

Particular Solutions for Boolean Algebra Systems.

Franco Sabino Stoianoff Lindstron

Universidad Nacional de La Plata

Email: stoianoff2000@gmail.com

Abstract

Any system of ‘big’ Boolean equations can be reduced to a single Boolean equation $\{g(\mathbf{Z}) = 1\}$. We propose a novel method for producing a general parametric solution for such a Boolean equation without attempting to minimize the number of parameters used, but instead using independent parameters belonging to the two-valued Boolean algebra B_2 for each asserted atom that appears in the discriminants of the function $g(\mathbf{Z})$. We sacrifice minimality of parameters and algebraic expressions for ease, compactness and efficiency in listing all particular solutions. These solutions are given by additive formulas expressing a weighted sum of the asserted atoms of $g(\mathbf{Z})$, with the weight of every atom (called its contribution) having a number of alternative possible values equal to the number of appearances of the atom in the discriminants of $g(\mathbf{Z})$. This allows listing a huge number of particular solutions within a very small space and the possibility of constructing solutions of desirable features. The new method is demonstrated via three examples over the ‘big’ Boolean algebras, B_4 , B_{16} , and B_{256} , respectively. The examples demonstrate a variety of pertinent issues such as complementation, algebra collapse, incremental solution, and handling of equations separately or jointly.

1 Introduction

Any system of ‘big’ Boolean equations can be reduced to a single Boolean equation $\{g(\mathbf{Z}) = 1\}$ or $\{\bar{g}(\mathbf{Z}) = 0\}$ [4, 5, 10, 11]. The main types of solutions of such a Boolean equation are the subsumptive general solutions, the parametric general solutions and the particular solutions. In a subsumptive general solution,

each of the variables is expressed as an interval based on successive conjunctive or disjunctive eliminants of the original function [4, 5, 7-12, 14-16, 21-23]. In a parametric general solution each of the variables is expressed via arbitrary parameters which are freely chosen elements of the underlying Boolean algebra. A particular solution is an assignment from the underlying Boolean algebra to every pertinent variable that makes the Boolean equation an identity [4, 5, 15, 16]. An exclusive enumeration of particular solutions is obtained from any of the two types of general solutions via an expansion tree in which a parent node might have as many children nodes as the elements in the underlying Boolean algebra [4, 5, 15, 16, 21, 22]. In particular, in the conventional method for producing a general parametric solution, the number of parameters used is minimized producing compact algebraic solutions with parameters belonging to the underlying Boolean algebra. Contrarily to this convention, we propose a novel method for producing a general parametric solution that does not attempt to minimize the number of parameters used, but instead uses independent parameters belonging to the two-valued Boolean algebra B_2 for each asserted atom that appears in the discriminants of the function $g(\mathbf{Z})$. The parametric solution obtained sacrifice minimality of parameters and algebraic expressions for ease, compactness and efficiency in listing all particular solutions. These solutions are given by additive formulas expressing a weighted sum of asserted atoms of $g(\mathbf{Z})$, with the weight of every atom (called its contribution) having a number of alternative possible values equal to the number of appearances of the atom in the discriminants of $g(\mathbf{Z})$. These alternatives are based on a set of orthonormal tags, and hence are listed in a rectangle divided into disjoint cells. This rectangle resembles a Karnaugh map, and is, in fact, a Karnaugh map, possibly with some adjacent cells combined. The representation suggested allows the possibility of listing a huge number of particular solutions within a very small space. The reason of this possibility is that an arbitrarily-selected contribution of a particular atom can be combined with any of the possible contributions of each of the other atoms. The combination via the additive (ORing) operation is simple and straightforward.

The methods presented herein could be cast in pure algebraic form [12, 16, 21], but become much easier to visualize and comprehend when presented via the natural map of a big Boolean algebra, which (for historical reasons) is called the Variable-Entered Karnaugh Map (VEKM) [13-20]. A Boolean function of n variables has $2n$ VEKM representations (depending on the choice of map and entered variables) ranging from a Conventional Karnaugh Map (CKM) which is a VEKM of n map variables and 0 entered variables, to a purely-algebraic expression which is a VEKM of 0 map variables and n entered variables [13]. The VEKM methods therefore include purely-algebraic methods as a special case [16, 21]. Hence, they can always take full advantage of the results provided by the algebraic theory. Moreover, they have a better control on the minimality of pertinent function representations.

The organization of the remainder of this paper is as follows. Section 2 gives a detailed description of both the conventional and novel methods for deriving parametric solutions of ‘big’ Boolean equations. The section starts by pointing out the unavoidability of ‘big’ Boolean algebras, and then it outlines the derivation of parametric solutions before detailing ways of generating all particular solutions from them. Features of the new method are demonstrated via three examples over the ‘big’ Boolean algebras B_4 , B_{16} , and B_{256} . The first example demonstrates the utility of the natural map (VEKM) of B_4 for expressing and complementing a 4-variable function and for producing the auxiliary function needed in deriving parametric solutions and subsequently in generating all particular solutions. The second example demonstrates the incremental generation of solutions of a certain equation in terms of those of another related equation. The third example lists a huge set of particular solutions (that number slightly less than one million solutions) in a very compact space. The example also relates the individual solution of two separate equations to the joint solutions of the equations.

2 Steps of the Method

2.1 Derivation of Parametric Solutions

We seek solutions of the Boolean equation

$$g(\mathbf{X}, \mathbf{Z}) = 1, \quad (1)$$

where $g(\mathbf{X}, \mathbf{Z}): B_2^{k+n} \rightarrow B_2$, is a two-valued Boolean function of k two-valued variables $\mathbf{X} = [X_1 X_2 \dots X_k]^T$ and n two-valued variables $\mathbf{Z} = [Z_1 Z_2 \dots Z_n]^T$. However, we do not need a listing of binary solutions for \mathbf{X} and \mathbf{Z} , but instead we want to express \mathbf{Z} in terms of \mathbf{X} . This is a prominent case when the use of ‘big’ Boolean algebras (ones other than the two-valued algebras) is unavoidable. We view $g(\mathbf{X}, \mathbf{Z})$ as $g(\mathbf{X}; \mathbf{Z})$ or simply $g(\mathbf{Z})$ and rewrite (1) as

$$g(\mathbf{Z}) = 1, \quad (1a)$$

where $g(\mathbf{Z}): B_{2^k}^n \rightarrow B_{2^k}$, and B_{2^k} is the free Boolean algebra $FB(X_1, X_2 \dots \dots X_k)$ with $K = 2^k$ atoms and 2^k elements [4, 20]. Now we express $g(\mathbf{Z})$ by its Minterm Canonical Form (MCF) [4]

$$g(\mathbf{Z}) \equiv \vee_{\mathbf{A} \in \{0,1\}^n} g(\mathbf{A}) \mathbf{Z}^{\mathbf{A}}. \quad (2)$$

For $\mathbf{Z} = [Z_1 Z_2 \dots Z_n]^T \in B_{2^k}^n$, $\mathbf{A} = [a_1 a_2 \dots a_n]^T \in \{0,1\}^n$, the symbol $\mathbf{Z}^{\mathbf{A}}$ is defined as

$$\mathbf{Z}^{\mathbf{A}} = Z_1^{a_1} Z_2^{a_2} \dots Z_n^{a_n}, \quad (3)$$

where $Z_i^{a_i}$ takes the value \bar{Z}_i (complemented literal) if $a_i = 0$, and takes the value Z_i (uncomplemented literal) if $a_i = 1$. For $\mathbf{A} \in \{0,1\}^n$, the symbol $\mathbf{Z}^{\mathbf{A}}$ spans the minterms of \mathbf{Z} , which are the 2^n elementary or primitive products

$$\bar{Z}_1 \bar{Z}_2 \dots \bar{Z}_{n-1} \bar{Z}_n, \bar{Z}_1 \bar{Z}_2 \dots \bar{Z}_{n-1} Z_n, \dots, Z_1 Z_2 \dots Z_{n-1} Z_n. \quad (4)$$

