coordinates in any dimension are derived from the powers of each letter in a word. The commutative monomials form a simplex in the noncommutative cube of paths, so that each point in a simplex sits at the end of all possible paths to it.

An elementary square, or homotopy between the paths XY and YX, is assigned a 2-arrow

$$[X, Y] = XY - YX, \tag{1}$$

an elementary commutator which is a boundary of the square. Given general words w and a noncommutative polynomial $\sum c_w w$, define commutative coefficients

$$a_w \equiv \sum_w c_w \tag{2}$$

over all words representing the same commutative monomial. For convenience, now let XY stand for the square, which is both XY and YX. By definition, a boundary

$$d(XY) = [X, Y] \tag{3}$$

has a higher dimensional analogue

$$d(XYZ) = [XY, Z] + [Y, XZ] + [YZ, X] \tag{4}$$

associated to the faces of a three dimensional cube. There is, for instance, a term $(XY)Z$, corresponding to a square whiskered with the edge Z. An arrow $XY \Rightarrow YX$ is typically a categorical braiding for objects X and Y in a category with tensor product. In this case, a whiskered square in dimension 3 stands for a generator of the braid group B_3. Below we will also assign three strands to the vertices of a cube, with twists but no braiding.

Consider words with more than one copy of a letter, on subdivided cubes. The associated commutative simplices are subdivided Postnikov simplices [9], in which the associahedra and permutohedra will be defined. Every point on a simplex carries a monomial.

3 Spatial Points

A Heyting algebra [10] is a not necessarily distributive poset lattice with a 0 and 1 and implication $x \Rightarrow y$. Objects in the lattice are idempotent with respect to meet and join

$$x \vee x = x, \quad x \wedge x = x, \tag{5}$$

but only alternatively distributive, satisfying

$$x \wedge (y \vee x) = x = (x \wedge y) \vee x. \tag{6}$$

Implication satisfies

$$(x \Rightarrow y) \wedge x = x \wedge y \tag{7}$$

and the distributivity

$$x \Rightarrow (y \wedge z) = (x \Rightarrow y) \wedge (x \Rightarrow z). \tag{8}$$

A lattice is non distributive if it contains, in particular, a pentagon, where the poset order defines the arrows of the category.

The duality underlying the non commutative geometry approach is a version of Stone's theorem [11] for categorical lattices. Another important example of this theorem comes from harmonic analyis, where the Fourier transform fixes only the circle, as either the group $U(1)$ or the dual space $\mathbb{R}/\mathbb{Z}$. In motivic quantum gravity, we start instead with the discrete Fourier transform for qubits and qutrits, the latter based on a three point space.

Let ω be the primitive cubed root of unity. The 3×3 Fourier transform

$$F_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{pmatrix} \tag{9}$$

diagonalizes 1-circulants, as in

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = F_3 \begin{pmatrix} r & \theta & \overline{\theta} \\ \overline{\theta} & r & \theta \\ \theta & \overline{\theta} & r \end{pmatrix} {F_3}^{\dagger}, \tag{10}$$

where the circulants are elements of the group Hopf algebra $\mathbb{C}C_3$ and the diagonals belong to a larger Hopf algebra. The circulant idempotents are

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \omega & \overline{\omega} \\ \overline{\omega} & 1 & \omega \\ \omega & \overline{\omega} & 1 \end{pmatrix}, \quad \overline{H} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \overline{\omega} & \omega \\ \omega & 1 & \overline{\omega} \\ \overline{\omega} & \omega & 1 \end{pmatrix}. \tag{11}$$

As Hermitian matrices, these circulants belong to the Jordan algebra $J_3(\mathbb{C})$. We are particularly interested in idempotents for the exceptional algebra $J_3(\mathbb{O})$ over the octonions [12][13], and its extensions by an arbitrary division algebra. Its off diagonal integral points are known to give the Leech lattice [14][15] as three copies of the root lattice for $\mathbf{e_8}$. Our motivic geometries will be discrete in all dimensions out to ∞, bearing in mind that even the full division algebras over $\mathbb{R}$ are ill defined in a (generalized) topos.

