Effect of Magnet Geometry on the Magnetic Component of the Lorentz Force Equation

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Abstract

All forces in the universe are created from changes in energy levels that result from changes in the separation of bodies, whether electromagnetic or otherwise. For example, two electrons separated by a finite distance will experience repulsive forces that urge them to separate further. When they are free to move this will cause them to accelerate away from each other, replacing part of the original energy levels with kinetic energy. Even when they are constrained so that they cannot move they still experience those same forces, but there is no energy exchange. Either way, for the force to exist the energy system must exist, even if only potentially. Now consider the second part of the Lorentz Force Equation, which looks at the forces experienced by an electron travelling through a fixed magnetic field. Here there is a lateral force on the electron normal to the direction of travel, and the electron's path is deflected into a curve, with no change in energy levels. However, the existence of the force requires an energy mechanism and in this paper I set out to identify it. There are enough clues to reach a sound conclusion, such as the fact that a neutron, with a bounded electric field, is not deflected, whereas an electron, with an infinite electric field, *is* deflected. With the energy mechanism clearly defined, we find that the Lorentz Force Equation fails to take in an important aspect of geometry and hence if we use electron deflection measurement and this Equation as a means of determining magnetic field strength we will virtually always calculate a field strength that is lower than the actuality.

The appendix gives the basic field interactions and point energy density equations. These are no longer included in some textbooks so can take time to locate.

Charges moving in Magnetic fields

Two conundrums

In this paper we consider the second component of the Lorentz Force Equation, which describes the forces on a charge moving through a magnetic field. It affects how electrons move in magnetic fields and becomes important in more complex atoms. It also affects how we measure the strength of magnetic fields.

It states that...

$$\mathbf{F} = \mathrm{q}(\mathbf{v}x\mathbf{B})$$

Force is therefore equal to charge times the cross product of velocity and magnetic field. How is this equation derived? - is it precise, or are there conditions under which it fails to describe the true situation? This is really a question of how the electron's distributed field reacts with the overall magnetic field, rather than simply analyse a point interaction. We can analyse it from first principles by using the integrated potential energy density in the same way as we have done for the neutron-neutron interaction. All we have to do is place ourselves in the rest frame of the moving electron, and instead of using the magnetic field directly, use the electric field induced by that motion through the magnetic field. Then we can use the same technique as we used for the neutron. So when an electron passes through a square magnetic field we could compute the induced electric field, then we could calculate the potential energy density for each point in the interaction between the electron and the induced field, finally integrating the result to get the total potential energy. We could differentiate this energy with respect to the potential movement along the induced electric field to derive the forces involved. However, there is a much simpler way to determine these forces without the need to compute the whole potential energy first and then take its derivative.

We have two conundrums to solve:

1. We know that the neutron has a much stronger electric field than the electron even though it is bounded, yet is not deflected inside a magnetic field, so the magnetic component of the Lorentz Force Equation seems to work with electrons, but not with neutrons. Our theory needs to show why this anomalous behaviour happens.
2. The Lorentz Force Equation suggests that if we replace the magnetic field with an electric field (in order that the electric field always pointed in the same direction rather than remaining normal to **B**x**v** and thus causing the electron to follow a curved path) an infinite number of electrons could traverse the field, increasing their kinetic energy inside the field, without any expenditure of energy anywhere in the system; when they left the field with their augmented kinetic energy they would have gained something for nothing; this violates the Principle of Conservation of Energy. Again, our replacement theory must demonstrate there is no such violation. While this is a thought experiment rather than a practical one it nevertheless highlights an important issue.

Identifying the energy system involved

Some energy systems must drive every force in the universe. So let us start with a simple square magnetic field with the electron travelling in the field, parallel to one of the sides, as shown in plan in Figure 1...