The constant values $g(\mathbf{A})$ in equation (2) are elements of B_{2^k} called the discriminants of $g(\mathbf{Z})$ [4]. These discriminants are the entries of the natural map of $g(\mathbf{Z})$ which has an input domain $\{0,1\}^n \subseteq B_{2^k}^n$. The Boolean algebra $B_{2^k} = FB(X_1, X_2 \dots \dots X_k)$, has generators X_i ($1 \leq i \leq k$) which look like variables (In fact, they were originally our input variables before we changed their roles to generators). Therefore, we can accept the name assigned (for historical reasons) to the natural map of $g(\mathbf{Z})$, namely the name of the Variable-Entered Karnaugh Map (VEKM). We now observe that the minterms of \mathbf{X} , which are the $2^k = K$ elementary or primitive products

$$\bar{X}_1 \bar{X}_2 \dots \bar{X}_{k-1} \bar{X}_k, \bar{X}_1 \bar{X}_2 \dots \bar{X}_{k-1} X_k, \dots, X_1 X_2 \dots X_{k-1} X_k, \quad (5)$$

are exactly the atoms of the underlying Boolean algebra. For convenience, we call these atoms T_i ($0 \leq i \leq (K - 1)$), and hence $g(\mathbf{A})$ can be written as

$$g(\mathbf{A}) = \vee_{i=0}^{K-1} (e_i(\mathbf{A}) \wedge T_i), \quad (6)$$

where we use the symbol $e_i(\mathbf{A})$ to denote an indicator of the event that atom T_i appears in the expression of $g(\mathbf{A})$, i.e.,

$$e_i(\mathbf{A}) = \begin{cases} 1, & \text{if } T_i \rightarrow g(\mathbf{A}) \\ 0, & \text{otherwise} \end{cases} = g(\mathbf{A})/T_i, \quad (7)$$

where the symbol $(r/s) = (r)_{s=1}$ denotes the Boolean quotient of r by s [4]. Equation (7) means that $e_i(\mathbf{A})$ indicates whether atom T_i appears in the cell \mathbf{A} of the natural map for $g(\mathbf{Z})$. Now, we define n_i as the total number of actual appearances of T_i in the expression (6) for $g(\mathbf{A})$, i.e.,

$$n_i = \sum_{\mathbf{A} \in \{0,1\}^n} e_i(\mathbf{A}). \quad (8)$$

The total number $N_{\text{unconditional}}$ of unconditional particular solutions of (1a) over B_{2^k} (as it is) is given by

$$N_{\text{unconditional}} = \prod_{i=0}^{K-1} n_i. \quad (9)$$

This number is zero if some $n_i = 0$, i.e., if an atom T_i never makes its way to any expression $g(\mathbf{A})$ where $\mathbf{A} \in \{0,1\}^n$ (i.e., if T_i does not appear in any cell of the map for $g(\mathbf{Z})$). To avoid such a situation, one must insist on the consistency condition that any atom T_i such that $n_i = 0$ must be forbidden or nullified [5, 20]. This means that the underlying Boolean algebra loses these atoms and hence collapses to a smaller algebra, i.e., to one of its strict sub algebras. The number of solutions over this new Boolean algebra is

$$N_{\text{conditional}} = \prod_{\substack{i=0 \\ n_i \neq 0}}^{K-1} n_i. \quad (10)$$

Now we introduce a set of parameters \mathbf{p}_i ($0 \leq i \leq (K-1)$, $n_i \neq 0$) to construct an orthonormal set of tags to attach to instances of appearances of the asserted atom T_i in the discriminants $g(\mathbf{A})$ (i.e., in the cells $\mathbf{A} \in \{0,1\}^n$ of the natural map of $g(\mathbf{Z})$). The number of parameters for atom T_i (the length of vector \mathbf{p}_i) is given by

$$l(\mathbf{p}_i) = \lceil \log_2 n_i \rceil, \quad 0 \leq i \leq (K-1), n_i \neq 0. \quad (11)$$

Here, $\lceil x \rceil$ denotes the ceiling of the real number x , i.e., the smallest integer greater than or equal to x . The parameters \mathbf{p}_i can be used to generate a set of $n_i \leq 2^{l(\mathbf{p}_i)}$ orthonormal tags $\{t_1, t_2 \dots t_{n_i}\}$, such that

$$t_1 \vee t_2 \vee \dots \vee t_{n_i} = 1, \quad (12a)$$

$$t_{j_1} \wedge t_{j_2} = 0 \quad \forall j_1, j_2 \in \{1, 2, \dots, n_i\}. \quad (12b)$$

We now consider the simple case of $n_i = 2^{l(\mathbf{p}_i)}$, specifically when $n_i = 8$ and $l(\mathbf{p}_i) = 3$. Figure 1(a) demonstrates how a Karnaugh map of three variables p_1, p_2 , and p_3 can be conveniently used to construct an element orthonormal set which consists of minterms over $\{p_1, p_2, p_3\}$, namely

$$\{ \overline{p_1} \overline{p_2} \overline{p_3}, \overline{p_1} \overline{p_2} p_3, \overline{p_1} p_2 \overline{p_3}, \overline{p_1} p_2 p_3, p_1 \overline{p_2} \overline{p_3}, p_1 \overline{p_2} p_3, p_1 p_2 \overline{p_3}, p_1 p_2 p_3 \}. \quad (13)$$

The specific value of this set is shown for each of the map cells in Fig. 1(a) in a vivid illustration of the concept of orthonormality given by (12). Now we consider the case $2^{l(\mathbf{p}_i)-1} < n_i < 2^{l(\mathbf{p}_i)}$. Figures 1(b) and 1(c) demonstrate the case $n_i = 5$, for which $l(\mathbf{p}_i) = 3$. We have many possibilities for constructing an orthonormal set. Two notable examples are shown in Figs. 1(b) and 1(c) and represent, respectively the two orthonormal sets.

$$\{ \overline{p_1}, p_1 \overline{p_2} \overline{p_3}, p_1 \overline{p_2} p_3, p_1 p_2 \overline{p_3}, p_1 p_2 p_3 \}. \quad (14)$$

$$\{ \overline{p_1} \overline{p_2}, \overline{p_1} p_2, p_1 \overline{p_2}, p_1 p_2 \overline{p_3}, p_1 p_2 p_3 \}. \quad (15)$$

When each appearance of an atom T_i is tagged by a particular member of its orthonormal set of tags, an auxiliary function $G(\mathbf{Z}, \mathbf{p}_i)$ ($0 \leq i \leq (K-1)$, $n_i \neq 0$) results. The parametric solution is now given by [5, 19, 20].

$$Z_l = \vee_{\{\mathbf{A} \in \{0,1\}^n | A_l = 1\}} G(\mathbf{A}, \mathbf{p}_i). \quad 1 \leq l \leq n, (0 \leq i \leq (K-1), n_i \neq 0) \quad (16)$$

The total number of parameters used in (16) is given by

$$E = \sum_{i=1}^k l(\mathbf{p}_i) = \sum_{i=1}^k \lceil \log_2(n_i) \rceil. \quad (17)$$

The conventional method is to select the parameter vectors from a shared pool of parameters so as to minimize the number of parameters used, which then becomes

$$E' = \max_i l(\mathbf{p}_i) = \max_i \lceil \log_2 n_i \rceil = \lceil \log_2 (\max_i n_i) \rceil. \quad (18)$$

However, parameters used must then belong to the underlying Boolean algebra (possibly collapsed due to the consistency condition). We now propose to use independent parameters \mathbf{p}_i for each atom T_i ($0 \leq i \leq K-1$, $n_i \neq 0$). The expressions (16) will not be as compact as they are in the conventional case, but the independent parameters \mathbf{p}_i now belong to the two-valued Boolean algebra B_2 [5, 20], a fact that facilitates the generation of all particular solutions as we will see shortly in the next subsection.

$\{1,0,0,0,0,0,0,0\}$	$\{0,0,1,0,0,0,0,0\}$	p_1		
		$\{0,0,0,0,0,0,1,0\}$	$\{0,0,0,0,1,0,0,0\}$	
$\{0,1,0,0,0,0,0,0\}$	$\{0,0,0,1,0,0,0,0\}$	$\{0,0,0,0,0,0,0,1\}$	$\{0,0,0,0,0,1,0,0\}$	
				p_3
p_2				

(a)

$\{1,0,0,0,0\}$	p_1		
	$\{0,0,0,1,0\}$	$\{0,1,0,0,0\}$	
	$\{0,0,0,0,1\}$	$\{0,0,1,0,0\}$	
			p_3
p_2			

(b)

$\{1,0,0,0,0\}$	$\{0,1,0,0,0\}$	p_1		
		$\{0,0,0,1,0\}$	$\{0,0,1,0,0\}$	
		$\{0,0,0,0,1\}$		
				p_3
p_2				

(c)

Fig. 1. Use of Karnaugh-map and Karnaugh-map like structures to construct orthonormal sets of tags.

2.2 Listing of All Particular Solutions

The parametric solutions (16) can be used to generate all particular solutions through the use of an expansion tree [5, 16, 19-21]. Generally, in the conventional method, this expansion tree is a complete tree that entails the assignment of $2^{K'}$ values to each of E' parameters where $K' \leq K$ is the final number of atoms of the underlying Boolean algebra (possibly after some collapse due to the consistency condition). Each parent node has $2^{K'}$ children nodes and the tree has E' levels beyond its root. Therefore, the tree has $(2^{K'})^{E'} = 2^{K'E'}$ leaves. These leaves constitute the whole set of particular solutions, possibly with repetitions. However, to avoid repetitions, we make sure, right from the first expansion level, to combine any sibling nodes that share the same solution value. With this kind of combining, the tree ceases to be a complete one, and its leaves become exactly the particular solutions, i.e., without repetitions [16].

