Question 446 : π as sum of arctangents

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abstract

This note presents some formulas of pi in terms of sum of arctangents.

1. Introduction.

Examples of π as sum of arctangents (Machin's formulas)

$$\frac{\pi}{2} = \tan^{-1}\left(\left(\tan\frac{2\pi}{5}\right)^2\right) + \tan^{-1}\left(\left(\tan\frac{4\pi}{5}\right)^2\right) - \tan^{-1}\frac{2}{5} \tag{1}$$

$$\pi = \tan^{-1}\left(\left(\tan\frac{2\pi}{7}\right)^2\right) + \tan^{-1}\left(\left(\tan\frac{4\pi}{7}\right)^2\right) + \tan^{-1}\left(\left(\tan\frac{6\pi}{7}\right)^2\right) + \tan^{-1}\frac{7}{17} \tag{2}$$

$$\begin{aligned}\pi = {} & \tan^{-1}\left(\left(\tan\frac{2\pi}{9}\right)^2\right) + \tan^{-1}\left(\left(\tan\frac{4\pi}{9}\right)^2\right) \\ & + \tan^{-1}\left(\left(\tan\frac{6\pi}{9}\right)^2\right) + \tan^{-1}\left(\left(\tan\frac{8\pi}{9}\right)^2\right) - \tan^{-1}\frac{12}{29}\end{aligned} \tag{3}$$

2. General formula

Let $n \in \mathbb{N} = \{1,2,3,4,\ldots\}$, then

$$S_n = \sum_{k=1}^{n} \tan^{-1}\left(\left(\tan\frac{2k\pi}{2n+1}\right)^2\right) = \left[\frac{n+1}{2}\right]\frac{\pi}{2} + (-1)^n \tan^{-1}\left(\frac{\min\{|x_n|,|y_n|\}}{\max\{|x_n|,|y_n|\}}\right) \tag{4}$$

where

$$x_n = \mathrm{Re}(Z_n) \quad , \quad y_n = \mathrm{Im}(Z_n) \tag{5}$$

$$Z_n = \frac{1}{2}\left(\left(1+\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^{2n+1}+\left(1-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^{2n+1}\right) \quad ,i=\sqrt{-1} \tag{6}$$

Remark 1: $\left[x\right]$ is the integer part of x , $\left|x\right|$ is the absolute value of x .

Remark 2: $\mathrm{Re}\left(z\right),\mathrm{Im}\left(z\right)$ are the real and imaginary part of z .

3. Particular examples

$$S_5 = \frac{3\pi}{2} - \tan^{-1}\frac{41}{99} \tag{7}$$

$$S_6 = \frac{3\pi}{2} + \tan^{-1}\frac{70}{169} \tag{8}$$

$$S_7 = 2\pi - \tan^{-1}\frac{239}{577} \tag{9}$$

$$S_8 = 2\pi + \tan^{-1}\frac{408}{985} \tag{10}$$

$$S_9 = \frac{5\pi}{2} - \tan^{-1}\frac{1393}{3363} \tag{11}$$

$$S_{10} = \frac{5\pi}{2} + \tan^{-1}\frac{2378}{5741} \tag{12}$$

$$S_{11} = 3\pi - \tan^{-1}\frac{8119}{19601} \tag{13}$$

$$S_{12} = 3\pi + \tan^{-1}\frac{13860}{33461} \tag{14}$$

4. The sequence $Z_n = x_n + i\,y_n$

$$Z_{n+2} = 2\left(1+i\right)Z_{n+1} + 2iZ_n \quad ,Z_1 = 1+3i, Z_2 = -4+10i, n\in\mathbb{N} \tag{15}$$

5. The sequences x_n, y_n

$$\begin{cases} x_{n+2} = 2x_{n+1} - 2y_{n+1} - 2y_n \\ y_{n+2} = 2x_{n+1} + 2y_{n+1} + 2x_n \end{cases} \quad ,x_1 = 1, x_2 = -4, y_1 = 3, y_2 = 10, n\in\mathbb{N} \tag{16}$$

$$x_{n+4} = 4x_{n+3} - 8x_{n+2} - 8x_{n+1} - 4x_n \quad , x_1 = 1, x_2 = -4, x_3 = -34, x_4 = -116 \tag{17}$$

$$y_{n+4} = 4y_{n+3} - 8y_{n+2} - 8y_{n+1} - 4y_n \quad , y_1 = 3, y_2 = 10, y_3 = 14, y_4 = -48 \tag{18}$$

$$x_n = \frac{1}{4}\left(\left(1+\frac{1+i}{\sqrt{2}}\right)^{2n+1} + \left(1+\frac{1-i}{\sqrt{2}}\right)^{2n+1} + \left(1-\frac{1+i}{\sqrt{2}}\right)^{2n+1} + \left(1-\frac{1-i}{\sqrt{2}}\right)^{2n+1}\right) \quad , n \in \mathbb{N} \tag{19}$$

$$y_n = \frac{1}{4i}\left(\left(1+\frac{1+i}{\sqrt{2}}\right)^{2n+1} - \left(1+\frac{1-i}{\sqrt{2}}\right)^{2n+1} + \left(1-\frac{1+i}{\sqrt{2}}\right)^{2n+1} - \left(1-\frac{1-i}{\sqrt{2}}\right)^{2n+1}\right) \quad , n \in \mathbb{N} \tag{20}$$

6. The sequences Z_n, x_n, y_n

$$Z_n = \sum_{k=0}^{n} \binom{2n+1}{2k} i^k \quad , n \in \mathbb{N} \tag{21}$$

$$x_n = \sum_{k=0}^{[n/2]} \binom{2n+1}{4k} (-1)^k \quad , n \in \mathbb{N} \tag{22}$$

$$y_n = \sum_{k=0}^{[(n-1)/2]} \binom{2n+1}{4k+2} (-1)^k \quad , n \in \mathbb{N} \tag{23}$$

References

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