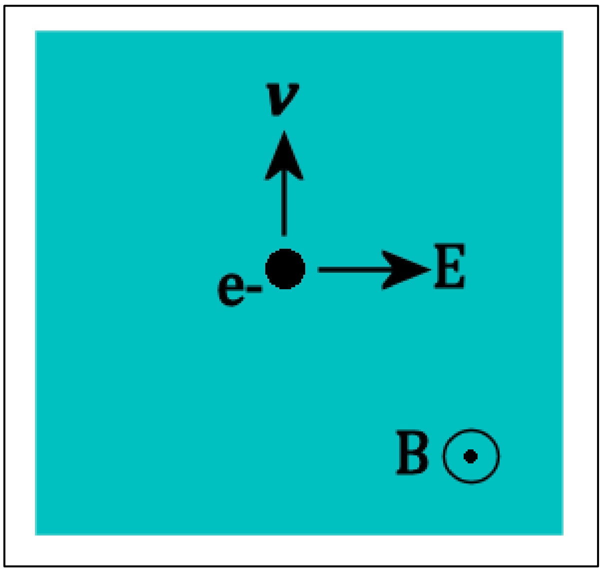

Figure 1

Figure 1 shows a magnetic field **B** in blue with the magnetic field vector pointing up out of the page. A test electron 'e-' is travelling through this field with a velocity ***v***, and perceives an electric field **E** induced in its own frame of reference. The electron will perceive forces that direct it towards the left in this picture. These forces must be derived from a reduction in energy somewhere in the system that creates attractive forces to the left via the equation F=dE/dl (Force F equals rate of change of Energy E by distance l) and/or repulsive forces from the right that will have the same effect. The fact that energy is not consumed but the electron merely deflected into a curved path by these forces does not change this argument. So where is the energy system that drives this deflection?

Although Figure 1 refers to a simple slice through the magnetic field, it is important not to make the mistake of analysing only one slice. Looking at the magnet side-on in Figure 2 shows how the field extends in three dimensions...

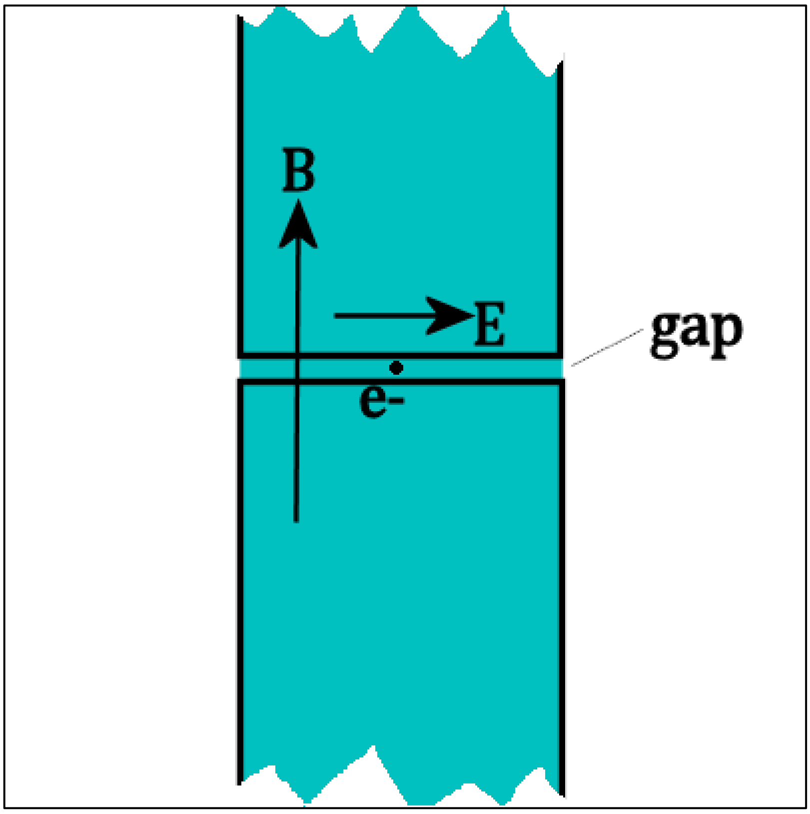

Figure 2

As before, the magnetic field is shown in blue and its vector by **B**, and the induced electric field by **E**. The electron is shown travelling into the page inside the gap between the two poles of the magnet. As can be seen, the magnetic field, and therefore the induced electric field, continues through the poles of the magnet. The moving electron interacts with the whole field, including that inside the magnet poles. This is because the electron's electric field penetrates even into the atoms of the poles. Hence the computation must be over all interacting space, not just the gap between the poles.

