Related to Fermat’s Last theorem: The quadratic formula of the equation

\[ X^{n-1} \mp X^{n-2}Y + X^{n-3}Y^2 \mp \ldots + Y^{n-1} = Z_2^n(nZ_2^n) \]
in the cases \( n = 3, 5 \) and \( 7 \)

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Abstract

We give some quadratic formulas (including Euler’ and Dirichlet’s formula in [1],[2]) of the equation \( X^{n-1} \mp X^{n-2}Y + X^{n-3}Y^2 \mp \ldots + Y^{n-1} = Z_2^n(nZ_2^n) \) in the cases \( n = 3,5 \) and \( 7 \) for finding a solution in integer.

The equation \( X^{n-1} \mp X^{n-2}Y + X^{n-3}Y^2 \mp \ldots + Y^{n-1} = Z_2^n \) always has a solution such as:

\[ X = a(a^{n-1} \mp a^{n-2}b + a^{n-3}b^2 \mp \ldots + b^{n-1}) \]
\[ Y = b(a^{n-1} \mp a^{n-2}b + a^{n-3}b^2 \mp \ldots + b^{n-1}) \]
\[ Z_2 = a^{n-1} \mp a^{n-2}b + a^{n-3}b^2 \mp \ldots + b^{n-1} \]

\( X \), \( Y \) and \( Z_2 \) have a common factor \( a^{n-1} \mp a^{n-2}b + a^{n-3}b^2 \mp \ldots + b^{n-1} \)

Below we consider the case \( X \), \( Y \) and \( Z_2 \) are relative prime and \( X \), \( Y \) are odd for \( n = 3,5 \) and \( 7 \).

1 The quadratic formulas of the equation \( X^2 - XY + Y^2 = Z_2^3 \) (related equation \( X^3 + Y^3 = Z_3^3; Z_1 = X + Y \))

1a.

\[ x^2 - xy + y^2 = z_2^3 \] \hspace{1cm} (1)

(\( x + y \) is not divisible by 3)

Write \( x = u + v, y = u - v \), then:

\[ x^2 - xy + y^2 = u^2 + 3v^2 \]
\[ u = \frac{x + y}{2} \] is not divisible by 3, consider the equation \( u^2 + 3v^2 = z_2^3 \)

and \( z_2 \) can be written as \( z_2 = a + 3b \)

\[ z_2^3 = (a + 3b)^3 = a(a - 9b)^2 + 27b(a - b)^2 \]

select \( u_1^2 = a(a - 9b)^2, v_1^2 = 9b(a - b)^2 \)
then \( a \) and \( b \) are the square, write \( a = c^2, b = d^2 \)

it gives:

\[ u_1^2 = c^2(c^2 - 9d^2), v_1^2 = 9d^2(c^2 - d^2)^2 \]
then:

\[ u_1 = c(c^2 - 9d^2) \] \hspace{1cm} (2)
\[ v_1 = 3d(c^2 - d^2) \] \hspace{1cm} (3)

and \( z_2 = (c^2 + 3d^2) \)

1b.

\[ x^2 - xy + y^2 = 3z_2^3 \] \hspace{1cm} (4)
x + y is divisible by 3

\[ u = \frac{x + y}{2} \] is divisible by 3, write \( u = 3u' \) then: \( u^2 + 3v^2 = 3^2u'^2 + 3v^2 \)

\[ x^2 - xy + y^2 = u^2 + 3v^2 = 3^2u'^2 + 3v^2 = 3(3u'^2 + v^2) \]

Consider the equation \( 3(3u'^2 + v^2) = 3z_2^2 \)

So \( (3u'^2 + v^2) = z_2^2 \)

And by the same way, we obtain:

\[ v_1' = c(c^2 - 9d^2) \] as \( u_1 \), \( u_1' = 3d(c^2 - d^2) \) as \( v_1 \) and \( z_2 = (c^2 + 3d^2) \)

\( x = u + v = 3u' + v, \ y = u - v = 3u' - v. \)

If \( u, v \) is the one solution, then:

\[ u' = \frac{(m^2 - 3n^2)u + 6mnv}{3n^2 + m^2} \]  

\[ v' = \frac{2mn + (3n^2 - m^2)v}{3n^2 + m^2} \]

are also solution: \( u'^2 + 3v'^2 = u^2 + 3v^2 \)

