Intuitive explanation of the Riemann hypothesis

I. Characterisation of the nontrivial zeroes of ζ.

There is a unique (canonical) one-form α on $\mathbb{H}$ invariant under $\Gamma(2)$ with a pole of residue 1 at the image of $i\infty$ and a pole of residue -1 at the image of 1. Under the embedding $\mathbb{H} \to \mathbb{C}$ with τ the coordinate on $\mathbb{C}$ the ratio $[\alpha : i\pi d\tau]$ tends to one at the upper end of the interval $(0, i\infty)$. Let T be the connected real multiplicative group and consider the multiplication actions

$$\begin{cases} \mu_+ : T \times \mathbb{H} \to \mathbb{H} \\ (g, z) \mapsto \sqrt{g} z \end{cases}$$

$$\begin{cases} \mu_- : T \times \mathbb{H} \to \mathbb{H} \\ (g, z) \mapsto \frac{1}{\sqrt{g}} z \end{cases}$$

1. Theorem. For each unitary character ω of T and each real number c with $0 < c < 1$, the differential two-form

$$g^{2c-2}\omega(g)\mu_+^*(\alpha - i\pi d\tau) \wedge \mu_-^*(\alpha - i\pi d\tau)$$

is real and integrable (rapidly decreasing, that is 'Schwartz') on $T \times (0, i\infty)$. Among rapidly decreasing forms, it is exact if and only if $\zeta(c + i\omega_0)$ is zero where ζ is Riemann's zeta function and ω_0 is the real number corresponding to ω under the rule $\omega(g) = g^{i\omega_0}$.

Proof. It is real because the factors besides $\omega(g)$ are anti-symmetric with respect to interchanging μ_+ and μ_- which matches the reversal of orientation of T. The two-form integrates to the squared absolute value of a holomorphic integral, namely $\int g^{c-1}\omega(g)(\alpha - i\pi d\tau)$. In turn, it is easy to calculate the holomorphic *definite* integral; it is $-L(s, \chi)\Gamma(s)\pi^{1-s}$ L is the L series for sums of four squares, χ is the Dirichlet sign character and $s = c + i\omega_0$. The rule $\omega(g_1 g_2^{-1}) = \omega(g_1)\omega(g_2)^{-1}$ is all that is needed.

II. Remark about the dynamical interpretation.

Here is an intuitive way of integrating the two-form let, us call it A_s for $s = c + i\omega_0$. Let $\tau = ie^t$. By 'integration by parts'

$$\int e^{(c-1)t+i\omega_0 t} d\ log(\frac{\lambda}{q}) = -(c-1+i\omega_0)\int e^{(c-1)t+i\omega_0 t} log(\frac{\lambda}{q}(ie^t))dt.$$

Therefore

$$\int\int A_s = |(s-1)|\ ^2|\int_{-\infty}^{\infty} e^{i\omega_0 t} e^{(c-1)t} log(\frac{\lambda}{q}(ie^t))dt|^2.$$

The second term on the right is the squared magnitude of the Fourier tranform value at frequency ω_0 of the real function

$$e^{(c-1)t} log(\frac{\lambda}{q}(ie^t)).$$

A disk spinning with angular rate ω_0 with pivot point held by a pair of opposing movable bearings, if we move the bearings in a line according to this function (of time), the limiting radius of the circle traced by the initial pivot point will be the magnitude and

2. Theorem.

$$\frac{\pi}{|s-1|^2}\int A_s = \text{ area inside final circle.}$$

III. Lie actions.

Whenever A_s is a Lie derivative, meaning $A_s = \delta B$ for some rapidly decreasing B, under the action of a vector-field δ, then A_s can be obtained by multiplying B by a suitable divergence ratio; put differently $A_s = d\ i_\delta B$ which is an exact form; this can only happen if $\zeta(s) = 0$ (still assuming $0 < 1 < c$).

IV. The action of $\frac{\partial}{\partial c}$.

A vector field which does not preserve $T \times (0, i\infty)$ is the partial derivative with respect to c. If $g = e^t$ it sends A_s to $2tA_s$.

