AN EXTENDED RING OF THE REALS

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ABSTRACT

This work expounds upon a theory of peripheral-integers and peripheral-reals, integers and reals that in a modular number line mirror their counterparts. It illustrates the properties of these numbers in hopes to breathe life into research of numbers that go beyond infinity.

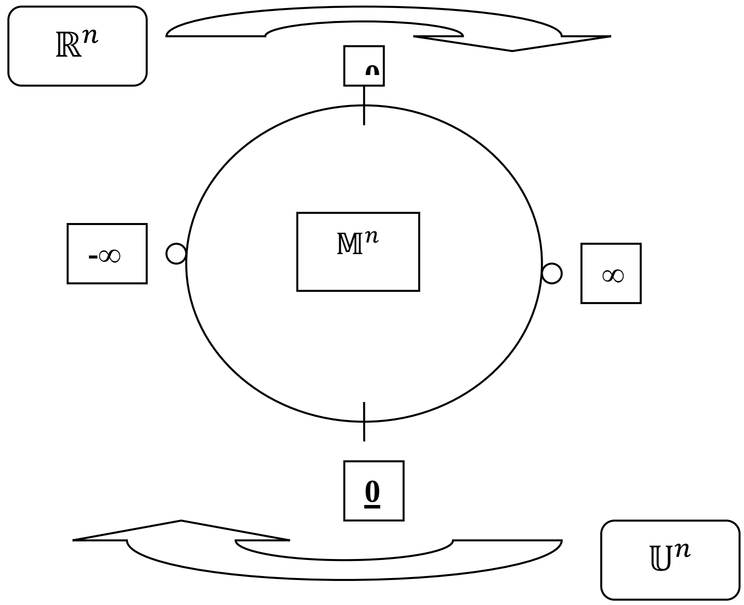


Figure 1.0

Often we may ask, is infinity the very limit of analysis. But infinity is not a number, it is a tool, an idea invented to assist in calculation. Infinity minus infinity is undefined. Because infinity operated on infinity often absorbs resulting in undefined situations is it really a reliable measure of the limit of all real numbers? The **combined modular number line** is a modular number line consisting of the reals and a set of numbers known as the **peripheral-reals**. The peripheral-reals extend the traditional number line past infinity to negative infinity.

Suppose there exists an infinite set of numbers that extends beyond the largest real number. Let that set of numbers be denoted by $\mathbb{U}^n$. Then let the union $\mathbb{R}^n \cup \mathbb{U}^n \mathbb{A}^n$ where $\mathbb{A}^n$ is the **combined modular numbers** and the circle formed by $\mathbb{A}^n$ the **combined modular number line.**

If x is an element of $\mathbb{R}^{\boldsymbol{n}}$, the following holds true:

$$\lim_{x \to \infty^-} x = \infty$$

$$\lim_{x \to -\infty^+} x = -\infty$$

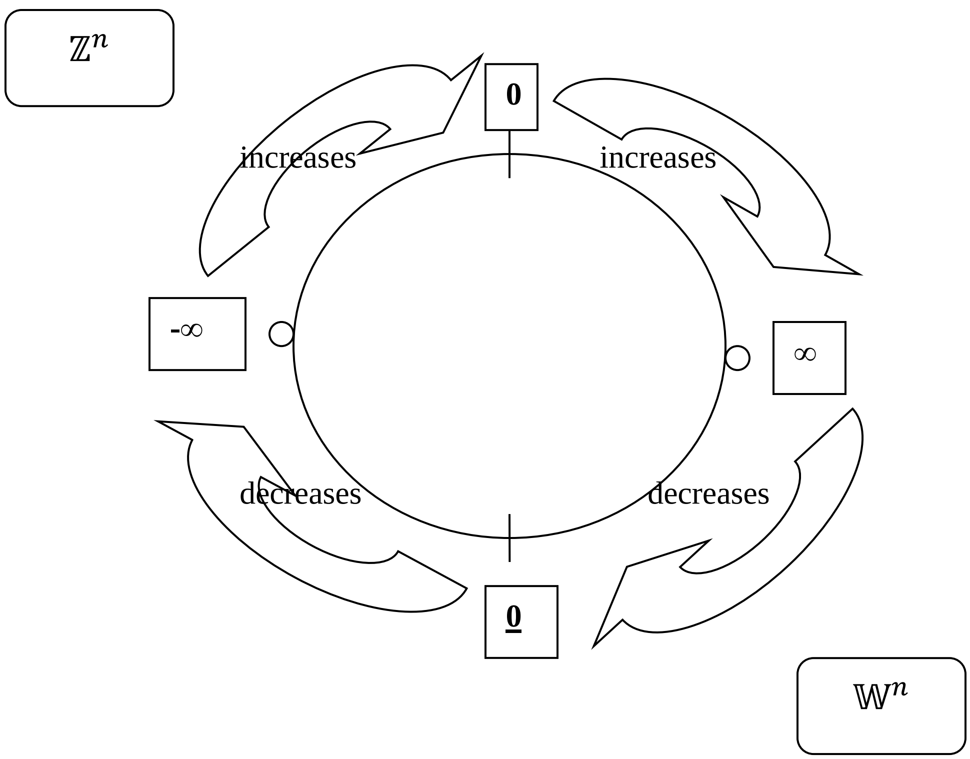
But what if a number system existed where infinity could be approached from the right? Or if negative infinity could be approached from the left. In very short, this is what we will expand upon in this paper. A general illustration of this number system is visible in figure 1.0