A unifying configuration of points for (Lie algebra) root lattices is the magic star [16][17][18] in the plane. For dimension 8, the decomposition

$$\mathbf{e_8} = su(3) + \mathbf{e_6} + (3, 27) + (3, \overline{27})$$

introduces a gravitational $SU(3)$ lattice of the plane, in which F_3 may act on mass operators. The Jordan pairs may be constructed from the descent Hopf algebra [19][20] for cubes, as noted below.

We consider cubes, and the related permutohedra and associahedra, as more fundamental building blocks than ordinary simplices, which are based on a set theoretic view of the ordinals. In quantum mechanics, an ordinal should be the dimension of a state space. Without labeled vertices, a triangle simplex contains three equivalent points and three edges (lines), along with the empty set and the plane. Together, these elements give the vertices of a *parity cube*. In any dimension, the cube has 2^n vertices, graded as for the elements of a Clifford algebra. The magic star is a projection of the three dimensional cube when we include edge midpoints for the $\mathbf{d_3}$ lattice.

A rest mass operator $\sqrt{M}$ obeying the Koide relation [21][22][23] is a circulant of the form (10). The diagonal of eigenvalues also gives a sum of three idempotents for $J_3(\mathbb{C})$, where for us $\mathbb{C}$ is densely filled with the half integers $\mathbb{Z}^8/2$, using the Penrose quasilattice of [24]. Here points (a, b, c, d) on the real line are in the golden ring of integers for $\mathbb{Q}(\sqrt{5})$,

$$a + b\phi + c\sqrt{\phi + 2} + d(\phi\sqrt{\phi + 2}), \tag{12}$$

where $\phi = (1+\sqrt{5})/2$. In other words, $J_3(\mathbb{C})$ is derived from the integral points of $J_3(\mathbb{O})$, forgetting the non associativity. Charts for the Leech lattice [15] can be chosen as associative (alternative) subsets of $\mathbb{O}^3$. $SO(8)$ is a little group for the $SO(9,1)$ in a 2×2 component of a $J_3(\mathbb{O})$ matrix.

The four dimensional Fourier transform comes from the eigenvectors of the chiral operator γ_5 in the Dirac representation,

$$F_4 = F_2 \otimes F_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \tag{13}$$

An alternative form for F_4 is the matrix

$$\frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \tag{14}$$

which is a quaternionic representation of ω, and this transform appears in the construction of a quasicrystal spacetime projection [25] from the $\mathbf{e_8}$ lattice.

4 Algebraic data

The algebras over $\mathbb{R}$ are extended by tensoring with another division algebra, starting with the 64 dimensions of the $\mathbb{C} \otimes \mathbb{O}$ ideal algebra [26][27][28], which assigns $U(1)_Q$ and $SU(3)_C$ color charges to the quarks and leptons of the Standard Model. Selecting one octonion unit for the lepton doublet, define the $\mathbb{C}\otimes\mathbb{O}$ idempotent $\nu = (1 + ie_7)/2$. The other six units $e_1, \cdots, e_6$ define

$$\alpha_1 = \frac{1}{2}(-e_5 + ie_4), \quad \alpha_2 = \frac{1}{2}(-e_3 + ie_1), \quad \alpha_3 = \frac{1}{2}(-e_6 + ie_2). \tag{15}$$

The Lie algebra generators Λ_a for $SU(3)_C$ occur in three different ways. Taking $(I, \nu, \overline{\nu})$ in $\mathbb{C} \otimes \mathbb{O}$, with $I = 1$,

$$\frac{1}{4}[\Lambda_a, \Lambda_b] = \frac{i}{2} f_{abc}\Lambda_c \tag{16}$$

$$\frac{1}{4}[\Lambda_a\nu, \Lambda_b\nu] = \frac{i}{2} f_{abc}\Lambda_c\nu$$

$$\frac{1}{4}[-\overline{\Lambda}_a\overline{\nu}, -\overline{\Lambda}_b\overline{\nu}] = -\frac{i}{2} f_{abc}\overline{\Lambda}_c\overline{\nu}.$$