3. Question. For $0 < c < 1/2$, is the partial derivative $\frac{\partial}{\partial c} \int A_{c+i\omega_0}$ non-positive?

An affirmative answer would imply that A_s is non-exact, and $\zeta(c + i\omega_0) \neq 0$, for all c in the same range. The reason is that for each value of ω_0 the dependence on c would be a non-increasing real analytic function $(0, 1/2) \to [0, \infty)$. Such a function cannot take the value of zero.

Let's attempt to estimate the partial derivative to see if we can start to answer the question. Let

$$h(r, v) = e^{2(c-1)v} log(\frac{\lambda}{q}(v + r/2)) log(\frac{\lambda}{q}(v - r/2)$$

This has the properties that for $0 < c < 1/2$

$$\begin{cases} h(r, v) > 0 & \text{for all } r, v \\ h(r, v) - h(r, -v) < 0 & \text{for all } r \text{ and all } v > 0 \end{cases}$$

.

For each fixed c and r let

$$\gamma(c, r) = \frac{\int v h(r, v) dv}{\int h(r, v) dv}$$

This is the mean value of $h(r, v)$ as a function of v.

Now

$$\frac{\partial}{\partial c}\int A_s = \frac{\partial}{\partial c}\ ((c-1)^2+\omega_0^2)\int\int cos(r\omega_0)h(r,v)dvdr.$$

$$= (2c-2)\int cos(r\omega_0)\int h(r,v) + ((c-1)^2+\omega_0^2)(2v)h(r,v)dvdr$$

$$= (2c-2)\int cos(r\omega_0)\int h(r,v)dvdr$$

$$+((c-1)^2+\omega_0^2)\int 2\gamma(c,r)cos(r\omega_0)\int h(r,v)dvdr$$

The integral $\int\int cos(r\omega_0)h(r,v)dvdr$ is semi-positive since it is the squared magnitude of a complex number. Each of the coefficients $2c-2$ and $\gamma(c,r)$ are negative when $0 < c < 1/2$.

Let

$$\rho(c+i\omega_0) = \frac{\int \gamma(c,r)cos(\omega_0 r)\int h(r,v)dvdr}{\int cos(\omega_0 r)\int h(r,v)dvdr}$$

so our integral is

$$= (2c-2) + 2((c-1)^2+\omega_0^2)\rho(s))\int cos(r\omega_0)\int h(r,v)dvdr$$

$$= 2(\frac{c-1}{(c-1)^2+\omega_0^2} + \rho(s)\ )\int A_s.$$

Removing the leading factor of -1 in $-\ L(s,\chi)\Gamma(s)\pi^{1-s}$ which has no effect, and removing our leading factor of 2 which relates the real part of the logarithmic derivative with our integral, we obtain

$$Re\ \frac{d}{ds}\ log\ (L(s,\chi)\Gamma(s)\pi^{1-s}) = \frac{Re(s-1)}{|s-1|^2} + \rho(s).$$