In the method proposed herein, a complete version of the expansion tree requires the assignment of binary values $\{0,1\}$ to each of E independent parameters. Since the complete binary tree has E levels beyond its root, it has 2^E leaves. With merging of sibling nodes of equal solution values, the tree is no longer complete, and its leaves are just the particular solutions without repetitions. The size of the expansion tree in the proposed method is typically less than that in the conventional method since typically $E < K'E'$ (though $E > E'$). However, the true advantage of the proposed method is that it allows us to avoid the use of an expansion tree altogether. The key of this is the observation that the parametric solution (16) can be rewritten as the weighted sum of the atoms T_i that appear in the discriminants $g(\mathbf{A})$ (as expressed in (6)) of the function $g(\mathbf{Z})$, viz.

$$\mathbf{Z} = \bigvee_{\substack{i=0 \\ n_i \neq 0}}^{K-1} (\mathbf{Co}(T_i) \wedge T_i), \quad (19)$$

where we call the vector $\mathbf{Co}(T_i)$ the ‘contribution’ of the asserted atom T_i and call the conjunction $(\mathbf{Co}(T_i) \wedge T_i)$ the ‘total contribution’ of that atom. We now note that $\mathbf{Co}(T_i)$ {or $\mathbf{Co}(T_i) \wedge T_i$ } has exactly n_i possible values, which can be conveniently listed via the same Karnaugh-map-like structure used in the representation of the associated tags. Therefore, we interpret (19) as a method of conveniently listing all particular solutions as a disjunction of total contributions of asserted atoms T_i , where the total contribution is given in all its n_i possibilities. To obtain a specific particular solution, one has simply to pick up arbitrarily one of the possibilities of the total contribution for every atom, and then add the selected total contributions together. The total number of particular solutions obtained this way agrees with that given by (10).

2.3 Picking up a Particular Solution of Specific Features

Equation (19) is of a paramount importance, as it provides a listing of a (possibly huge) number of all particular solutions in a compact space. As such, it allows picking up certain solutions enjoying particular desirable features simply by a quick inspection of the aforementioned listing. This point will be clarified further by way of examples in the next section.

3 Examples

In the following, we illustrate the method of Sec. 2 via three detailed examples. The first example applies our novel method to a 4-variable problem over B_4 . The second example considers two related problems over B_{16} , solves the second incrementally in terms of the solution of the first, and further solves the inverse problem for one of them over B_{256} . The third example tackles the solutions of two equations over B_{256} when taken separately or collectively.

3.1 Example 1

Consider the Boolean function $f(a; X_1, X_2, X_3, X_4): B_4^4 \rightarrow B_4$, where $B_4 = FB(a) = \{0, a, \bar{a}, 1\}$. A solution of the equation ($f = 0$) expresses the dependent variable vector $\mathbf{X} = [X_1 \ X_2 \ X_3 \ X_4]^T$ in terms of the independent variable a (used herein as a generator of B_4). Suppose f is given by the product-of-sums (pos) expression

$$f = (a \vee X_2)(a \vee X_3)(a \vee X_4)(\bar{X}_1 \vee \bar{X}_2). \quad (20)$$

We construct the natural map (VEKM) for f by noting that f is given by [13]

$$f = (0 \vee \text{Co}(0)) (a \vee \text{Co}(a)), \quad (21)$$

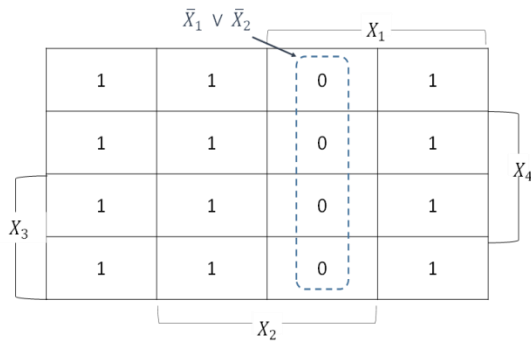
where $\text{Co}(0)$ and $\text{Co}(a)$ are called the dual or conjunctive contributions of alterms 0 and a , respectively, and are presented by the Conventional Karnaugh Maps (CKMs) in Figs. 2(a) and 2(b). These two figures are used to produce a natural map for f in Fig. 2(c). The entries of this map are then complemented in Fig. 2(d) to obtain the inverse $g = \bar{f}$ of f , so that our problem becomes to solve $\{g = 1\}$. Since the entry 1 stands for $(a \vee \bar{a})$, Fig. 2(d) shows that the two atoms a and \bar{a} of B_4 appear 4 and 15 times, respectively, in the natural map for g . This means that the consistency condition for the Boolean equation $\{g = 1\}$ is the identity $\{0 = 0\}$ and the number of particular solutions is $4 \cdot 15 = 60$. The instances of appearances of a needs 15 orthonormal tags which are defined in terms of four parameters p_1, p_2, p_3 , and p_4 as shown in Fig. 3(a). Likewise, Fig. 3(a) uses two additional parameters p_5 and p_6 to define 4 orthonormal tags to be associated with the instances of appearances of atom \bar{a} . Figure 3(b) shows the auxiliary function $G(a, \mathbf{X}, \mathbf{p})$ for this example. The parametric solution for $\{f = 0\}$ or $\{g = 1\}$ is given by

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \bar{a} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4(p_1 \vee \bar{p}_2 \vee \bar{p}_3) \end{bmatrix} \vee a \begin{bmatrix} 1 \\ 1 \\ p_5 \\ p_6 \end{bmatrix}. \quad (22)$$

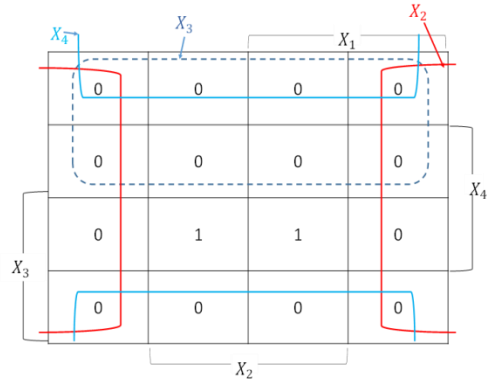
The parametric solution (22) leads to the compact listing of all 60 particular solutions as follows:

$$\begin{array}{c}
 \begin{matrix} p_1 \\ \hline \begin{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ \bar{a} \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \bar{a} \\ \bar{a} \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \bar{a} \\ \bar{a} \\ \bar{a} \\ 0 \end{bmatrix} \\
 \hline \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bar{a} \end{bmatrix} & \begin{bmatrix} 0 \\ \bar{a} \\ 0 \\ \bar{a} \end{bmatrix} & \begin{bmatrix} \bar{a} \\ \bar{a} \\ 0 \\ \bar{a} \end{bmatrix} & \begin{bmatrix} \bar{a} \\ \bar{a} \\ 0 \\ \bar{a} \end{bmatrix} \\
 \hline \begin{bmatrix} 0 \\ 0 \\ \bar{a} \\ \bar{a} \end{bmatrix} & \begin{bmatrix} 0 \\ \bar{a} \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \bar{a} \\ \bar{a} \\ \bar{a} \\ \bar{a} \end{bmatrix} & \begin{bmatrix} \bar{a} \\ \bar{a} \\ 0 \\ \bar{a} \end{bmatrix} \\
 \hline \begin{bmatrix} 0 \\ 0 \\ \bar{a} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ \bar{a} \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \bar{a} \\ \bar{a} \\ \bar{a} \\ 0 \end{bmatrix} & \begin{bmatrix} \bar{a} \\ \bar{a} \\ 0 \\ \bar{a} \end{bmatrix} \\
 \hline \end{matrix} \\
 \hline \end{matrix} \\
 \begin{matrix} p_2 \\ \hline \end{matrix} \\
 \hline \end{array}
 \begin{matrix} p_3 \\ \hline \end{matrix}
 \begin{matrix} p_4 \\ \hline \end{matrix}
 \mathbf{V}
 \begin{matrix} p_5 \\ \hline \begin{matrix} \begin{bmatrix} a \\ a \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} a \\ a \\ a \\ 0 \end{bmatrix} \\
 \hline \begin{bmatrix} a \\ a \\ 0 \\ a \end{bmatrix} & \begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix} \\
 \hline \end{matrix} \\
 \hline \end{matrix}
 \begin{matrix} p_6 \\ \hline \end{matrix}
 \end{matrix}
 \quad (23)$$

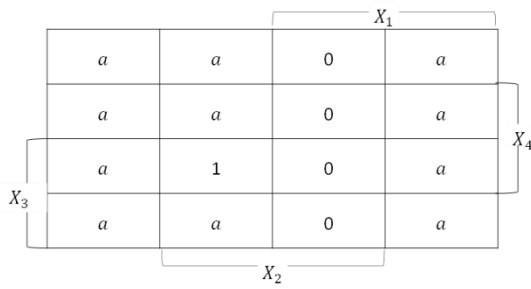
Equation (23) shows that there are 15 possibilities of total contributions ($\bar{a} \text{ Co } (\bar{a})$) of \bar{a} , exhausting all 16 combinations of 4 tuples belonging each to either 0 or \bar{a} , with the sole exception of $[0 \ \bar{a} \ \bar{a} \ \bar{a}]^T$. There are also 4 possibilities of total contribution ($a \text{ Co } (a)$) of a , which are one quarter of the 16 combinations of the 4 tuples belonging each to either 0 or a . A particular solution is obtained by ORing an arbitrarily selected possibility of ($\bar{a} \text{ Co } (\bar{a})$) with an arbitrarily selected possibility of ($a \text{ Co } (a)$). For example, one might implement the ORing operation of $[\bar{a} \ \bar{a} \ \bar{a} \ \bar{a}]^T$ with $[a \ a \ a \ a]^T$, to produce the solution $[1 \ 1 \ 1 \ 1]^T$, which can be immediately verified to produce a value of 0 for f in (20).



