ON Q-LAPLACE TRANSFORMS AND MITTAG-LEFFLER TYPE FUNCTIONS

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Abstract. In the present paper, the author derived the results based on q-Laplace transform of the K-Function introduced by Sharma[7]. Some special cases of interest are also discussed.

2010 Mathematics Subject Classification. 44A10, 44A20.

Keywords and phrases. q-Laplace transforms, M-series, Generalized M-series, Mittag-Leffler function, Generalized Mittag-Leffler function, K-Function.

1. INTRODUCTION

The q-Laplace transform was defined by Hahn[18] as

\[ L(f(t); s) = \int_0^\infty e^{-st} f(t) dt \]  

by means of the two q-integrals given below

\[ qL_s(f(t)) = \frac{1}{(1 - q)} \int_0^{s^{-1}} E_q(qst) f(t) dt; q, \]

and

\[ qL_s(f(t)) = \frac{1}{(1 - q)} \int_0^\infty e_q(-st) f(t) dt; Re(s) > 0. \]

The q-integrals(see Gasper and Rahman[4]) of a function is defined by

\[ \int_0^t f(x) d(x; q) = t(1 - q) \sum_{k=0}^\infty q^k f(tq^k), \]  

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\[ \int_{t}^{\infty} f(x) d(x; q) = t(1 - q) \sum_{k=1}^{\infty} q^{-k} f(tq^{-k}), \]  
\[ \int_{0}^{\infty} f(x) d(x; q) = (1 - q) \sum_{k=-\infty}^{\infty} q^{k} f(q^{k}). \] 
(1.5) (1.6)

In the theory of q-calculus [5], for \(0 < |q| < 1\), the q-shifted factorial or q-analogue of Pochhammer symbol is defined by

\[ (a; q)_{k} = \begin{cases} \prod_{j=0}^{k-1} (1 - aq^{j}), & \text{if } k > 0 \\ 1, & k = 0 \\ \prod_{j=0}^{\infty} (1 - aq^{j}), & \text{if } k \to \infty \end{cases} \]

Or equivalently

\[ (a; q)_{k} = \frac{(a; q)_{\infty}}{(aq^{k}; q)_{\infty}}, k \in \mathbb{N} \] 
(1.7)

and for any complex number \(\alpha\),

\[ (a; q)_{\alpha} = \frac{(a; q)_{\infty}}{(aq^{\alpha}; q)_{\infty}} \] 
(1.8)

where the principal value of \(q^{\alpha}\) is taken.

\[ (q^{\alpha}; q)_{n} = \frac{(q^{\alpha}; q)_{\infty}}{(q^{\alpha+n}; q)_{\infty}} \]

The q-analogue of the power function is defined as

\[ (a-b)_{\alpha} = a^{\alpha}(b/a; q)_{\alpha} \]

\[ = a^{\alpha} \prod_{j=0}^{\infty} \frac{1 - (b/a)q^{j}}{1 - (b/a)q^{j+\alpha}} = a^{\alpha} \frac{(b/a; q)_{\infty}}{(q^{\alpha}b/a; q)_{\infty}}, a \neq 0. \] 
(1.9)
The $q$-gamma function is defined as

$$\Gamma_q(\alpha) = \frac{G(q^{\alpha})}{G(q)}(1-q)^{1-\alpha} = (1-q)_{\alpha-1}(1-q)^{1-\alpha}, \alpha \in \mathbb{R}/0,-1,-2,... \quad (1.10)$$

where

$$G(q^{\alpha}) = \frac{1}{(q^{\alpha}; q)_{\infty}}. \quad (1.11)$$

$$\Gamma_q(\alpha) = \frac{(q; q)_{\infty}}{(q^{\alpha}; q)_{\infty}}(1-q)^{1-\alpha}$$

where

$$\alpha \neq 0, -1, -2,...$$

The $q$-binomial series [5] is given by

$$1\phi_0[-:-; q, x] = \frac{1}{(x; q)_{\infty}}$$

$$1\phi_0[\alpha; -; q, x] = \sum_{n=0}^{\infty} \frac{(\alpha; q)_n}{(q; q)_n} x^n = \frac{(\alpha x; q)_{\infty}}{(x; q)_{\infty}}. \quad (1.12)$$

The $q$-Binomial coefficients are given by [4] as

$$C_q(n, k) = \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}}. \quad (1.13)$$

The K-function was introduced by Sharma[7] as

$$AK^n_{\beta}(\gamma)(x)$$

$$= \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{A} (a_i r e^{-\gamma})_r x^r}{\prod_{j=0}^{B} (b_j)_r r! \Gamma(\alpha r + \beta)} \quad (1.15)$$
where

\[ \alpha, \beta, \gamma \in \mathbb{C}; \text{Re}(\alpha), \text{Re}(\beta), \text{Re}(\gamma) > 0. \]

Further details of this function are given in [7].