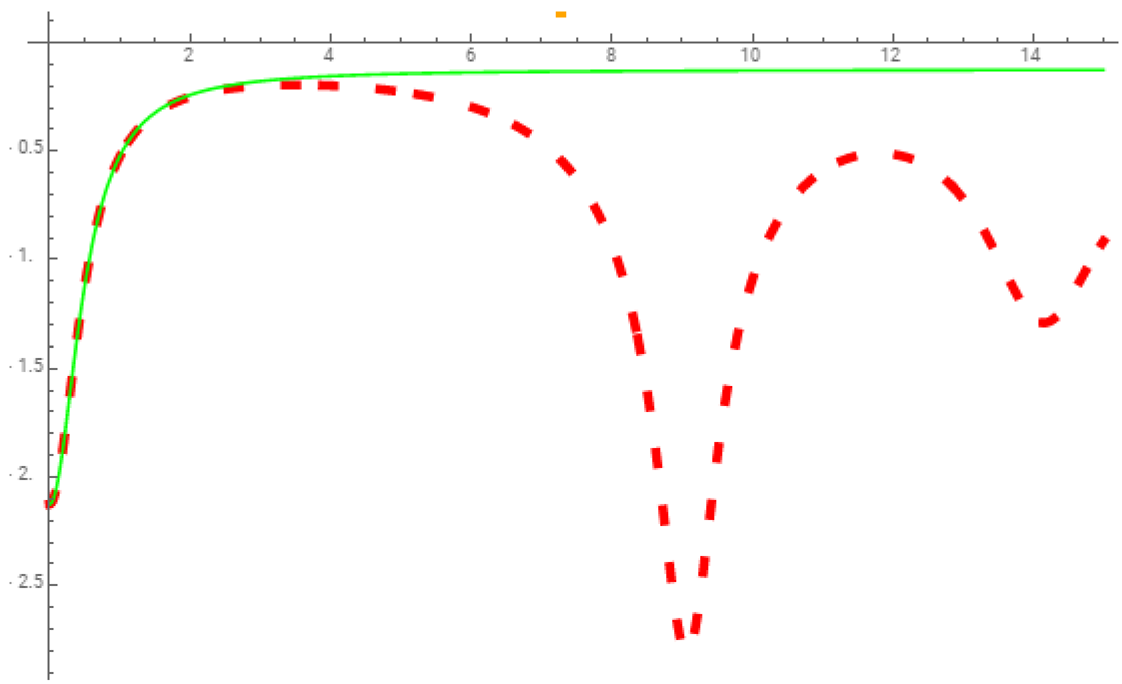
The logarithmic derivative of the gamma function is the digamma function $\Psi(s)$. With this notation then

4. Theorem.

$$Re\ \frac{d}{ds}\ log\ L(s,\chi) = \frac{Re(s-1)}{|s-1|^2} + \rho(s) + log(\pi) - Re\ \Psi(s).$$

From this we can write an expression for the real part of the logarithmic derivative of $L(s)$ itself and $\zeta(s)\zeta(s-1)$.

When $c = 1/2$ a first approximation of $\rho(s)$ would just be the constant $log(\frac{log(16)}{\pi})$. Here the real part of the logarithmic derivative of $L(s,\chi)\Gamma(s)\pi^{1-s}$ as a red graph, and the $\frac{Re(s-1)}{|s-1|^2} + log(\frac{log(16)}{\pi})$ as a green graph as a function of ω_0 when $c = 1/2$.



This graph is not much evidence as we don't know why the actual value departs from the approximation.

V. Holomorphic interpratation

Recall we let α be the unique $\Gamma(2)$ invariant holomorphic one-form on $\mathbb{H}$ which has a simple pole at the image of $i\infty$ of residue $+1$ and a simple pole at the image of 1 of residue -1, and we consider for each constant s with $0 < Re(s) < 1$ the holomorphic one-form

$$(-i\tau)^{s-1}(\alpha - i\pi d\tau) = u^{s-1}(\alpha + \pi du)$$

where $u = -i\tau$. The restriction to $u \in (0, \infty)$ is rapidly decreasing or 'Schwartz' once one makes the definition in an appropriately symmetric way (e.g. on the Lie algebra of the multiplicative group).

Our earlier arguments gave equivalent conditions for the existence of f rapidly decreasing. Since they were based on integration by parts, we lost the actual underlying symmetry on the level of forms.

Denote by $0, 1, \infty$ in order be the images in the Riemann sphere of the bounary points $0, 1, i\infty$ of $\mathbb{H}$, so that α can be interpreted as the meromorphic one-form on the Riemann sphere with a pole of residue $1, -1$ at ∞ and 0.

Let Z be the rational function which takes the value 0 at 0 with a simple pole of residue 1 at ∞. The permutations of the three-element set $\{0, 1, \infty\}$ extend uniquely to automorphisms, for instance the transform of α under interchanging 0 and ∞ is the same as the multiple of α by $\frac{1}{Z}$. That is to say, $\frac{1}{Z}$ is the eigenfunction for this involution. In fact $\alpha = \frac{-dZ}{Z-1}$.

The real points of the projective line form a real circle inside the Riemann sphere, which contains all three of the points $0, 1, \infty$. Writing $\tau = iu$, the line where u takes positive real values actually covers the line where Z takes *negative* real values.