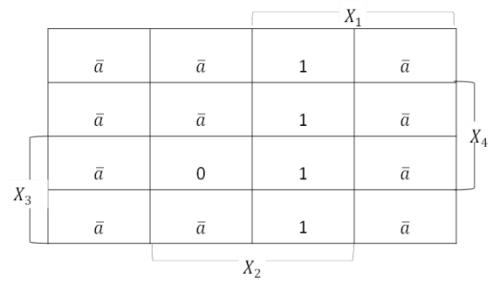
(a) $Co'(0)$



(b) $Co'(a)$



(c) $f = (0 \vee Co'(0)) (a \vee Co'(a))$



(d) $g = \bar{f}$

Fig. 2. Construction of the function f in Example 1 in terms of the conjunctive contributions $Co'(a)$ and $Co'(0)$, and derivation of its inverse g via cellwise complementation

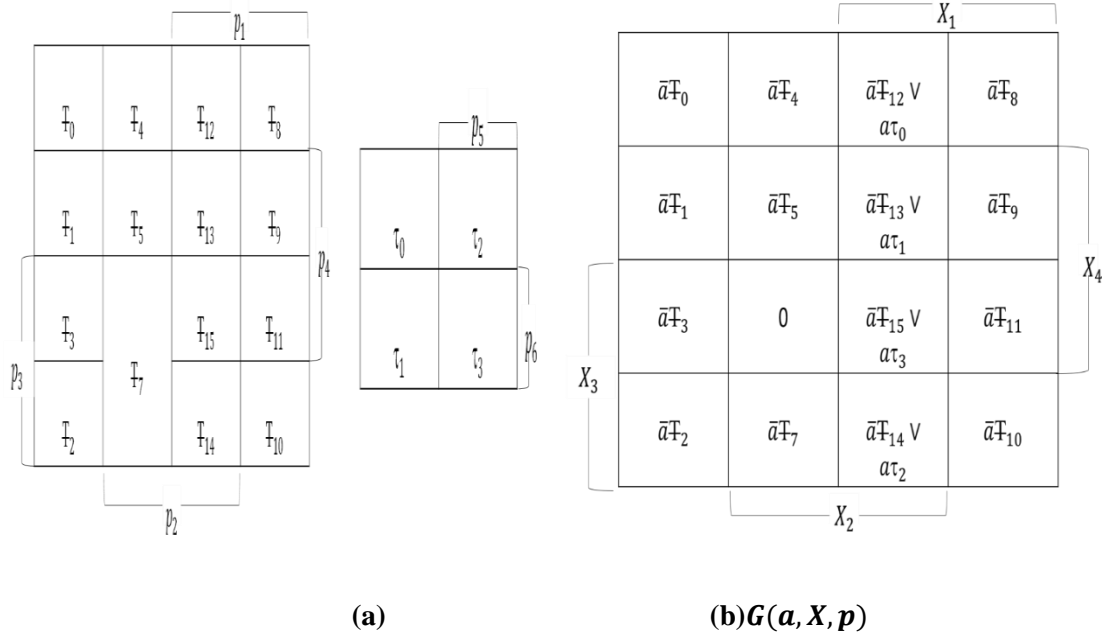


Fig. 3. Definition of sets of orthonormal tags for atoms a and \bar{a} in Example 1. Here, for example, T_7 stands for $\bar{p}_1 p_2 p_3$, T_{10} depicts $p_1 \bar{p}_2 p_3 \bar{p}_4$, while τ_2 equals $p_5 \bar{p}_6$. This definition results in the auxiliary function in (b).

3.2 Example 2

This example has two related versions that are very popular in the literature of Boolean equations. The first version was solved by Brown [5] and Rushdi and Amashah [19], while the second version was solved (repeatedly) by Brown [4, 5] and then by Rushdi [14, 15], Rudeanu [9], and Rushdi and Amashah [20]. The first version deals with the solution of $g_1(X_1, X_2, X_3) = 1$, where

$$g_1(Y_1, Y_2; X_1, X_2, X_3) = \bar{Y}_1 \bar{Y}_2 (X_1 X_2 \vee \bar{X}_1 \bar{X}_2 X_3) \vee Y_1 Y_2 (\bar{X}_1 X_2 \vee \bar{X}_1 \bar{X}_3) \vee Y_1 \bar{Y}_2 (X_1 \bar{X}_2 \vee \bar{X}_2 \bar{X}_3), \quad (24)$$

is a Boolean function $g = B_{16}^3 \rightarrow B_{16}$, where $B_{16} = FB(Y_1, Y_2)$ is a Boolean carrier of 16 elements constituting the switching functions of two variables Y_1 and Y_2 . Though our equation is formulated on a big Boolean algebra, it could be thought of as one of expressing three dependent variables X_1, X_2 and X_3 in terms of two independent switching variables Y_1 and Y_2 . Figure 4(a) represents the natural map (VEKM) for g_1 . The three atoms $\bar{Y}_1 \bar{Y}_2$, $Y_1 Y_2$, and $Y_1 \bar{Y}_2$ make their appearances 3, 3, and 4 times in the 8 cells of the map of Fig. 4(a), while atom $\bar{Y}_1 Y_2$ is never entered in that map. Therefore, the number of particular solutions of the equation $g_1 = 1$, is $3 \cdot 3 \cdot 4 = 36$, and the consistency condition is $\bar{Y}_1 Y_2 = 0$. The three asserted atoms demand the use of the three sets of parameters $\{p_1, p_2\}$, $\{p_3, p_4\}$, and $\{p_5, p_6\}$, so as to produce the three orthonormal sets of tags $\{p_1, \bar{p}_1 p_2, \bar{p}_1 \bar{p}_2\}$, $\{p_3, \bar{p}_3 p_4, \bar{p}_3 \bar{p}_4\}$, and $\{p_5 p_6, p_5 \bar{p}_6, \bar{p}_5 p_6, \bar{p}_5 \bar{p}_6\}$, of cardinalities 3, 3, and 4 respectively. The various tags are associated with the respective atoms to construct the auxiliary function $G_1(X_1, X_2, X_3; p_1, p_2, p_3, p_4, p_5, p_6)$ shown in Fig. 4(b). The set of parametric solutions is given by

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \bar{Y}_1\bar{Y}_2 \begin{bmatrix} p_1 \vee \bar{p}_1 p_2 \\ p_1 \vee \bar{p}_1 p_2 \\ p_1 \vee \bar{p}_1 \bar{p}_2 \end{bmatrix} \vee Y_1 Y_2 \begin{bmatrix} 0 \\ p_3 \vee \bar{p}_3 p_4 \\ p_3 \end{bmatrix} \vee Y_1 \bar{Y}_2 \begin{bmatrix} p_5 p_6 \vee \bar{p}_5 p_6 \\ 0 \\ p_5 p_6 \vee p_5 \bar{p}_6 \end{bmatrix} \vee \begin{bmatrix} d(\bar{Y}_1 Y_2) \\ d(\bar{Y}_1 Y_2) \\ d(\bar{Y}_1 Y_2) \end{bmatrix} \quad (25)$$

and could be translated into a form listing all 36 solutions, namely

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ \bar{Y}_1 \bar{Y}_2 \end{bmatrix} \\ \begin{bmatrix} \bar{Y}_1 \bar{Y}_2 \\ \bar{Y}_1 \bar{Y}_2 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} \overbrace{p_1} \\ \overbrace{p_2} \end{array} \vee \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ Y_1 Y_2 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} \overbrace{p_3} \\ \overbrace{p_4} \end{array} \vee \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} Y_1 \bar{Y}_2 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} \overbrace{p_5} \\ \overbrace{p_6} \end{array} \vee \begin{bmatrix} d(\bar{Y}_1 Y_2) \\ d(\bar{Y}_1 Y_2) \\ d(\bar{Y}_1 Y_2) \end{bmatrix} \quad (26)$$