In Figure 3 we show the interaction between the electron's field and the induced electric field. The electron's field is shown in blue and the induced field is shown in red. The cutaways show only the effect on the horizontal component and the resultant electric field vector is shown in black.

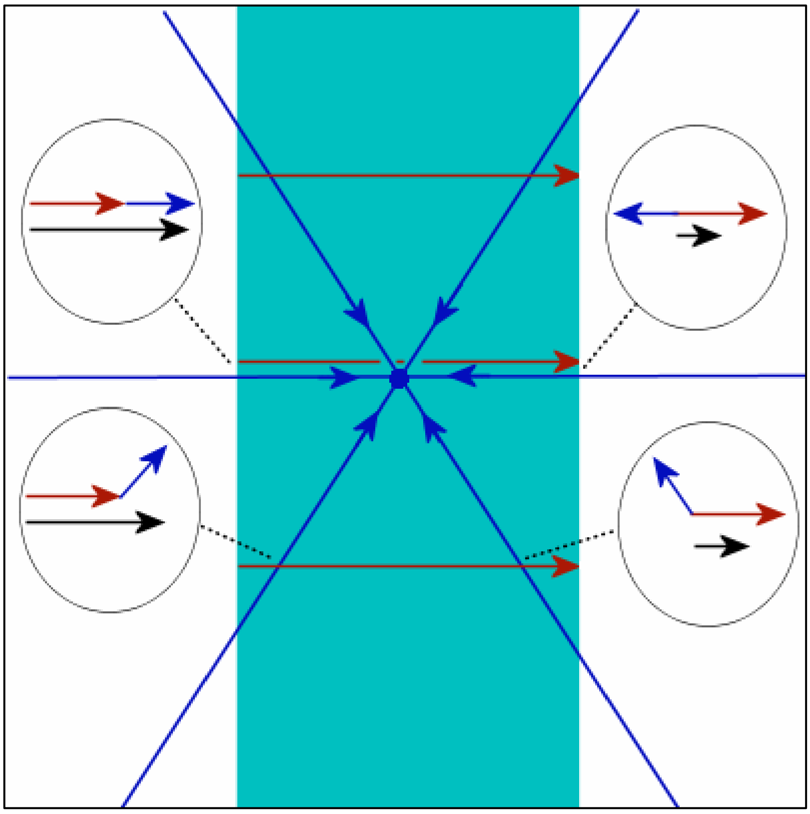

Figure 3

The energy system that drives the deflection of the electron inside the magnet is in fact the interaction of the electron's fields with the edges of the magnet. If the electron moves a little to the left those parts of the fields of the electron that remain *inside* the magnetic field see no change in their energy density as the induced electric field is constant everywhere inside the magnet, so those parts of the field produce no force. However, those horizontal components of the electron's field that enter at the right edge of the magnet as a result of this movement will go from their normal field to a partial cancellation with the magnet's induced electric field and hence to a reduced energy density, while those horizontal components leaving the magnet on the left will go from the increased energy density of interaction within the magnet to their normal energy density outside the magnet. Hence there is a drop in energy density at both edges leading to (from **F**=dU/d**l**) forces that tend to force the electron to the left.

So how does this theory compare with the Lorentz Force Equation?

Now for the theory – how does this compare with the Lorentz Force Equation? Let us use the following co-ordinate system...

1. The x-axis is parallel with the lines of the induced electric field, that is, to the right in Figures 2 and 3.
2. The y-axis is parallel with the lines of the magnetic flux, that is, vertically in Figure 2.
3. The z-axis is parallel with the electron's velocity vector, that is, vertically in Figure 3.

The whole of the induced electric field from the magnet is constant, and normal to the direction of motion. Then there is no component of the induced field in any other axis and the potential energy lies along only the x-axis component field of the electron. Hence all forces will be along the x-axis.

Next, if we consider that the induced electric field is constant within the magnet and zero elsewhere, then for any point on the electron's field the potential energy density will be constant inside the induced electric field and zero outside. There are changes in potential energy density *only* on the external boundary of the induced field.

We need therefore consider only the potential energy associated with a plane of infinitesimal thickness at the induced-field boundary plane normal to the x-axis as shown in Figure 4.