Complex conjugation $i \mapsto -i$ sends particles to antiparticles. Charges for one generation [28] come from the number operator

$$N = \sum_{j=1}^{3} \alpha_j^\dagger \alpha_j, \tag{17}$$

with values in $\{0, 1, 2, 3\}$. Writing out the α_j components of N, a set of eight charges on a three qubit parity cube gives the set $\{\nu, d(3), \overline{u}(3), e^-\}$ from

$$A^\dagger A, \;\; \alpha_j A^\dagger A, \;\; \alpha_j\alpha_k A^\dagger A, \;\; \alpha_j\alpha_k\alpha_l A^\dagger A, \tag{18}$$

where

$$A \equiv \alpha_1\alpha_2\alpha_3 = i\nu. \tag{19}$$

The three copies of 64 for the generations suggest a (massless form of) triality for $\mathbf{e_8}$. Moving in this dirction, in (16) I represents LL^{-1}, where L and $R = L^{-1}$ are chiralities for the massless neutrino ν. Since ribbon diagrams will characterise chirality, by adding a braiding to the fundamental representation of S_3, we start with the basis for $\mathbb{C}C_3$,

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{20}$$

In every prime power dimension, the Fourier transform F_{p^r} diagonalizes 1-circulants, and there exists $p^r + 1$ mutually unbiased bases [29][30][31] generalizing the 2×2 Pauli matrices, and providing a canonical matrix representation for multiplication.

The charges on the parity cube are equivalent to the ribbon twists in the Standard Model spectrum of [4] and [32]. Integral octonion units also have a ribbon representation [33], using both twists and half loops, so that ribbons are now associated to both cubical geometries and canonical algebraic data.

5 Motivic cubes and Solomon's descent algebra

A parity cube is inscribed in one corner of a larger cube, in which each edge is halved to define the $\mathbf{d_3}$ lattice. The magic star [16][17][18] sits in the inscribed hexagon around six edge centres, so that the three points on the diagonal (say $(0,0,0)$, $(1,1,1)$ and $(2,2,2)$) project to the central $\mathbf{e_6}$ point in the plane. This magic plane is tiled by a tetractys simplex, as follows.

Gluon gluon scattering amplitudes are computed motivically as integrals over the associahedra polytopes. The required dimension of this operad polytope increases with the number of particles, and such polytopes should appear as axioms for higher dimensional categories. Each associahedron lives inside a subdivided simplex [9], carrying the monomials and integer coordinates from the cubic lattice in which it is embedded. For example, the square of paths ending in $(2,2)$ carries an associator edge in the diagonal from $(0,2)$ to $(1,1)$. As usual, a planar rooted tree labels a cell in the associahedron, and a trivalent vertex in a tree stands for a nonassociative product, such as a Lie bracket. If a series of interactions is expressed in brackets, then the tree concretely represents an interaction diagram.

Consider the pentagon. It sits inside the ten point tetractys simplex of three letter words in a three letter alphabet, using Postnikov's [9] restriction of integral coordinates (x_1, x_2, x_3), where $x_1 + x_2 + x_3 = 3$. Take the five points (x_1, x_2, x_3) such that in the i-th coordinate the sum $\sum^{j=i} x_j$ is greater than or equal to i. These numerical coordinates directly encode the noncommutative forest representation of vertices on the pentagon.

The tetractys simplex tiles the magic star plane, in place of a triangle. A parity cube sits on each side of the simplex, representing three generations. The cubic lattice is labeled by the 27 words in three letters. Since the neutrino number ν multiplies every other charge, its states may be removed from the three sides of the tetractys and associated to paths in the centre. A generic point in the magic plane now carries two labels: the 27 dimensions of $J_3(\mathbb{O})$ and the 6 objects in the center of a tetractys. As it happens, $6/27 = 2/9$ is the value of the θ parameter for the lepton Koide $\sqrt{M}$ mass matrices.