Now, we consider the second version of our current problem which is $\{g_2 = 1\}$, where g_2 is specified by Fig. 4 of Rushdi and Amashah [20], reproduced here, for convenience, as Fig. 5. A quick visual comparison of the natural maps g_1 (Fig. 4(a)) and g_2 (Fig. 5) shows that the two maps are identical except for an extra entry $Y_1 \bar{Y}_2$ in the $\bar{X}_3 X_2 X_1$ cell of the map of g_2 , i.e., g_2 is related to g_1 via the relation

$$g_2 = g_1 \vee Y_1 \bar{Y}_2 \bar{X}_3 X_2 X_1. \quad (27)$$

The atoms $\bar{Y}_1 \bar{Y}_2$, $Y_1 Y_2$, $Y_1 \bar{Y}_2$ and $\bar{Y}_1 Y_2$ now appear 3, 3, 5, and 0 times respectively in the cells of the map in Fig. 5. Hence the equation $\{g_2 = 1\}$ again has the consistency condition $\{\bar{Y}_1 \bar{Y}_2 = 0\}$, but its number of particular solutions is now $3 \cdot 3 \cdot 5 = 45$. Note that the two functions g_1 and g_2 match with the single exception that the latter function has an extra instance of appearance of atom $Y_1 \bar{Y}_2$ in the natural map of the function. Hence, the set of solutions of $\{g_2 = 1\}$ is a strict superset for the set of solutions of $\{g_1 = 1\}$. The remaining 9 solutions of $\{g_2 = 1\}$ are the solutions of the equation $\{g_3 = 1\}$ where g_3 is specified by the natural map in Fig 6(a). That map has a single instance of appearance of the atom $Y_1 \bar{Y}_2$ in the cell $\bar{X}_3 X_2 X_1$, and the same appearances of the atoms $\bar{Y}_1 \bar{Y}_2$ and $Y_1 Y_2$ as in either of Figs. 4(a) or 5. We now proceed to solve $\{g_2 = 1\}$ incrementally by observing that the 36 solutions of $\{g_1 = 1\}$ in (26) are also solutions of $\{g_2 = 1\}$, and that the remaining 9 solutions of $\{g_2 = 1\}$ are simply the solutions of $\{g_3 = 1\}$. To solve $\{g_3 = 1\}$, we use an orthonormal tag of (1) for the sole appearance of $Y_1 \bar{Y}_2$ and use the same sets of parameters and tags previously used in Fig. 4(b) to obtain the auxiliary function $G_1(X_1, X_2, X_3; p_1, p_2, p_3, p_4)$. The parametric solutions of $\{g_3 = 1\}$ are now given by

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \bar{Y}_1 \bar{Y}_2 \begin{bmatrix} p_1 \vee \bar{p}_1 p_2 \\ p_1 \vee \bar{p}_1 p_2 \\ p_1 \vee \bar{p}_1 \bar{p}_2 \end{bmatrix} \vee Y_1 Y_2 \begin{bmatrix} 0 \\ p_3 \vee \bar{p}_3 p_4 \\ p_3 \end{bmatrix} \vee Y_1 \bar{Y}_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \vee \begin{bmatrix} d(\bar{Y}_1 Y_2) \\ d(\bar{Y}_1 Y_2) \\ d(\bar{Y}_1 Y_2) \end{bmatrix} \quad (28)$$

Note that the Eq. (28) can be obtained from (5) by simply changing the contribution of atom $Y_1 \bar{Y}_2$ (to represent its new single instance), while keeping all other terms intact. Equation (28) produces the following set of 9 particular solutions for $\{g_3 = 1\}$.

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \bar{Y}_1 \bar{Y}_2 \end{bmatrix} \\ \begin{bmatrix} \bar{Y}_1 \bar{Y}_2 \\ \bar{Y}_1 \bar{Y}_2 \\ 0 \end{bmatrix} \end{bmatrix} \vee \begin{bmatrix} \begin{bmatrix} \bar{Y}_1 \bar{Y}_2 \\ \bar{Y}_1 \bar{Y}_2 \\ \bar{Y}_1 \bar{Y}_2 \end{bmatrix} \\ \begin{bmatrix} \bar{Y}_1 \bar{Y}_2 \\ \bar{Y}_1 \bar{Y}_2 \end{bmatrix} \end{bmatrix} \vee \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ Y_1 Y_2 \\ 0 \end{bmatrix} \end{bmatrix} \vee \begin{bmatrix} \begin{bmatrix} 0 \\ Y_1 Y_2 \\ Y_1 Y_2 \end{bmatrix} \\ \begin{bmatrix} 0 \\ Y_1 Y_2 \end{bmatrix} \end{bmatrix} \vee \begin{bmatrix} \begin{bmatrix} Y_1 \bar{Y}_2 \\ Y_1 \bar{Y}_2 \\ 0 \end{bmatrix} \\ \begin{bmatrix} Y_1 \bar{Y}_2 \\ Y_1 \bar{Y}_2 \end{bmatrix} \end{bmatrix} \vee \begin{bmatrix} \begin{bmatrix} d(\bar{Y}_1 Y_2) \\ d(\bar{Y}_1 Y_2) \\ d(\bar{Y}_1 Y_2) \end{bmatrix} \\ \begin{bmatrix} d(\bar{Y}_1 Y_2) \\ d(\bar{Y}_1 Y_2) \end{bmatrix} \end{bmatrix} \quad (29)$$

The union of the set of 36 particular solutions in (25) and the set of 9 particular solutions in (29) constitutes the set of 45 particular solutions of $\{g_2 = 1\}$. This is in agreement with earlier listings obtained via subsumptive solutions in [15] or via conventional parametric solutions in [20].

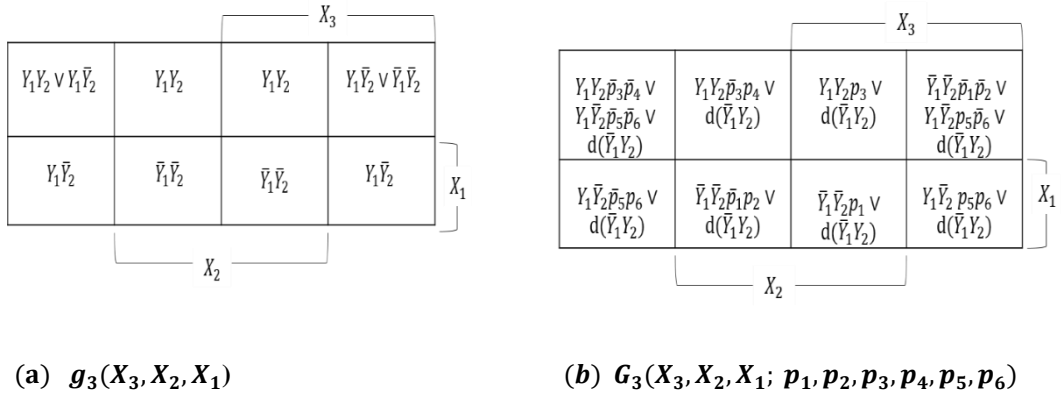


Fig. 4. Natural maps for the function g_1 of Example 2 and its associated auxiliary function G_1

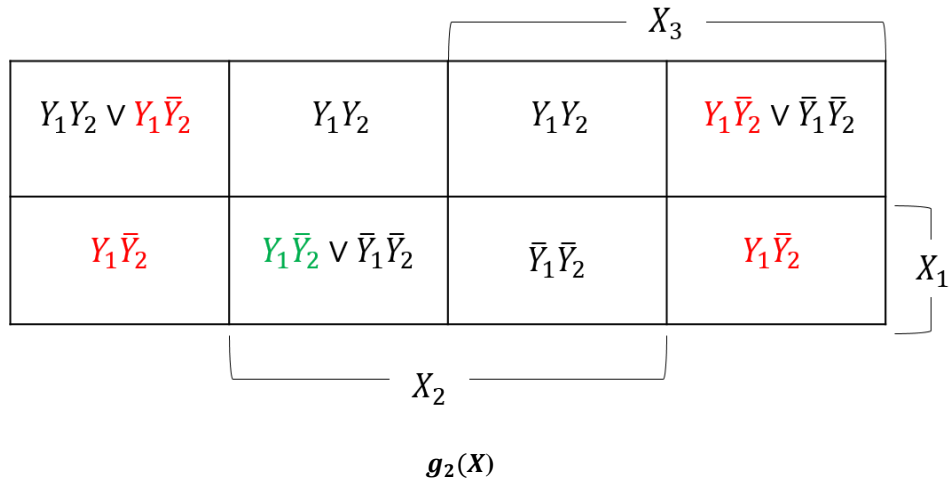


Fig. 5. The natural map for the function g_2 in Example 2.