The ideal triangle in $\mathbb{H}$ which contains the boundary points $0, 1, i\infty$ maps topologically isomorphically to the circle of real points of the projective line.

Suppose now that f is a function on the $\mathbb{H}$ such that

$$df = \frac{-u^{s-1}dZ}{Z-1} + \pi u^{s-1}du.$$

We can restrict f to the ideal triangle and therefore to the real points of the projective line.

The form df is rapidly vanishing at the points 0 and ∞. At the point 1 the form $\alpha = \frac{-dZ}{Z-1}$ has a simple pole of residue -1 (as we know) and the function u^{s-1} takes the value $(-i)^{s-1}$. Since the point $\tau = 1$ is invariant under

$$\frac{-1}{\tau - 1} \mapsto \frac{-1}{\tau-1} + 2$$

the function $Z = \frac{\lambda - 1}{\lambda}$ can be expressed by a 'q-expansion' in terms of $e^{i\pi/(1-\tau)}$ about $Z = 1$

$$Z = 1 - 16e^{i\pi/(1-\tau)}....$$

Here $i\pi/(1-\tau) = -\pi/(u+i)$. Note then, to the first approximation

$$\frac{dZ}{Z-1} = -\pi\frac{du}{u+i}.$$

We can expand u in terms of Z giving

$$u = -i - \frac{\pi}{log(\frac{Z-1}{-16})}...$$

To the first approximation

$$\pi du = -(u+i)\frac{dZ}{Z-1}.$$

This means our form is to the first approximation

$$-(u+i+1)u^{s-1}\frac{dZ}{Z-1}.$$

We already knew that including the terms $u+i$ in the first bracket would not affect the 'residue.'

Then our form is approximately merely

$$-(-i - \frac{\pi}{log(\frac{1-Z}{16})})^{s-1}\frac{dZ}{Z-1}$$

which agrees closely with

$$-(-i)^{s-1}\frac{dZ}{Z-1}.$$

This means that we may use the residue calculation, we may work as if the form has a simple pole at the point 1 of the Riemann sphere, and integrate up to a point near 1, then on the other side from a corresponding point, and as the gap is made smaller, the omitted difference will approach $i\pi$ times $-(-i)^{s-1}$. That is, the counter-clockwise integral about the ideal triangle makes sense, and evaluates to

$$(i\pi)(-(-i)^{s-1}) = \pi e^{i\pi(\frac{3}{2}s+1)}$$

Strangely, the condition $Re(s) = 1/2$ corresponds to the square of the residue having zero real part.

The relation with the condition that $\zeta(s) = 0$ is that this is equivalent to the value at 0 and ∞ being equal, in which case the residue equals the integral over the arc from 0 to ∞ which passes through 1 (now where Z runs over the positive real numbers). Here we must interpret this as an improper integral, leaving a gap about 1 which is symmetrical with respect to the involution, and taking the limit as the gap tends to zero.

Thus

5. Theorem. The form $u^{s-1}(\alpha+\pi du)$ corresponds to a well-defined one-form on the real projective line except the point 1, where the form is locally meromorphic with residue $-(-i)^{s-1}$. Thus the integral of the form over the real projective line evaluates to $\pi e^{i\pi(\frac{3}{2}s+1)}$. The condition for $\zeta(s) = 0$ is that the integral takes equal value at 0 and ∞ while the condition for $Re(s) = 1/2$ is that the real part of the square of the integral is zero. Thus the Riemann hypothesis is equivalent to asserting that when the integral from 0 to ∞ which does not pass 1 is trivial, the square of the integral along the other arc, passing through 1, has real part zero.

References

1. Marcus Du Sautoy, The music of the primes: why an unsolved problem in Mathematics matters, 2004

2. Keith Ball, Rational approximations to the zeta function, June, 2017

3. Robert Mackay, email, 19 October, 2017

4. Robert Mackay, email, 20 October, 2017

5. Robert Mackay, email, 22 October, 2017

6. Robert Mackay, email, 14 December, 2017

7. Nayo Reid, conversation, Euston Place, Leamington, 23 January, 2018

8. Robert Mackay, email, 28 February, 2018

9. Robert Mackay, email, 20 March, 2018

John Atwell Moody
20 April 2018