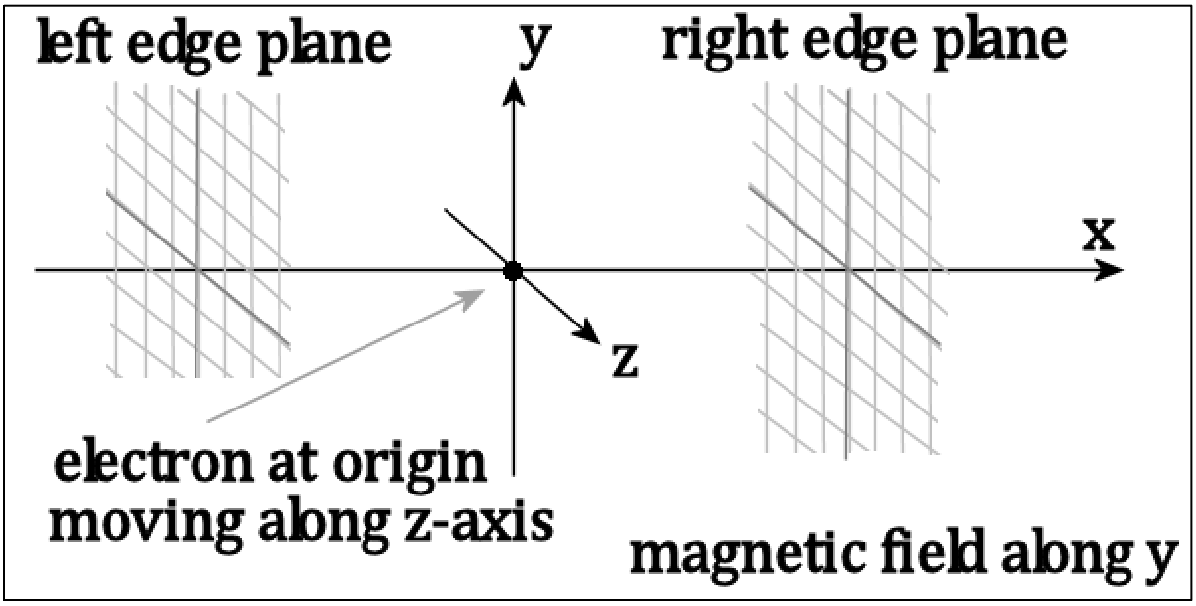

Figure 4

The magnetic field **B** lies vertically along the y-axis and the motion ***v*** of the electron is instantaneously along the z-axis. Positive z is up out of the plane of the page.

This potential energy density integrated over this boundary plane, being of infinitesimal thickness, is effectively the derivative dU/d**x** = **F**, where dU/d**x** is the derivative of the potential energy with respect to motion of the electron along the x-axis, giving force **F** directly and causing the electron to accelerate along the x-axis, curving its path. There are two such planes, one on positive-x and one on negative-x with respect to the centre of the magnet.

Using Figure 4 we can derive the equation for the energy density at a point on each of these two planes as the electric field strength from the electron at the boundary, times the ratio of that which lies parallel to the x-axis, times the induced electric field strength...

$$\frac{\mathrm{dU}}{\mathrm{ds}} = \varepsilon \frac{q}{4\pi\varepsilon(x^2 + y^2 + z^2)} \frac{x}{\sqrt{(x^2 + y^2 + z^2)}} \ (\mathbf{v}x\mathbf{B})$$

$$= \frac{\varepsilon q(\mathbf{v}x\mathbf{B})}{4\pi\varepsilon} \frac{x}{(x^2 + y^2 + z^2)^{3/2}}$$

Then to get the energy density of the sheet we have...

$$\frac{dU}{d\mathbf{x}} = \frac{\varepsilon q(\mathbf{v}x\mathbf{B})}{4\pi\varepsilon} \iint_{-\infty}^{\infty} \frac{x}{(x^2 + y^2 + z^2)^{3/2}} \, dy \, dz$$

We can simplify this integration by recognising that the term $(y^2+z^2) = r^2$ is constant for a circle around the x-axis, where 'r' is the radius of the circle. So substitute semi-polar co-ordinates and multiply the function by $2\pi r$, then integrate over radius 'r' from $r=\infty$ to $r=0$...