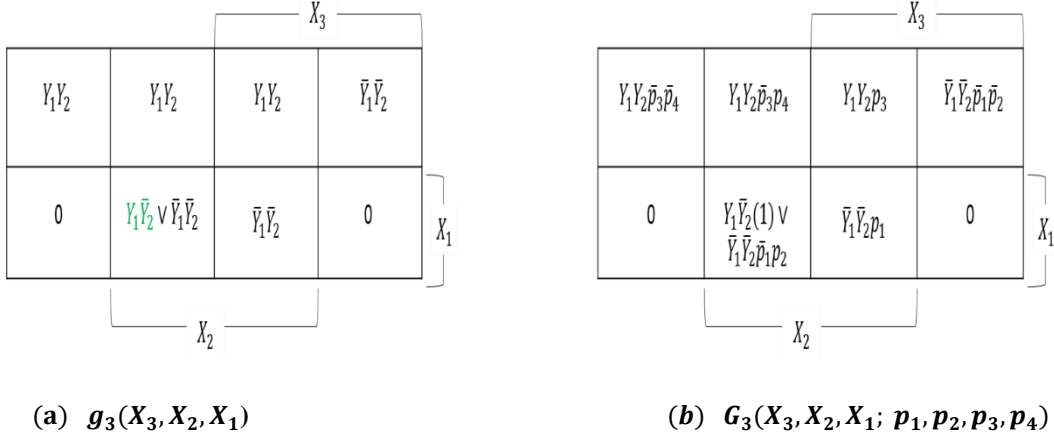


Fig. 6. Natural maps for the function g_3 and its associated auxiliary function G_3 in Example 2

Now, we consider the inverse problem of solving $\{g_1 = 1\}$ to express Y in terms of X . We view g_1 as $g_1 = B_{256}^2 \rightarrow B_{256}$, where $B_{256} = FB(X_1, X_2, X_3)$, and represent g_1 by the natural map in Fig. 7(a). The atoms of B_{256} are $T_0 = \bar{X}_1 \bar{X}_2 \bar{X}_3$, $T_1 = \bar{X}_1 \bar{X}_2 X_3$, $T_2 = \bar{X}_1 X_2 \bar{X}_3$, $T_3 = \bar{X}_1 X_2 X_3$, $T_4 = X_1 \bar{X}_2 \bar{X}_3$, $T_5 = X_1 \bar{X}_2 X_3$, $T_6 = X_1 X_2 \bar{X}_3$, and $T_7 = X_1 X_2 X_3$. Their appearances as entries in the natural map of g_1 in Fig. 7(a) are 2, 2, 1, 1, 1, 1, 2, and 1 respectively. Hence, the consistency condition is the identity $\{0=0\}$ and the number of particular solutions is $2^3 = 8$. The atoms T_0 , T_1 , and T_6 need the two element sets of tags (\bar{p}_0, p_0) , (\bar{p}_1, p_1) and (\bar{p}_6, p_6) , while each of the remaining atoms have a single-element set of tag (1), each. The auxiliary function G_1 is shown in Fig. 7(b), with the parametric solutions given by

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} T_0 \vee T_2 \vee T_3 \vee T_4 \vee T_5 \vee T_1 p_1 \vee T_6 p_6 \\ T_0 p_0 \vee T_2 \vee T_3 \end{bmatrix}, \quad (30)$$

and the set of eight particular solutions is listed by

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \bar{X}_1 \bar{X}_2 \bar{X}_3 \end{bmatrix}}_{p_0} \vee \underbrace{\begin{bmatrix} 0 \\ 0 \\ \bar{X}_1 \bar{X}_2 X_3 \\ 0 \end{bmatrix}}_{p_1} \vee \underbrace{\begin{bmatrix} 0 \\ 0 \\ X_1 X_2 \bar{X}_3 \\ 0 \end{bmatrix}}_{p_6} \vee \begin{bmatrix} \bar{X}_1 \bar{X}_3 \vee \bar{X}_1 X_2 \vee X_1 \\ \bar{X}_1 X_2 \end{bmatrix} \quad (31)$$

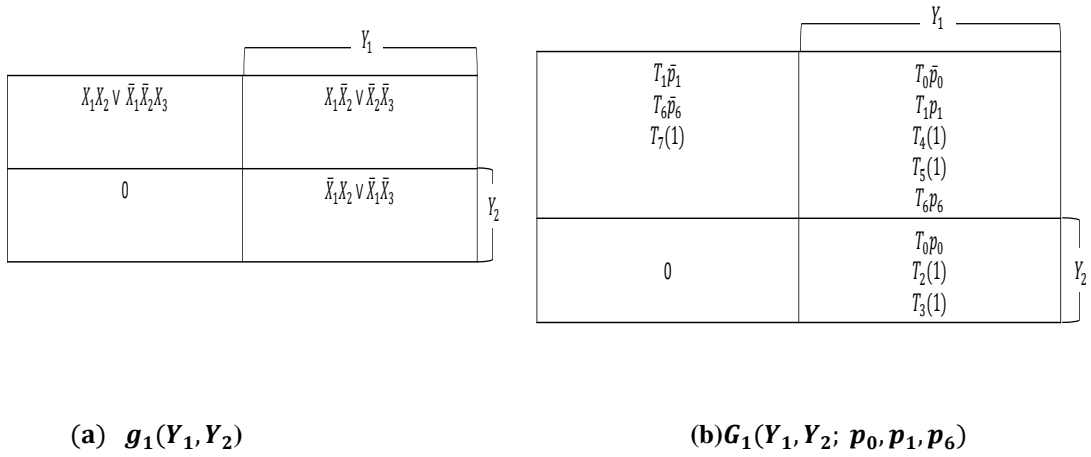


Fig. 7. The natural map for the function g_1 of Example 2, now viewed as a function of Y_1 and Y_2 in terms of X_1, X_2 , and X_3 , together with the associated auxiliary function G_1 .

3.3 Example 3

Suppose we want to solve the system of Boolean equations

$$X \vee Y \vee Z = a \vee b \vee c, \quad (32a)$$

$$\bar{X} \vee \bar{Y} \vee \bar{Z} = \bar{a} \vee \bar{b} \vee \bar{c}. \quad (32b)$$

for the dependent two-valued Boolean variables X, Y , and Z in terms of the independent two-valued Boolean variables a, b , and c . An obvious solution is one that assigns to the ordered set $\{X, Y, Z\}$ the value $\{a, b, c\}$ or any of its 6 permutations. We will find shortly that other solutions exist and that the number of solutions is a huge one, indeed. We now treat the independent variables as generators of the free Boolean algebra $B_{256} = FB(a, b, c)$, which has 256 elements that represent the two-valued Boolean functions of the 3 variables a, b , and c . Clearly, this Boolean algebra has 8 atoms, viz., $T_0 = \bar{a}\bar{b}\bar{c}$, $T_1 = \bar{a}b\bar{c}$, $T_2 = \bar{a}b\bar{c}$, $T_3 = \bar{a}bc$, $T_4 = a\bar{b}\bar{c}$, $T_5 = a\bar{b}c$, $T_6 = ab\bar{c}$, and $T_7 = abc$. The above system of equations reduces to a single equation of the form

$$g_t(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = ((X \vee Y \vee Z) \odot (a \vee b \vee c)) \wedge ((\bar{X} \vee \bar{Y} \vee \bar{Z}) \odot (\bar{a} \vee \bar{b} \vee \bar{c})) = 1, \quad (33)$$

where $g_t(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) : B_{256}^3 \rightarrow B_{256}$. We represent $g_t(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ by its natural map in Fig. 8. The discriminants of $g_t(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ entered in the map cells of Fig. 8 are obtained by noting that

$$0 \odot (a \vee b \vee c) = \overline{(a \vee b \vee c)} = \bar{a}\bar{b}\bar{c}, \text{ for } X = Y = Z = 0, \quad (34a)$$

$$1 \odot (a \vee b \vee c) = (a \vee b \vee c), \text{ otherwise.} \quad (34b)$$

Figure 9 is a replication of Fig. 8 with map entries given as minterm expansions, i.e., as disjunctions of the atoms of the underlying Boolean algebra. We note immediately that no atom failed to appear in Fig. 9, thereby resulting in the consistency condition becoming an identity ($0 = 0$) and leaving the underlying Boolean algebra intact. Each of the two atoms $T_0 = \bar{a}\bar{b}\bar{c}$ and $T_7 = abc$ appears once, while each of the six

atoms $T_1 = \bar{a}\bar{b}c, T_2 = \bar{a}b\bar{c}, T_3 = \bar{a}bc, T_4 = a\bar{b}\bar{c}, T_5 = a\bar{b}c,$ and $T_6 = ab\bar{c}$ appears six times. The total number of particular solutions is given (according to (9) or (10)) by

$$N_t = (1)^2 * (6)^6 = 46656. \quad (35)$$

It is now our task to display this large number of solutions economically. First, we construct the auxiliary function $G(X, Y, Z; \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_6)$, where each of the six vectors of parameters $\mathbf{p}_i = [p_{i1} \ p_{i2} \ p_{i3}]^T (1 \leq i \leq 6)$ is used to produce an orthonormal set of six tags

$$\{p_{i1}\bar{p}_{i2}, p_{i1}p_{i2}, \bar{p}_{i1}p_{i2}p_{i3}, \bar{p}_{i1}\bar{p}_{i2}\bar{p}_{i3}, \bar{p}_{i1}\bar{p}_{i2}p_{i3}, \bar{p}_{i1}p_{i2}\bar{p}_{i3}\}. \quad (36)$$