A blowup, or refinement, of each point to a tetractys creates larger copies of the magic star, until we see the geometry of Metatron's cube in the plane. Three dimensional space is tiled either by cubes or by permutohedra. Our parity cube is derived from the permutohedron as follows. The 24 vertices of

S_4 have integral coordinates, namely permutations of $(1, 2, 3, 4)$, which lie in $\mathbb{R}^3$. Translating the origin of $\mathbb{R}^3$ to the centre of the polytope, the alternative coordinates are permutations of

$$(\frac{-3}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{3}{2}). \tag{21}$$

Each $\sigma \in S_4$ is assigned a *signature*, which lists the shifts between entries in σ. For example, $(2, 3, 1, 4)$ has signature $+ - +$. The eight signature classes for S_4 define the vertices of the parity cube. The product of two signature classes for S_n is given by the product in the group Hopf algebra $\mathbb{C}S_n$, producing the descent Hopf algebra of Solomon [34]. In any dimension, this is exactly the Hopf algebra we need to construct Jordan pairs [19][20]. The Platonic solids are closely related to the three dimensional permutohedron, with the icosahedron selecting alternative points.

6 Application: masses and mixing for the Standard Model

Further details on the masses and mixings will appear in other papers. The motivic framework appears to justify Koide mass parameters and related mixing angles. The tetractys tiles of the magic plane pick out 6 of 27 paths at the center point, giving $\theta = 2/9$. The Koide rule [21][22] for the charged lepton mass eigenvalues is expressed as a 3×3 circulant matrix

$$\sqrt{M} = \sqrt{\mu} \begin{pmatrix} \sqrt{2} & e^{i\theta} & e^{-i\theta} \\ e^{-i\theta} & \sqrt{2} & e^{i\theta} \\ e^{i\theta} & e^{-i\theta} & \sqrt{2} \end{pmatrix} \tag{22}$$

for the square roots of the masses m_i, with μ a scale. The determinant function [40]

$$1 + \sqrt{m_1 m_2 m_3} \equiv r \cos(3\theta) \tag{23}$$

goes to zero when (i) the masses are normalized and (ii) one eigenvalue of $\sqrt{M}$ is negative. This occurs for Brannen's extension of the Koide rule to the neutrinos [23], which pick up the basic arithmetic $\pi/12$ phase. The diagonal parameter $r = \sqrt{2}$ in $\sqrt{M}$ centers the determinant around zero for $\theta = \pm\pi/12$. Observationally, the charged lepton phase θ is very close to $2/9$, while the active neutrino triplet fits oscillation data with a phase of $\theta = 2/9 + \pi/12$ [23]. Similarly, quark rest mass triplets are obtained with phases $2/27$ and $4/27$. The charged lepton scale μ is a simple multiple of the proton mass, and the mirror neutrino phase is $2/9 - \pi/12$ [2].

The first Euler angle in the CKM mixing matrix [35][36], the Cabibbo angle δ_{12}, is roughly approximated by

$$\delta_{12} + \delta_{13} = \frac{\pi}{4} - \arctan\frac{1}{\phi} = 13.28°. \tag{24}$$

The other two irrationals in the golden ring give

$$\delta_{23} = \frac{\pi}{6} - \arctan \frac{1}{\sqrt{\phi+2}} = 2.3^\circ, \quad \frac{\pi}{2} - \arctan \frac{1}{\phi\sqrt{\phi+2}} = 72^\circ, \tag{25}$$

where 72° is an angle in the Penrose tiling. The phases $\pi/6$ and $\pi/4$ define the tribimaximal matrix [37], which is known to approximate the PMNS neutrino mixing matrix [38][39], indicating the quark lepton complementarity that is expected from the cosmological Higgs mechanism. For the small quark phase $\delta_{13} \simeq 0.2^\circ$ we look to the breaking of tribimaximal mixing in the neutrino sector. The small mixing angle $\theta_{13} = 8.5^\circ$ is close to $4/27$, which is obtained as two thirds of $2/9$ from a triality action on the complex phase X in

$$\begin{pmatrix} a & X & \overline{X} \\ \overline{X} & a & X \\ X & \overline{X} & a \end{pmatrix} \tag{26}$$

in $J_3(\mathbb{C})$ [40]. Now in the CKM matrix, $\delta_{12} = 13.01^\circ$ and δ_{13} is close to 0.27. Since Euler angles are expressed as circulants in the Hopf algebra $\mathbb{C}S_3$, complex phases are automatically included and the CKM and PMNS matrices are completely determined [2] by these Euler angles along with a CP phase.

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