$$\frac{dU}{d\mathbf{x}} = \frac{\varepsilon q(\mathbf{v}x\mathbf{B})}{4\pi\varepsilon} \int_0^{\infty} \frac{2\pi r x}{(x^2 + r^2)^{3/2}} \, dr$$

$$= \frac{q(\mathbf{v}x\mathbf{B})}{2} \int_0^{\infty} \frac{rx}{(x^2 + r^2)^{3/2}} \, dr$$

$$= -\frac{q(\mathbf{v}x\mathbf{B})x}{2\sqrt{(x^2 + r^2)}} \quad \text{for } r = \infty, 0$$

$$\frac{dU}{d\mathbf{x}} = \mathbf{F} = \frac{q(\mathbf{v}x\mathbf{B})}{2}$$

The same result holds for the edge plane at the opposite side of the magnet. The force in both cases is in the same direction as there is repulsion from one edge and attraction from the other, so the total force is twice the above...

$$\mathbf{F}_{\mathbf{total}} = q(\mathbf{v}x\mathbf{B})$$

This is simply the Lorentz result, and so it can be seen that the Lorentz force for an induced electric field is caused by the interaction of the electron's field with the edges of the induced field.

A Summary of the discrepancies

However, the integration is between plus and minus infinity – the edges of the field parallel to the motion of the electron (and normal to the deflecting force) are assumed to extend to infinity; anything less and we do not get the Lorentz result. It can therefore be seen that the result agrees with Lorentz *only* where the areas of the left and right sides of the magnet (taking the electron motion as up as in Figure 4) are *infinite* in extent and a *finite* distance from the centre of the electron, and the particle's electric field extends across the edges of the magnetic field (as it does for the electron, but not the neutron). Cutting the sides down in size to some finite area will reduce the integration sum and so one can expect significant discrepancies for short magnets between the actual magnetic field strength and that strength as measured by the deflecting force on a charged particle. Any measurement of a practical magnetic field made by looking at the deflective force on a moving charged particle will therefore be artificially low in value, although it will obviously be in full agreement with any experiment involving the deflection of moving charged particles.

We can approximate this condition for a single point on the electron's path in the magnetic field, by making 'x' very small and 'y' and 'z' very large for the point shown, but it is impossible to create a geometry that maintains this condition around the full path of the electron.

Also, you may have noticed that the term in 'x' disappears for the infinite-area integration. It does not matter what finite width the magnet is in 'x' if the length of the sides is infinite. Why is this? Well, consider what happens if we double the value of 'x' – that is, make the electron twice as far from the magnet's edge as before. Take any solid angle that intersects the plane. The electric field strength drops by four, while the induced field remains constant, so the potential energy density (given by ε times the dot product) also drops by four. However it now covers four times the area, exactly balancing the drop in the potential energy density. As a result, for any specific solid angle projected from the electron onto the edge of the magnetic field, there will be a constant potential energy density across that the projection surface. Hence 'x' disappears from the equation when measuring potential energy in terms of solid angles (not the way derived here). Since the solid angle projected from a finite distance onto an infinite surface must always be a hemisphere, it follows that the *length* of the normal projection to that surface, passing through the centre of the electron, is immaterial.

Therefore if we make both the magnet's y- and z-axes very much larger than its x-axis, and ignore the return flux paths from the far ends of the poles, we find that the resultant computed force on the electron agrees with the Lorentz Force Equation. But as the y-axis and/or the z-axis reduce in size to the same dimension as the x-axis, the resultant force drops dramatically below the Lorentz value. It should be clear that in such circumstances the deflection of an electron by the field is reduced. Since many measurements of magnetic field strength are made by measuring the effect of the field on the motion of an electron, they often report a lower magnetic field than they should; however this is not usually a problem as the field is then generally used to provide a deflective force to electron motion, whether in an electric motor or a particle accelerator. However, under-reporting of the magnetic field strength can have significant consequences.