The solution in vector form is

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} T_7 \vee \bigvee_{i=1}^6 T_i(p_{i1}\bar{p}_{i2} \vee p_{i1}p_{i2} \vee \bar{p}_{i1}p_{i2}p_{i3}) \\ T_7 \vee \bigvee_{i=1}^6 T_i(\bar{p}_{i1}\bar{p}_{i2}p_{i3} \vee \bar{p}_{i1}p_{i2}\bar{p}_{i3} \vee \bar{p}_{i1}p_{i2}p_{i3}) \\ T_7 \vee \bigvee_{i=1}^6 T_i(p_{i1}p_{i2} \vee \bar{p}_{i1}\bar{p}_{i2}p_{i3} \vee \bar{p}_{i1}p_{i2}\bar{p}_{i3}) \end{bmatrix} \quad (37)$$

The following equation (38) displays all 46656 particular solutions of the system (25) as a disjunction of possible contributions of atoms $T_i (1 \leq i \leq 7)$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \bigvee_{i=1}^6 \begin{array}{|c|c|c|c|} \hline & & \mathbf{p}_{i1} & \\ \hline \begin{array}{|c|} \hline \begin{bmatrix} 0 \\ 0 \\ T_i \end{bmatrix} \\ \hline \end{array} & \begin{array}{|c|} \hline \begin{bmatrix} 0 \\ T_i \\ 0 \end{bmatrix} \\ \hline \end{array} & \begin{array}{|c|} \hline \\ \hline \end{array} & \begin{array}{|c|} \hline \\ \hline \end{array} \\ \hline \mathbf{p}_{i3} \left[\begin{array}{|c|} \hline \begin{bmatrix} 0 \\ T_i \\ T_i \end{bmatrix} \\ \hline \end{array} \right. & \left. \begin{array}{|c|} \hline \begin{bmatrix} T_i \\ T_i \\ 0 \end{bmatrix} \\ \hline \end{array} \right] & \begin{array}{|c|} \hline \begin{bmatrix} T_i \\ 0 \\ T_i \end{bmatrix} \\ \hline \end{array} & \begin{array}{|c|} \hline \begin{bmatrix} T_i \\ 0 \\ 0 \end{bmatrix} \\ \hline \end{array} \\ \hline & & \mathbf{p}_{i2} & \\ \hline \end{array} \vee \begin{array}{|c|} \hline \begin{bmatrix} T_7 \\ T_7 \\ T_7 \end{bmatrix} \\ \hline \end{array} \quad (38)$$

Table 1 demonstrates a few samples of the particular solutions that can be obtained via (38). The first of these is the obvious solution.

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \quad (39)$$

Each of the solutions in Table 1 can be easily shown to satisfy each of the separate equations (32a) and (32b), or equivalently to satisfy the combined equation (33). The fact that each of these equations is invariant to a permutation of $\{a, b, c\}$ or of $\{X, Y, Z\}$ is reflected on a similar invariance for the set of particular solutions in equations (37) or (38). This means that if a permutation of $\{a, b, c\}$ or a permutation of

$\{X, Y, Z\}$ is applied to any particular solution, then it remains a particular solution (albeit possibly a different one).

In passing, we note that the number of particular solutions of each of the individual equations (32a) and (32b) is given by

$$n = (1)^2 * (7)^7 = 823543, \quad i = 1, 2, \quad (40)$$

i.e., it is about 17.65 times the number of solutions of the system of equations (38). The following equation (41) displays all 823543 particular solutions of the equation (32a) as a disjunction of possible contributions of atoms T_i ($1 \leq i \leq 7$).

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \vee_{i=1}^7 \begin{array}{|c|c|c|c|} \hline & & p_{i1} & \\ \hline \begin{bmatrix} 0 \\ 0 \\ T_i \end{bmatrix} & \begin{bmatrix} 0 \\ T_i \\ 0 \end{bmatrix} & & \begin{bmatrix} T_i \\ 0 \\ 0 \end{bmatrix} \\ \hline p_{i3} & & \begin{bmatrix} T_i \\ 0 \\ T_i \end{bmatrix} & \begin{bmatrix} T_i \\ T_i \\ T_i \end{bmatrix} \\ \hline & p_{i2} & & \\ \hline \end{array} \quad (41)$$

Similarly, The following equation (42) displays all 823543 particular solutions of the system (32b) as a disjunction of possible contributions of atoms T_i ($1 \leq i \leq 7$).

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \vee_{i=0}^6 \begin{array}{|c|c|c|c|} \hline & & p_{i1} & \\ \hline \begin{bmatrix} 0 \\ 0 \\ T_i \end{bmatrix} & \begin{bmatrix} 0 \\ T_i \\ 0 \end{bmatrix} & & \begin{bmatrix} T_i \\ 0 \\ 0 \end{bmatrix} \\ \hline p_{i3} & & \begin{bmatrix} T_i \\ 0 \\ T_i \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \hline & p_{i2} & & \\ \hline \end{array} \vee \begin{array}{|c|} \hline \begin{bmatrix} T_7 \\ T_7 \\ T_7 \end{bmatrix} \\ \hline \end{array} \quad (42)$$

The intersection of the set of solutions in Equation (41) with the set of solutions in Equation (42) is the set of solutions in Equation (38). In fact, the set of solutions in Equation (38) do not contain a contribution of T_0 and must contain a contribution (to each variable) of T_7 . This means that this set is the intersection of the $(6)^7 = 117649$ solutions in Fig. 6 that has a contribution of T_7 of the form $[T_7 \ T_7 \ T_7]^T$, with the $(6)^7$ solutions in Equation (42) that have zero contribution of T_0 of the form $[0 \ 0 \ 0]^T$. When we add the restriction that solutions of the form $[T_i \ T_i \ T_i]$ for $1 \leq i \leq 6$ in Equation (41) should be excluded, we retain only the 46656 solutions in Equation (38). Likewise when we add the restriction that no contribution of T_i for $(1 \leq i \leq 6)$ in Equation (42) should be of the form $[0 \ 0 \ 0]^T$, we again retain only the 46656 solutions in Equation (38). Since the two sets of solutions in Equation (41) and (42) are now restricted to match the set in Equation (38), their intersection is also equal to this set.

Before closing this example, we include its general subsumptive solution, obtained in the most compact form via the don't-care method [13, 22]. This solution includes the consistency condition ($0 = 0$) together with the subsumptive relations

$$abc \leq Z \leq (a \vee b \vee c), \quad (43a)$$

$$abc \leq Y \leq (a \vee b \vee c), \quad (43b)$$

$$abc \vee (a \vee b \vee c)\bar{Y}\bar{Z} \leq X \leq abc \vee (a \vee b \vee c)(\bar{Y} \vee \bar{Z}). \quad (43c)$$

which might be rewritten in terms of atoms $T_i (0 \leq i \leq 7)$ as follows

$$T_7 \leq Z \leq (\bigvee_{i=1}^7 T_i), \quad (44a)$$

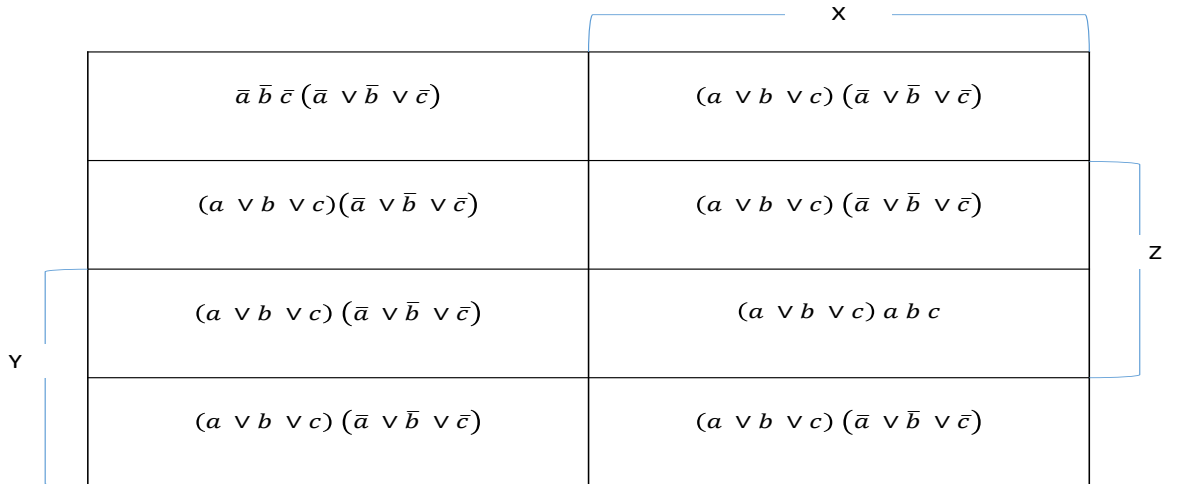
$$T_7 \leq Y \leq (\bigvee_{i=1}^7 T_i), \quad (44b)$$

$$T_7 \vee (\bigvee_{i=1}^6 T_i)\bar{Y}\bar{Z} \leq X \leq T_7 \vee (\bigvee_{i=1}^6 T_i)(\bar{Y} \vee \bar{Z}). \quad (44c)$$

A straightforward (albeit lengthy) analysis shows that (44) is in exact agreement with Equation (38).

