An "algebraic" method for the application of the constructive proof of classification theorem for closed and connected surfaces

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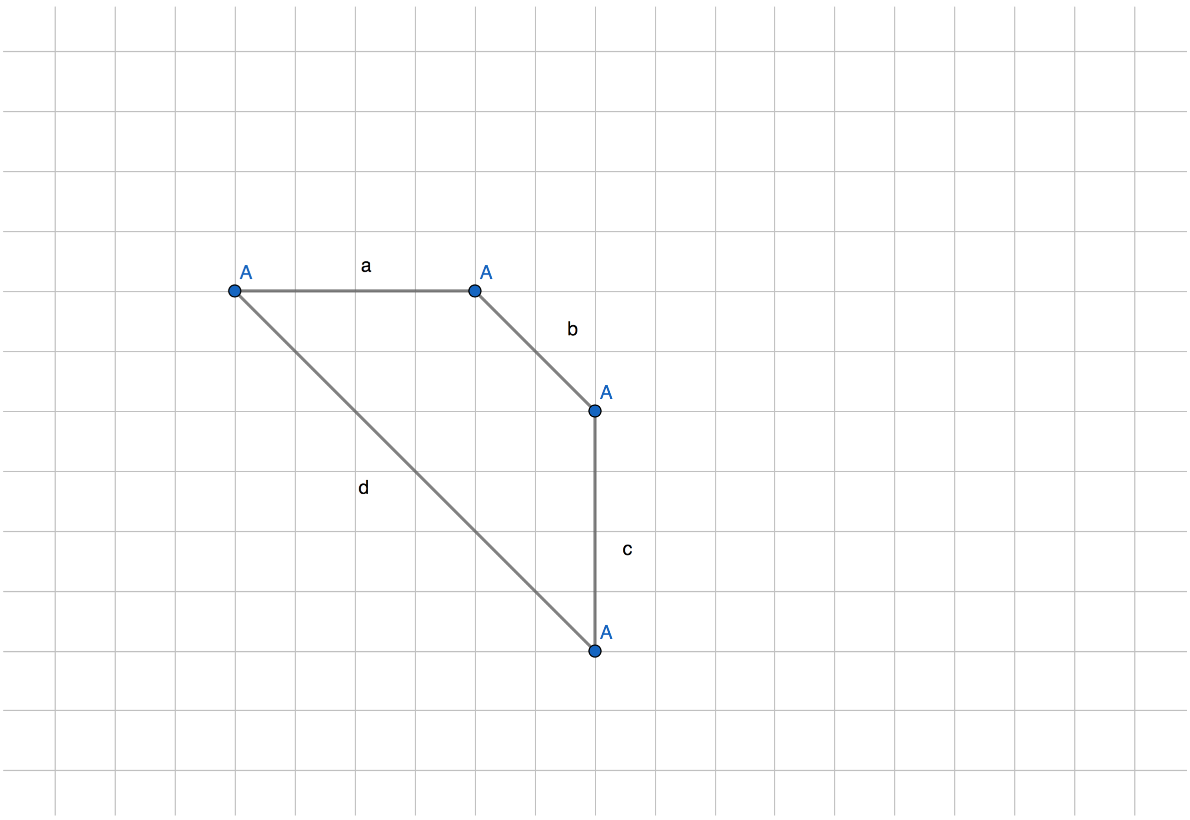
Abstract

For a given planar diagram of a closed & connected surface, we establish an "algebraic" method for cutting and gluing operations on the edges of the diagram. By this, by just manipulating the name of the edges with the given rules, with the guidance of the classification of closed and connected surface theorem given in [1], we can determine the type of the surface without having need to draw any diagram.

Motivation

Provided that all the vertices in the given diagram are the same, such a diagram



can be represented as one of the following way: $abcd$, $bcda$, $cdab$, $dabc$, or $d^{-1}c^{-1}b^{-1}a^{-1}$, $c^{-1}b^{-1}a^{-1}d^{-1}$, $b^{-1}a^{-1}d^{-1}c^{-1}$, $a^{-1}d^{-1}c^{-1}b^{-1}$, i.e as long as the order of the edges are the same, there is not preferred specific vertex that we need to start reading, nor there is a preferred orientation for how to read the labels of

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edges.Therefore, if we were to represent the given surface with an expression containing its edges with their order as one of those given above, $abcd$ and $d^{-1}c^{-1}b^{-1}a^{-1}$ would yield the same surface.

Now consider the surface given in Figure 1.1,

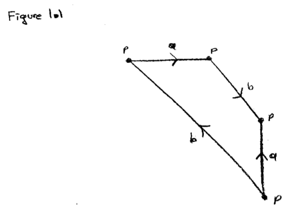

Figure 1.1

and let first choose the counterclockwise direction as the direction which we will read our labels of the edges.Then, if we were to cut a given surface along some new edge e as it is shown in figure 1.2

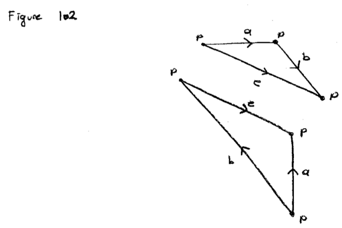

Figure 1.2

if we read the labels of these two pieces of diagrams in the same orientation that we read our orijinal diagram, we should read them as abe^{-1} and $ea^{-1}b$.