Definition:
If x is an element of $\mathbb{U}^n$, the following holds true:

$$\lim_{x \to \infty^-} x = -\infty$$

$$\lim_{x \to \infty^+} x = +\infty$$

Let us name the integers spanning from +∞ to −∞ as the peripheral-integers $\mathbb{W}^n$. There behavior is illustrated in Figure 2.0.



FUNDAMENTAL THEOREM OF THE PERIPHERAL NUMBERS

Given a peripheral integer k, k decreases as the limit of k approaches -∞

PROOF

Let

$$\lim_{x \to \infty^-} x = -\infty$$
$$\lim_{x \to \infty^+} x = \infty$$

Let k be at infinity. If the limit of x to the left of infinity is approaching infinity, k-1 is closer to infinity and thus greater than k. In reverse, if k is at infinity, k+1 is further from infinity. If k+1 is further from infinity than k, k must be decreasing.

THEOREM 1.0

$\mathbb{A}^{\boldsymbol{n}}$ is discontinuous at $[-\infty, \infty]$

PROOF

Let

$$\lim_{x \to \infty^-} x = \infty$$

$$\lim_{x \to -\infty^+} x = -\infty$$

There is an asymptote at ∞ and -∞ and is therefore discontinuous. The rest of the proof is trivial.

MODULAR OPERATION

The modular operation on an integer divides the integer and returns the remainder. The modular operation operates in a cycle, repeating according to the number being divided. For example:

1 mod 5 = 1
2 mod 5 = 2
3 mod 5 = 3
4 mod 5 = 4
5 mod 5 = 0
6 mod 5 = 1
7 mod 5 = 2
And so on.

So what happens if we expand the modular operation to far beyond 5, up to infinity. Suppose we take x mod k, where x is any integer and k is the largest integer possible.

0 mod k= 0
1 mod k = 1
.
.
k-1 mod k = k-1

THEOREM 2.0

Any integer a mod k, where k is the largest integer, is equal to a.

COROLLLARY

The set of integers mod k, where k is the largest integer, is equal to the set of integers.

PROOF

Base Case:

0 mod 1 = 0

Inductive Case:

If n holds, then n+1 holds

n+1 mod k = n+1

let n+1 = m

m mod k = m

If there are k integers, m divided into the largest integer must return a remainder of m.\

Now, since the set of integers is equal to the set of integers retrieved from the mod operation, we can map the set of integers to the set of peripheral integers. Let an integer defined by the mod function k be a **modular integer.**

Letting k be the largest integer, and -k the smallest integer, for every integer there is a modular integer. There is a one-to-one relationship and an onto relationship, forming a function $G_i(x)$ mapping the Integers to the Modular Integers

Integers

-2	-1	0	1	2

Modular Integers

-2	-1	0	1	2

So we have established that the set of integeres can be mapped to the set of modular integers, but we still have yet to establish the existence of the peripheral-integers (and peripheral-reals).

To establish the peripheral-integers we start with peripheral-zero, $\underline{0}$. Peripheral-zero according to our figures 1.0 and 2.0 above occurs at the halfway point between -∞ and ∞**,** on the other side of the integers. The behavior of the peripheral-reals is the opposite and counter-intuitive. They decrease as we progress from infinity to negative infinity, opposite on the circle from the integers. So, taking the largest integer k, to reach $\underline{0}$ we must add k peripheral-integers. Therefore, to reach from the largest peripheral-integer k to the smallest peripheral-integer, let's call it z, we add k+z. Then to reach $\underline{0}$, we take k+z. k will be approximately infinity and z will approximately be negative infinity, giving us our term $\underline{0}$.

Let us define a successor function $succ_i(x)$ over $\underline{0}$.

$m + \underline{0} = m$

$m + S(n) = S(m) + n$

Before we can define the successor function, we have to prove that the rest of the peripheral-reals exist. This is a tough question.

THEOREM 3.0

Existence of the peripheral-integers

PROOF

Let the largest integer be denoted by k. Suppose k+1 is the largest peripheral-integer. Let k+1 = c. Then c-1=k which is the largest integer. If c-1 exists, then c must exist as well.