The treatment here has been simplified. The electron in Figure 4 interacts with all of the magnetic field, not just that inside the magnet. The field continues beyond the end of the magnetic poles and then loops around to meet up with the flux lines from the opposite pole in a return loop outside the magnet. We can ignore the return loop of magnetic flux in our calculations as every electric flux line from the electron intersects both the inside and outside edges of the return flux and the effects at the two edges therefore cancel out (the outer edge is often not well defined as the field intensity generally drops slowly as we move away from the magnet, but that does not affect the cancellation).

The solutions to our conundrums

The solutions to our two conundrums are then:

1. *Why is a neutron not deflected in moving through a magnetic field?* A neutron will not be subject to any forces inside the magnetic field despite having a strong electric field, because that electric field is wholly contained within the magnet, and it has no interaction with the edges of the magnetic field; there is therefore no change in potential energy density if it moves in the magnetic field, and hence no forces exist. It may however be deflected when entering the edge of a magnetic field at a shallow angle; this is not analysed in detail as the effect is immeasurably small because of the tiny radius of the neutron's electric field.

2. *What if we replace the magnetic field with an electric field (so that the electric field always pointed in the same direction rather than remaining normal to **B**x**v** and thus following a curved path)?* An electron falling from infinity is indeed accelerated across the field. But it is decelerated before entering the field by the interaction of the leading edge of the field with the electron's field, its effect being reversed by the centre of the electron being on the other side of the boundary, and likewise decelerated after exit from the field, the sum of the two decelerations matching the single acceleration. The overall effect is of no net change in the kinetic energy. The Principle of Conservation of Energy stands.

Where the magnetic field strength is *measured* by the deflection of an electron inside the field in equipment such as Hall-effect devices and *applied* involving the deflection of electrons in the field, there is no conflict as identical errors appear in both cases. However, discrepancies may appear when a magnetic field is calibrated by electron deflection and then used to measure a magnetic dipole, as in magnetic resonance.

Testing the concept

This concept makes other predictions at odds with the Lorentz Force Equation which can be tested. For example, consider an electron circling inside a magnet, where the plan view of the magnetic field gap is rectangular, as shown in Figure 5 (i.e. the magnetic field lines lie vertically up out of the page). Here the Lorentz Force Equation predicts a perfect circle. However our analysis predicts that the forces on the electron are greater when the electron is travelling parallel to the longer sides, causing the electron to follow an elliptical rather than a circular path. The major axis of the ellipse would therefore be parallel to the shorter sides.

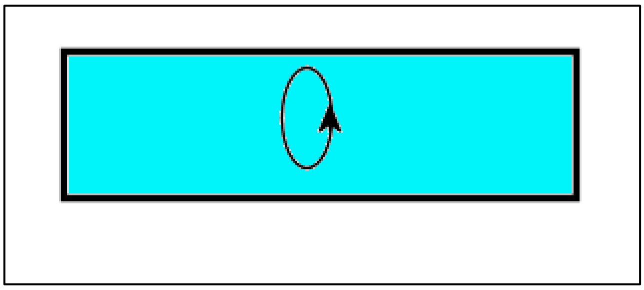

Figure 5

Appendix: The Basic Electromagnetic Field Equations

Introduction to the Electromagnetic Field

In this appendix we examine the fundamental energy and induction equations from which everything else in this book is derived. Little else is needed. They are presented here because over the latter half of the 20th century these equations have been increasingly dropped from textbooks. All equations are based on SI units.

Within Electromagnetic Field Theory, the interaction between charged particles is in fact the interaction of their distributed Electromagnetic Fields. Here we introduce those basic field equations. No matter how complex the problem we can analyse the behaviour of any assembly of charged particles by the interaction of their fields and the corresponding changes in total field energy as they move. This is often best done by Numerical Computation, also termed Finite Element Analysis, where the energy density at a point is integrated over all space to give a complete result.

The Electric Field

The electric field at a point in space is described by the electric field vector **E**, giving the magnitude and direction of the electric field. There is energy in any electric field, and to describe the amount of energy associated with that point in space we have to use the energy density, which describes the energy density in terms of energy per unit volume (such as Joules/comic metre). The total energy associated with a single point in space is of course zero as a point in space has zero length in all spatial dimensions; it is by integrating the energy density over all space that we get actual energy values.