2 The quadratic formulas of the equation \( X^4 - X^3Y + X^2Y^2 - XY^3 + Y^4 = Z_2^5 \) (related equation \( X^5 + Y^5 = Z_2^5; \ Z_1 = X + Y \))

2a.

\[ x^4 - x^3y + x^2y^2 - xy^3 + y^4 = z_2^5 \]  

x + y is not divisible by 5

Write \( x = p + q, \ y = p - q \), then:

\[ x^4 - x^3y + x^2y^2 - xy^3 + y^4 = p^4 + 10p^2q^2 + 5q^4 = p^4 + 10p^2q^2 + 25q^4 - 20q^4 = (p^2 + 5q^2)^2 - 5(2q^2)^2 \]

\[ p = \frac{x + y}{2} \] is not divisible by 5, consider the equation:

\[ (p^2 + 5q^2)^2 - 5(2q^2)^2 = z_2^2 \]

let \( u = p^2 + 5q^2 \) and \( v = 2q^2 \)

then \( z_2^2 = u^2 - 5v^2 \) and \( z_2 \) can be written as \( z_2 = a - 5b \)

\[ z_2^2 = (a - 5b)^2 = a(a^2 + 50ab + 25b^2) - 5b(a^2 + 10ab + 5b^2)^2 \]

select \( u_1^2 = a(a^2 + 50ab + 25b^2)^2 \), \( v_1^2 = 5b(a^2 + 10ab + 5b^2)^2 \)

then \( a \) and \( b \) are the square, write: \( a = c^2, b = d^2 \)

it gives:

\[ u_1^2 = c^2(c^4 + 50c^2d^2 + 125d^4)^2 \]

\[ v_1^2 = 5d^2(c^4 + 10c^2d^2 + 5d^4)^2 \]

then:

\[ u_1 = c(c^4 + 50c^2d^2 + 125d^4) \]  

\[ v_1 = 5d(c^4 + 10c^2d^2 + 5d^4) \]

and \( z_2 = c^2 - 5d^2 \)
2b.

\[ x^4 - x^3y + x^2y^2 - xy^3 + y^4 = 5z_2^5 \] (10)

x + y is divisible by 5

\[ p = \frac{x + y}{2} \] is divisible by 5, write \( p = 5p' \), so \( p^4 + 10p^2q^2 + 5q^4 = (5p')^4 + 10(5p')^2q^2 + 5q^4 = 5(125p'^4 + 50p'^2 + q^4) \)

\( 5(125p'^4 + 50p'^2 + q^4) = 5(625p'^4 + 50p'^2 + q^4 - 500p'^4) = 5((q^2 + 25p'^2)^2 - 5(10p'^2)^2) \)

Consider the equation: \( 5[(q^2 + 25p'^2)^2 - 5(10p'^2)^2] = 5z_2^5 \)

So \( (q^2 + 25p'^2)^2 - 5(10p'^2)^2 = z_2^5 \)

Let \( u = q^2 + 25p'^2, v = 10p'^2 \) then \( z_2 = c^2 - 5d^2 \) and \( u, v \) are given by (8);(9).

If \( u, v \) is the one solution, then:

\[ u' = \frac{(m^2 + 5n^2)u - 10mnu}{5n^2 - m^2} \] (11)

\[ v' = \frac{2mnu - (m^2 + 5n^2)v}{5n^2 - m^2} \] (12)

are also solution: \( u'^2 - 5v'^2 = u_0^2 - 5v_0^2 \)

The other one solution* of the equation \( u^2 + 5v^2 = (c^2 - 5d^2)^5 \) is:

\[ u = u_2 = c(c^4 + 10c^2d^2 - 75d^4) \] (13)

\[ v = v_2 = d(3c^4 - 10c^2d^2 - 25d^4) \] (14)

(* was not considered by Dirichlet in his proof).

