Question 445 : Catalan's Constant and Some Integration Questions

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abstract

This note presents some formulas for Catalan's constant.

1. Introduction. Catalan's constant is defined by

$$G=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\ldots=0.915965594\ldots \tag{1}$$

2. Integral Identities

$$G=\int_1^{\infty}\frac{\ln x}{1+x^2}\,dx \tag{2}$$

$$G=\int_0^{\infty}\tan^{-1}e^{-x}\,dx \tag{3}$$

$$G=\frac{1}{2}\int_0^{\infty}\frac{x}{\cosh x}\,dx \tag{4}$$

$$G=\int_0^{1}\frac{\tan^{-1}x}{x}\,dx \tag{5}$$

3. Three Formulas

$$G=2\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_1^{\infty}\left(\frac{x^{\left(1+x^2\right)^{-1}}-1}{x^{\left(1+x^2\right)^{-1}}+1}\right)^{2n+1}dx \tag{6}$$

$$G=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_1^{\infty}\left(x^{\left(1+x^2\right)^{-1}}-1\right)^{n}dx \tag{7}$$

$$G=\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\left(1-x^{-\left(1+x^2\right)^{-1}}\right)^{n}dx \tag{8}$$

4. The Functions $F(n)$, $f(x,n)$, $u(x)$.

$$u(x)=\frac{x^{(1+x^2)^{-1}}-1}{x^{(1+x^2)^{-1}}+1} \quad x\ge 1 \tag{9}$$

$$f(x,n)=\left(u(x)\right)^{2n+1} \quad ,x\ge 1, n\in\mathbb{N}\cup\{0\} \tag{10}$$

$$F(n)=\int_1^\infty f(x,n)\,dx \quad ,n\in\mathbb{N}\cup\{0\} \tag{11}$$

$$0\le u(x)\le u_m=\max_{1\le x\le\infty} u(x)=0.069503...,(x=1.895025...) \tag{12}$$

$$F(n)=\int_1^\infty f(x,n)\,dx\le u_m^{2n}\int_1^\infty u(x)\,dx=u_m^{2n}F(0) \quad ,n\in\mathbb{N}\cup\{0\} \tag{13}$$

$$F(0)=\int_1^\infty u(x)\,dx=0.457763... \tag{14}$$

5. The Functions $K(n),k(x,n),v(x)$.

$$v(x)=x^{(1+x^2)^{-1}}-1 \quad ,x\ge 1 \tag{15}$$

$$k(x,n)=\left(v(x)\right)^{n} \quad ,x\ge 1, n\in\mathbb{N} \tag{16}$$

$$K(n)=\int_1^\infty k(x,n)\,dx \quad ,n\in\mathbb{N} \tag{17}$$

$$0\le v(x)\le v_m=\max_{1\le x\le\infty} v(x)=0.149391...,(x=1.895025...) \tag{18}$$

$$K(n+1)\le v_m^{n}\int_1^\infty v(x)\,dx=v_m^{n}K(1) \quad ,n\in\mathbb{N} \tag{19}$$

$$K(1)=\int_1^\infty v(x)\,dx=0.943358... \tag{20}$$

6. The Functions $H(n),h(x,n),w(x)$.

$$w(x)=1-x^{-(1+x^2)^{-1}} \quad ,x\ge 1 \tag{21}$$

$$h(x,n)=\left(w(x)\right)^n \quad ,x\geq 1,n\in\mathbb{N} \tag{22}$$

$$H(n)=\int_1^\infty h(x,n)\,dx \quad ,n\in\mathbb{N} \tag{23}$$

$$0\leq w(x)\leq w_m=\max_{1\leq x\leq\infty} w(x)=0.129974...,(x=1.895025...) \tag{24}$$

$$H(n+1)\leq w_m^n\int_1^\infty w(x)\,dx=w_m^n\,H(1) \quad ,n\in\mathbb{N} \tag{25}$$

$$H(1)=\int_1^\infty w(x)\,dx=0.890327... \tag{26}$$

7. Generalizations

$$G=2k^2\sum_{n=0}^\infty\frac{1}{2n+1}\int_1^\infty\left(\frac{x^{x^{k-1}\left(1+x^{2k}\right)^{-1}}-1}{x^{x^{k-1}\left(1+x^{2k}\right)^{-1}}+1}\right)^{2n+1}dx \quad ,k>0 \tag{27}$$

$$G=k^2\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n}\int_1^\infty\left(x^{x^{k-1}\left(1+x^{2k}\right)^{-1}}-1\right)^n dx \quad ,k>0 \tag{28}$$

$$G=k^2\sum_{n=1}^\infty\frac{1}{n}\int_1^\infty\left(1-x^{-x^{k-1}\left(1+x^{2k}\right)^{-1}}\right)^n dx \quad ,k>0 \tag{29}$$

References

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