A note on a problem in Mishō Sampō

Hiroshi Okumura

Abstract. A problem involving an isosceles triangle with a square and three congruent circles is generalized.

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1. Introduction

In this note we generalize the following problem, which can be found in [1, 2, 3, 4, 5], where the sangaku with this problem in [4] is undated (see Figure 1).

Problem 1. $EFG$ is an isosceles triangle with base $EF$. $ABCD$ is a square such that $B$ and $C$ lie on the sides $FG$ and $GE$, respectively, and $D$ and $A$ lie on the side $EF$. The incircles of the triangles $ABF$ and $BCG$ are congruent and have radius $r$. Show that $4r = |AB|$.

We show that the isosceles triangle $EFG$ is formed by a 3-4-5 triangle with its reflected image in the side of length 4, i.e., the ratio of the sides of $EFG$ equals $5:5:6$.

2. Generalization

Let $EFG$ be a triangle. Let $\gamma_1, \gamma_2, \cdots, \gamma_n$ be circles of radius $r$ such that they touch the side $EF$ from the inside of $EFG$, $\gamma_1$ and $\gamma_2$ touch, $\gamma_i$ ($i = 3, 4, \cdots, n$) touches $\gamma_{i-1}$ from the side opposite to $\gamma_1$, $\gamma_1$ touches $GE$, $\gamma_n$ touches $FG$. In this case we say that $EF$ has $n$ circles of radius $r$ with respect to $G$ (see Figure 2). This is equivalent to the following equation being true:

$$|EF| = r \cot \frac{\angle E}{2} + r \cot \frac{\angle F}{2} + 2(n - 1)r.$$

Problem 1 is generalized as follows (see Figure 3).
Theorem 2.1. $EFG$ is an isosceles triangle with base $EF$. $ABCD$ is a square such that $B$ and $C$ lie on the sides $FG$ and $GE$, respectively, and $D$ and $A$ lie on the side $EF$. If $AB$ has $n$ circles of radius $r$ with respect to $F$ and $BC$ has $n$ circles of radius $r$ with respect to $G$, then the following statements hold.

(i) $|FG| : |EF| = 5 : 6$.

(ii) $2(n + 1)r = |AB|$.

(iii) If $n$ is odd and expressed as $n = 2k - 1$ for a natural number $k$, $EF$ has $5k - 1$ circles of radius $r$ with respect to $G$.

Proof. Let $2\theta = \angle ABF$. Then we have

\[ |AB| = r \cot \theta + (2n - 1)r. \]

While $\angle CBG + 2\theta = 90^\circ$ implies $|BC| = 2r \cot(45^\circ - \theta) + 2(n - 1)r$. Therefore we get $\cot \theta = 3$ by $|AB| = |BC|$. Hence $\tan 2\theta = 3/4$, i.e., $ABF$ is a 3-4-5 triangle. This proves (i). The part (ii) follows from (1). We assume $n = 2k - 1$. Let $s = |AB|$. Then $s = 4kr$ by (ii). The distance from $G$ to $BC$ equals $(s/2) \cdot (4/3) = 2s/3$. Therefore $|BC| : |EF| = 2s/3 : (s + 2s/3) = 2 : 5$, i.e., $|EF| = 5s/2 = 10kr$. Hence $|EF| = 2r \cot(\angle E/2) + 2(5k - 1 - 1)r$, since $\cot(\angle E/2) = 2$. This proves (iii). □

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References

Tohoku Univ. WDB is short for Tohoku University Wasan Material Database.