3 The quadratic formulas of the equation \( X^6 - X^5Y + X^4Y^2 - X^3Y^3 + X^2Y^4 - XY^5 + Y^6 = Z_2^7 \) (related equation \( X^7 + Y^7 = Z_2^7; Z_1 = X + Y \) )

3a.

\[ x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6 = z_2^7 \] (15)

x + y is not divisible by 7

Write \( x = p + q, y = p - q \), then:

\[ x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6 = p^6 + 21p^4q^2 + 35p^2q^4 + 7q^6 \]

\[ = p^6 + 14p^4q^2 + 49p^2q^4 + 7p^2q^4 - 14p^2q^4 + 7q^6 \]

\[ = p^2(p^2 + 7q^2)^3\frac{7}{2}q^2(p^2 - q^2)^2 \]

\[ p = \frac{x + y}{2} \] is not divisible by 7, consider the equation: \( z_2^7 = p^2(p^2 + 7q^2)^2 + 7q^2(p^2 - q^2)^2 \) let \( u = p(p^2 + 7q^2) \) and \( v = q(p^2 - q^2) \)

then \( z_2 = u^2 + 7v^2 \) and \( z_2 \) can be written as \( z_2 = a + 7b \)

and by the same way as above, we obtain:

\[ u_1 = c(c^2 - 21d^2)^2 - 7d^2(3c^2 - 7d^2)^2 - 14d^2(c^2 - 21d^2)(3c^2 - 7d^2)) \] (16)
\[ v_1 = d[2c^2(c^2 - 21d^2)(3c^2 - 7d^2) + c^2(c^2 - 21d^2)^2 - 7d^2(3c^2 - 7d^2)^2] \]  
\[ u_2 = c[c^2(c^2 - 21d^2)^2 - 7d^2(3c^2 - 7d^2)^2 + 14d^2(c^2 - 21d^2)(3c^2 - 7d^2)] \]  
\[ v_2 = d[2c^2(c^2 - 21d^2)(3c^2 - 7d^2) - c^2(c^2 - 21d^2)^2 + 7d^2(3c^2 - 7d^2)^2] \]

and \( z_2 = c^2 + 7d^2 \)

3b.

\[ x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6 = 7z_2^7 \]

\( x + y \) is divisible by 7

\[ p = \frac{x + y}{2} \] is divisible by 7, write \( p = 7p' \), so \( p^2(p^2 + 7q^2)^2 + 7q^2(p^2 - q^2)^2 = 7^2p'^2(7^2p'^2 + 7q^2)^2 + 7q^2(7^2p'^2 - q^2)^2 \)

Consider the equation \( 7[7^3p'^2(7p'^2 + q^2)^2 + q^2(7^2p'^2 - q^2)^2] = 7z_2^7 \)

So \( 7^3p'^2(7p'^2 + q^2)^2 + q^2(7^2p'^2 - q^2)^2 = z_2^7 \)

Let \( u = q(7^2p'^2 - q^2), v = 7p'(7p'^2 + q^2) \), then \( z_2^2 = u^2 + 7v^2 \), and \( u, v \) are given by (16); (17); (18); (19).

If \( u, v \) is the one solution, then:

\[ u' = \frac{(m^2 - 7n^2)u + 14mnv}{7n^2 + m^2} \] \hspace{1cm} (21)

\[ v' = \frac{2mnv + (7n^2 - m^2)v}{7n^2 + m^2} \] \hspace{1cm} (22)

are also solution: \( u'^2 + 7v'^2 = u^2 + 7v^2 \)

Notes:
- Different from case \( n = 3 \), for the case \( n = 5 \), \( u \) and \( v \) must satisfy \( u = p^2 + 5q^2 \) and \( v = 2q^2 \) \hspace{1cm} (2a), \( u = q^2 + 25p^2, v = 10p^2 \) \hspace{1cm} (2b), for the case \( n = 7 \), \( u \) and \( v \) must satisfy \( u = p(p^2 + 7q^2) \), \( v = q(p^2 - q^2) \) \hspace{1cm} (3a), \( u = q(7^2p'^2 - q^2), v = 7p'(7p'^2 + q^2) \) \hspace{1cm} (3b).
- \( u', v' \) are integer or not, depend on \( u, v, m, n \).
- For the equation \( X^{n-1} + X^{n-2}Y + X^{n-3}Y^2 + ... + Y^{n-1} = Z_2^n(nZ_2^n) \), the algorithm is the same as above.

References

[1] Quang N V, Euler’s proof of Fermat Last’s Theorem for \( n = 3 \) is incorrect Vixra:1605.0123v3(NT)

[2] Quang N V, Dirichlet’s proof of Fermat last’s theorem for \( n = 5 \) is flawed Vixra:1607.0400v2(NT)

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