Table 1. Samples of the Particular Solutions of Example 1

Selected total contribution of atom T_i ($0 \leq i \leq 7$)								Particular solutions
$T_0 = \bar{a} \bar{b} \bar{c}$	$T_1 = \bar{a} \bar{b} c$	$T_2 = \bar{a} b \bar{c}$	$T_3 = \bar{a} b c$	$T_4 = a \bar{b} \bar{c}$	$T_5 = a \bar{b} c$	$T_6 = a b \bar{c}$	$T_7 = a b c$	
$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ T_1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ T_2 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ T_3 \\ T_3 \end{bmatrix}$	$\begin{bmatrix} T_4 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} T_5 \\ 0 \\ T_5 \end{bmatrix}$	$\begin{bmatrix} T_6 \\ T_6 \\ 0 \end{bmatrix}$	$\begin{bmatrix} T_7 \\ T_7 \\ T_7 \end{bmatrix}$	$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$
$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ T_1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ T_2 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ T_3 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ T_4 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ T_5 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ T_6 \end{bmatrix}$	$\begin{bmatrix} T_7 \\ T_7 \\ T_7 \end{bmatrix}$	$\begin{bmatrix} abc \\ abc \\ a \vee b \vee c \end{bmatrix}$
$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ T_1 \\ T_1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ T_2 \\ T_2 \end{bmatrix}$	$\begin{bmatrix} 0 \\ T_3 \\ T_3 \end{bmatrix}$	$\begin{bmatrix} 0 \\ T_4 \\ T_4 \end{bmatrix}$	$\begin{bmatrix} 0 \\ T_5 \\ T_5 \end{bmatrix}$	$\begin{bmatrix} 0 \\ T_6 \\ T_6 \end{bmatrix}$	$\begin{bmatrix} T_7 \\ T_7 \\ T_7 \end{bmatrix}$	$\begin{bmatrix} abc \\ a \vee b \vee c \\ a \vee b \vee c \end{bmatrix}$
$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} T_1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ T_2 \\ 0 \end{bmatrix}$	$\begin{bmatrix} T_3 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ T_4 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ T_5 \end{bmatrix}$	$\begin{bmatrix} 0 \\ T_6 \\ 0 \end{bmatrix}$	$\begin{bmatrix} T_7 \\ T_7 \\ T_7 \end{bmatrix}$	$\begin{bmatrix} \bar{a}c \vee bc \\ b\bar{c} \vee ab \\ a\bar{b} \vee ac \end{bmatrix}$
$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} T_1 \\ T_1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} T_2 \\ 0 \\ T_2 \end{bmatrix}$	$\begin{bmatrix} T_3 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ T_4 \\ T_4 \end{bmatrix}$	$\begin{bmatrix} 0 \\ T_5 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ T_6 \end{bmatrix}$	$\begin{bmatrix} T_7 \\ T_7 \\ T_7 \end{bmatrix}$	$\begin{bmatrix} \bar{a}b \vee \bar{a}c \vee bc \\ \bar{a}\bar{b} \vee \bar{b}c \vee ac \\ \bar{a}\bar{c} \vee b\bar{c} \vee ab \end{bmatrix}$
$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ T_1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ T_2 \end{bmatrix}$	$\begin{bmatrix} T_3 \\ T_3 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ T_4 \end{bmatrix}$	$\begin{bmatrix} T_5 \\ T_5 \\ 0 \end{bmatrix}$	$\begin{bmatrix} T_6 \\ T_6 \\ 0 \end{bmatrix}$	$\begin{bmatrix} T_7 \\ T_7 \\ T_7 \end{bmatrix}$	$\begin{bmatrix} ab \vee bc \vee ac \\ ab \vee bc \vee ac \\ a \oplus b \oplus c \end{bmatrix}$
$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} T_1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} T_2 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} T_3 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ T_4 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ T_5 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ T_6 \end{bmatrix}$	$\begin{bmatrix} T_7 \\ T_7 \\ T_7 \end{bmatrix}$	$\begin{bmatrix} \bar{a}b \vee \bar{a}c \vee bc \\ abc \\ a \end{bmatrix}$
$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} T_1 \\ T_1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} T_2 \\ T_2 \\ 0 \end{bmatrix}$	$\begin{bmatrix} T_3 \\ T_3 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ T_4 \\ T_4 \end{bmatrix}$	$\begin{bmatrix} 0 \\ T_5 \\ T_5 \end{bmatrix}$	$\begin{bmatrix} 0 \\ T_6 \\ T_6 \end{bmatrix}$	$\begin{bmatrix} T_7 \\ T_7 \\ T_7 \end{bmatrix}$	$\begin{bmatrix} \bar{a}b \vee \bar{a}c \vee bc \\ a \vee b \vee c \\ a \end{bmatrix}$



$$g_t(X, Y, Z)$$

Fig. 8. The natural map for $g_t(X, Y, Z)$ in (33)

		x	
	$\bar{a} \bar{b} \bar{c}$		$\bar{a} \bar{b} c \vee \bar{a} b \bar{c} \vee \bar{a} b c \vee a \bar{b} \bar{c} \vee a \bar{b} c \vee a b \bar{c}$
	$\bar{a} \bar{b} c \vee \bar{a} b \bar{c} \vee \bar{a} b c \vee a \bar{b} \bar{c} \vee a \bar{b} c \vee a b \bar{c}$		$\bar{a} \bar{b} c \vee \bar{a} b \bar{c} \vee \bar{a} b c \vee a \bar{b} \bar{c} \vee a \bar{b} c \vee a b \bar{c}$
y	$\bar{a} \bar{b} c \vee \bar{a} b \bar{c} \vee \bar{a} b c \vee a \bar{b} \bar{c} \vee a \bar{b} c \vee a b \bar{c}$		$a b c$
	$\bar{a} \bar{b} c \vee \bar{a} b \bar{c} \vee \bar{a} b c \vee a \bar{b} \bar{c} \vee a \bar{b} c \vee a b \bar{c}$		$\bar{a} \bar{b} c \vee \bar{a} b \bar{c} \vee \bar{a} b c \vee a \bar{b} \bar{c} \vee a \bar{b} c \vee a b \bar{c}$
			z

$$g_t(X, Y, Z)$$

Fig. 9. The map in Fig. 8 with its entries given as minterm expansions (disjunctions of the atoms of the underlying Boolean algebra)

		x	
	$T_0(1)$		$\bigvee_{i=1}^6 T_i(p_{i1} \bar{p}_{i2})$
	$\bigvee_{i=1}^6 T_i(\bar{p}_{i1} \bar{p}_{i2} \bar{p}_{i3})$		$\bigvee_{i=1}^6 T_i(p_{i1} p_{i2})$
y	$\bigvee_{i=1}^6 T_i(\bar{p}_{i1} \bar{p}_{i2} p_{i3})$		$T_7(1)$
	$\bigvee_{i=1}^6 T_i(\bar{p}_{i1} p_{i2} \bar{p}_{i3})$		$\bigvee_{i=1}^6 T_i(\bar{p}_{i1} p_{i2} p_{i3})$
			z

$$G_t(X, Y, Z; p_1, p_2, p_3, p_4, p_5, p_6)$$

Fig. 10. The auxiliary function for Example 3

References

- [1] Banković, D. Some remarks on number of parameters of the solutions of Boolean equations, *Discrete Mathematics*. 1990;79(3): 229-234.
- [2] Brown, F.M.Reduced solutions of Boolean equations, *IEEE Transactions on Computers*. 1970;C-19: 976-981.
- [3] Brown, F.M. Segmental solutions of Boolean equations, *Discrete Applied Mathematics*. 1982;4: 87-96.
- [4] Brown, F. M. *Boolean Reasoning: The Logic of Boolean Equations*, Kluwer Academic Publishers, Boston, USA; 1990.
- [5] Brown, F. M.*Boolean Reasoning: The Logic of Boolean Equations*, 2nd Ed., Dover Publications, Mineola, NY, USA; 2003.
- [6] Deschamps, J.-P. Parametric solutions of Boolean equations, *Discrete Mathematics*. 1972; 3 (4): 333-342.
- [7] Leont'ev, V. K., and G. P. Tonoyan. On systems of Boolean equations, *Computational Mathematics and Mathematical Physics*. 2013; 53(5): 632-639.

- [8] Levchenkov, V. S. Boolean equations with many unknowns, *Computational Mathematics and Modeling*. 2000;11 (2): 143-153.
- [9] Levchenkov, V. S. Solution of equations in Boolean algebra, *Computational Mathematics and Modeling*, 2000;11 (2): 154-163.
- [10] Rudeanu, S. *Boolean Functions and Equations*, North-Holland Publishing Company & American Elsevier, Amsterdam, the Netherlands; 1974.
- [11] Rudeanu, S. *Lattice Functions and Equations*, Springer Verlag, London, UK; 2001.
- [12] Rudeanu, S. Algebraic methods versus map methods of solving Boolean equations, *International Journal of Computer Mathematics*. 2003;80 (7): 815-817.