Deriving electromagnetic equations from first principles is straightforward, and is based on the equation...

$$\frac{dW}{ds} = \frac{\varepsilon|\mathbf{E}|^2}{2}$$

$$= \frac{\varepsilon\, \mathbf{E} \bullet \mathbf{E}}{2}$$

...where dW/ds is the absolute electric energy density per volume 'ds' at a point in space where the vector electric field is **E**.

The *potential energy* is a measure of the work we can extract from a system, and is generally much more useful than the absolute energy. From the above equation we can work out the *potential* (as opposed to the absolute) energy density. When two electric fields E_1 and E_2 interact the absolute energy density at a point in space is derived from the vector sum of the two fields...

$$\frac{dW}{ds} = \frac{\varepsilon(|\mathbf{E_1} + \mathbf{E_2}|^2)}{2}$$

$$= \frac{\varepsilon(\mathbf{E}_1 \bullet \mathbf{E}_1 + 2\mathbf{E}_1 \bullet \mathbf{E}_2 + \mathbf{E}_2 \bullet \mathbf{E}_2)}{2}$$

$$= \frac{\varepsilon\mathbf{E}_1 \bullet \mathbf{E}_1}{2} + \boldsymbol{\varepsilon}(\mathbf{E}_1 \bullet \mathbf{E}_2) + \frac{\varepsilon\mathbf{E}_2 \bullet \mathbf{E}_2}{2}$$

Now since the first term and the third term are simply the absolute energy densities of the individual and separate fields $\mathbf{E}_1$ and $\mathbf{E}_2$, it follows that the second term is purely the interaction energy – that is, the *potential* energy associated by bringing the two fields together to interact. Hence, at a point in space where two fields interaction, the "potential energy density" dU/ds is the vector dot product of the fields times the permittivity...

$$\frac{dU}{ds} = \varepsilon(\mathbf{E}_1 \bullet \mathbf{E}_2)$$

The integral of this function over all space is the total potential energy. This is the potential energy seen when (say) bringing two electrons together from infinite separation and appears often in electrostatic field calculations. Differentiating either the total energy or the total potential energy with separation '**l**' gives the force, **F**=dU/d**l**. It is generally easier to do the latter.

The Magnetic Field and Potential

Magnetic fields operate in the same way. The energy density dW/ds for a magnetic field of field strength **B** is:-

$$\frac{dW}{ds} = \frac{\mathbf{B}^2}{2\mu}$$

The magnetic field, with its energy density, links regions of different magnetic potential. Regions with a constant magnetic potential have zero energy density.

It is trivial to derive the magnetic potential energy of two interacting magnetic fields in a similar way to the derivation of the electric field potential energy of two interacting electric fields. It is...

$$\frac{dU}{ds} = \frac{(\mathbf{B}_1 \bullet \mathbf{B}_2)}{\mu}$$

Moving fields

A magnetic field moving in our own stationary frame of reference induces an electric field in that stationary frame. The strength of the induced electric field is...

$$\mathbf{E} = \mathbf{v}x\mathbf{B}$$

Here **B** is the vector magnetic field strength and **v** is the vector velocity so that the induced electric field **E** is the cross product **v***x***B**.

Equally, an electric field moving in our own stationary frame of reference induces a magnetic field in that stationary frame. The strength of the induced magnetic field **B** is...

$$\mathbf{B} = \mu\varepsilon(\mathbf{v}x\mathbf{E})$$

$$= (\mathbf{v}x\mathbf{E})/\mathbf{c}^2$$

Force

Force is an effect that appears when energy is extracted from or entered into an energy storage mechanism and essentially couples different energy systems. Force is anonymous in that a pure force by itself carries no information as to what generated it; you have to look further into how it arose although you may get clues by its spatial distribution. Force is an energy conservation mechanism as without it no energy could ever be exchanged. Equally, without at least possible energy changes there can be no forces. It is a vector quantity defined by the equation:-

$$\mathbf{F} = \frac{\mathrm{dU}}{\mathrm{d}\mathbf{s}}$$

That is, force **F** is equal to rate of change of energy U with vector distance **s**. This is a universal equation and applies whatever the energy mechanism might be. You can use either absolute or potential energy (in the former case the symbol ‘W’ is commonly used), but the result is identical.