2. Relationship of the K-function with another special functions

The function defined by (1.15) related with the following special functions as: (i) If we set \( \gamma = 1 \), we get

\[ A K_{B}^{\alpha,\beta;\gamma}(x) = A M_{B}^{\alpha,\beta}(x) \]  

(2.1)

where \( A M_{B}^{\alpha,\beta}(x) \) is the generalized M-series introduced by Sharma and Jain[8].

(ii) If we take \( \beta = \gamma = 1 \), we arrive at

\[ A K_{B}^{\alpha,1;1}(x) = A M_{B}^{\alpha,1}(x) = A M_{B}(x) \]  

(2.2)

where \( A M_{B}(x) \) is the M-series given by Sharma[9].

(iii) If we put \( A = B = 0 \), we arrive at

\[ 0 K_{0}^{\alpha,\beta;\gamma}(x) = E_{\alpha,\beta}^{\gamma}(x) \]  

(2.3)

where \( E_{\alpha,\beta}^{\gamma}(x) \) is the generalized Mittag-Leffler function given by Prabhakar[15].

(iv) If we put \( A = B = 0, \gamma = 1 \), we arrive at

\[ 0 K_{0}^{\alpha,\beta;1}(x) = E_{\alpha,1}^{\gamma}(x) = E_{\alpha,1}^{\gamma}(x) \]  

(2.4)

where is the generalized Mittag-Leffler function given by Wiman[1].

(v) If we put \( A = B = 0, \beta = \gamma = 1 \), we arrive at

\[ 0 K_{0}^{\alpha,1;1}(x) = E_{0,1}^{\alpha}(x) = E_{\alpha,1}(x) = E_{\alpha}(x) \]  

(2.5)

where \( E_{\alpha}(x) \) is the Mittag-Leffler function given by Mittag-Leffler[5,6].

(vi) If we put \( A = B = 0, \alpha = \beta = \gamma = 1 \), we arrive at

\[ 0 K_{0}^{1,1;1}(x) = E_{1,1}^{1}(x) = E_{1,1}(x) = E_{1}(x) = e^{x} \]  

(2.6)

where \( e^{x} \) is the exponential function given by [2].
3. Main Results

In this section we investigate the q-Laplace transform of the K-function.

**Theorem 3.1.** Let $\alpha, \beta, \gamma \in \mathbb{C}; Re(\alpha), Re(\beta), Re(\gamma) > 0$, then

$$qL_s(x^\lambda_A K_B^{\alpha, \beta; \gamma}(x))$$

$$= \frac{(q;q)_\infty}{s^{1+\lambda}} \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{A} (a_i)_r(\gamma)_r}{\prod_{j=0}^{B} (b_j)_r r! \Gamma(\alpha r + \beta)s^{r}(q^{\lambda+r}; q)_\infty}$$

(3.1)

**Proof:**

With the help of (1.4), equation(1.2) can be written as

$$qL_s(f(t)) = \frac{(q;q)_\infty}{s} \sum_{k=0}^{\infty} \frac{q^k f(s^{-1}q^k)}{(q;q)_k}$$

(3.2)

Taking $f(x) = x^\lambda_A K_B^{\alpha, \beta; \gamma}(x)$ in the above equation and making the use of definition(1.15), we get

$$qL_s(x^\lambda_A K_B^{\alpha, \beta; \gamma}(x))$$

$$= \frac{(q;q)_\infty}{s^{1+\lambda}} \sum_{j=0}^{\infty} \frac{q^\lambda_j}{(q;q)_j} \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{A} (a_i)_r}{\prod_{j=0}^{B} (b_j)_r}$$

$$\times \frac{(\gamma)_r}{r! \Gamma(\alpha r + \beta)}(s^{-1}q^j)_r$$

(3.3)

On interchanging the order of summations and then summing the resulting with the help of (1.12), the right hand side of (3.3) converted to

$$\frac{(q;q)_\infty}{s^{1+\lambda}} \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{A} (a_i)_r}{\prod_{j=0}^{B} (b_j)_r}$$

$$\times \frac{(\gamma)_r}{r! \Gamma(\alpha r + \beta)s^{r}(q^{\lambda+r}; q)_\infty}$$

(3.4)

which is the desired result.
4. SPECIAL CASES

Theorem (3.1) leads to the q-Laplace transform of generalized M-series[8], M-series[9], generalized Mittag-Leffler functions[1,15], Mittag-Leffler function[5,6] and exponential function[2] after implementing the necessary changes in the values of $A, B, \alpha, \beta$ and $\gamma$ as mentioned in the section 2.

Acknowledgement

The author is highly grateful to the learned and renowned referee for making valuable suggestions and comments which led to a better presentation of the manuscript.

REFERENCES


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