With the existence of c and $\underline{0}$ we can now use the successor function, however, since the hyper-reals are decreasing in the positive direction of the integers, we have to modify the successor function:

$m - \underline{0} \quad = m$

$m - S(n) = S(m) - n$

We have established the existence of the peripheral-integers { c, c-1, c-2, …, z }
Now the next step is to expand the modulo for a complete modular integer plane.

THEOREM 4.0
The modular ring of integers can be expanded to include the peripheral-integers.

PROOF

We established before:

a mod k = a. Where a is some integer and k is the largest possible integer.

Now we propose:

a mod k = b

But what is b? And what is the new value for k since we are expanding beyond the integers?

We need to establish a new largest number. This largest number has to take into account the peripheral-integers. This means interactions with the integers is a must. We will keep k as the largest integer, but instead, keep it as the largest integer of well, we need a name for this. Let us call the union of $\mathbb{W}^n$ and $\mathbb{Z}^n$: $\mathbb{T}^n$ Let t be the largest value of $\mathbb{T}^n$. Then

Where $t \in \mathbb{T}$ and t is the largest of this set,
0 mod t = 0
1 mod t = 1
.
.
.
t mod t = 0
t+1=$\underline{t}$ mod t = $\underline{1}$
t+2=$\underline{t\ \text{mod}}$ t = $\underline{2}$

etc. around the modular circle

Finally, we need to prove the hardest part, that the modular number line of integers 𝕋 can be extended to the real number line, giving us the real numbers and the peripheral-reals.

Before we can do that, we must prove some properties of the peripheral-integers.

OPERATION OF ADDITION

Because of the mirroring effect, addition works like subtraction since we are decreasing in the positive integer direction.
$m + n = q$ where $m > n$
if $m = q$ and $m > n$, $m + n < q$

ASSOCIATIVE PROPERTY OF ADDITION

(underlines left out for brevity)

(a+b)+c=c+(a+b)=(b+a)+c

$(\underline{1} + \underline{2)} + \underline{3} = -5$
$\underline{3} + (\underline{1}+\underline{2)} = \underline{5}$
$(\underline{2} + \underline{1)} + \underline{3} = \underline{-2}$
This does not hold. Since the peripheral-reals are decreasing and addition of integers works like subtraction.

COMMUTATIVE PROPRETY OF ADDITION

(underlines left out for brevity)

1 + 2 =-5
2 + 1=-3

Again, this does not hold for the same reason.

COMMUTATIVE PROPERTY OF MULTIPLICATION

(underlines left out for brevity)

1*2 =1 + 1 = -2
2*1= 1 + 1 = -2

ADDITIVE IDENTITY

$1 + 0 = 1$
0, by definition of peripheral-integers

MULTIPLICATIVE IDENTITY

$2 * 1 = 2$
Self-evident. There is one instance of 2.

ASSOCIATIVE PROPERTY OF MULTIPLICATION

$1*(2*3) = 6$
$(1*2)*3= 6$

Holds.

INVERSE

$a*a^{-1}=a^{-1}a = e$

Where e is the multiplicative identity, for some number a^{-1}, multiplied by a, equals 1.
$2 * ½ = 1$.
But what is ½? That is what we are about to prove.

PROOF OF THE PERIPHERAL RATIONALS

Given the peripheral-integer $\underline{n}$, the mirroring map is as follows:

$M: \mathbb{U} \rightarrow \mathbb{R}$

THEOREM 5.0 (MIRRORING THEOREM)
Given any peripheral-integer, there is a corresponding integer. Given any integer there is a corresponding peripheral-integer.

PROOF

For every integer a, largest integer k, and integer b the operation

a mod k = b

has a one to one correspondence with mod function $\underline{a,b,k}$

$\underline{a}$ mod $\underline{k}$ =$\underline{b}$**.**

THEOREM 6.0
For every rational number there is a rational peripheral number. For every real number there is a real peripheral number.

PROOF
If $M: \mathbb{U} \rightarrow \mathbb{R}$ holds true, then since for every integer there is a peripheral-integer, then for every quotient m/n in the integers, there is a corresponding quotient in the peripheral-integers.

Then for every rational number in the integers there is one in the peripheral-integers and for every real number there is a peripheral-real number.

CONCLUSION

We have shown that it is possible to create a model that extends beyond infinity and negative infinity and we have shown the behavior of this new set of numbers. The peripheral integers mirror the integers in a sort of topsy-turvy way. We have shown that the modular operation can extend the integers and reals, transforming the number line into a circle that inverts the operations on the integers, transforming addition into subtraction and creating an environment where associativity of addition doesn't hold. We are certain that there is much more that can be discovered about this model of the integers and reals with further investigation.