Observe that the expressions abe^{-1} and $ea^{-1}b$ are in such a form that as if while we are cutting, we created ee^{-1} pair, and gave one piece of $e's$ to each piece.Conversely, if we were to glue the pieces along e, we should have e in one piece, and e^{-1} in the other piece and these terms should be in the end of the expressions for the pieces, such as ...e and e^{-1}.... .

Furthermore, if we reflect one of the pieces along one of the edges, as it is shown in the figure 1.3, nothing changes, but now observe that the representation of this piece changes from abe^{-1} to $eb^{-1}a^{-1}$; therefore given a representation of diagram with some edges, reversing the order of edges while changing it orientation does not change the diagram.

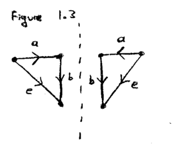

Figure 1.3

Rules

For a given planar diagram of a surface, let fix an orientation, and represent the edges of the surface with that orientation.

i-) Then cutting the surface along a new edge, say x, will introduce a new edge in the representation as xx^{-1} and give one of $x's$ to one of the pieces.

ii-) To glue to pieces along an edge, say x, you need to first read the edges of both pieces as $x's$ are in one of the ends of the representing expressions, such as $[A]x$ and $x^{-1}[B]$.Then gluing will "calcel" the edges x and x^{-1} and connect the rest of the diagram, such as $[A]xx^{-1}[B] = [A][B]$.

iii-) Any diagram can be reflected along one of its edges, which does not change the diagram, and the representation changes its order while direction of every edge is reversed, i.e $abc \to c^{-1}b^{-1}a^{-1}$ - algebraically taking the inverse of the whole expression corresponds to reflecting the diagram along one of its edges, i.e $(abc)^{-1} = c^{-1}b^{-1}a^{-1}$.

An Application

Consider the planar diagram of $\mathbb{K}^2\#\mathbb{T}^2$, and represent it as $cdc^{-1}d^{-1}ba^{-1}ba$.

With the rules that we described above, we can argue that:

$$cdc^{-1}d^{-1}ba^{-1}ba = bacdc^{-1}d^{-1}ba^{-1}$$

$$\Rightarrow bacdk \quad \textbf{and} \quad k^{-1}c^{-1}d^{-1}ba^{-1} \to k^{-1}d^{-1}c^{-1}a^{-1}b^{-1} \quad \textbf{and} \quad ba^{-1}k^{-1}c^{-1}d^{-1}$$

$$\Rightarrow k^{-1}d^{-1}c^{-1}a^{-1}b^{-1}ba^{-1}k^{-1}c^{-1}d^{-1} = k^{-1}d^{-1}c^{-1}a^{-1}a^{-1}k^{-1}c^{-1}d^{-1}$$

Name $a^{-1}a^{-1}$ pair as $[P_1]$, so we have

$$k^{-1}c^{-1}d^{-1}k^{-1}d^{-1}c^{-1}[P_1]$$

$$\Rightarrow k^{-1}c^{-1}e^{-1} \quad \textbf{and} \quad ed^{-1}k^{-1}d^{-1}c^{-1}(P_1)$$

$$\Rightarrow kec \quad \textbf{and} \quad c^{-1}[P_1]ed^{-1}k^{-1}d^{-1} \Rightarrow [P_1]ed^{-1}k^{-1}d^{-1}ke$$

$$\Rightarrow [P_1]er \quad \textbf{and} \quad r^{-1}d^{-1}k^{-1}d^{-1}ke$$

$$\Rightarrow r[p_1]e \quad \textbf{and} \quad e^{-1}k^{-1}dkdr \to r[P_1]k^{-1}dkdr$$

Name rr component as $[P_2]$, so we have

$$[P_2][P_1]k^{-1}dkd.$$

Moreover,

$$[P_2][P_1]k^{-1}dkd = [P_2][P_1]k^{-1}dr \quad \textbf{and} \quad r^{-1}kd$$

$$\Rightarrow r[P_2][P_1]k^{-1}d \quad \textbf{and} \quad d^{-1}k^{-1}r \to [P_2][P_1]k^{-1}k^{-1}rr$$

Name $k^{-1}k^{-1}$ as $[P_3]$ and rr as $[P_4]$, hence we have

$$[P_1][P_2][P_3][P_4],$$

a connected sum of 4 projective plane!

Another application can be found in [2].

References

[1] Kinsey, C., Topology of Surfaces, 2nd ed.

[2] http://users.metu.edu.tr/e214070/application.png