On The Proving Method of Fermat's Last Theorem

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Abstract: In this paper the author gives an elementary mathematics method to solve ***Fermat's Last Theorem*** (FLT), in which let this equation become an one unknown number equation, in order to solve this equation the author invented a method called "Order reducing method for equations", where the second order root compares to one order root, and with some necessary techniques the author successfully proved when *x*^(*n*-1)+*y*^(*n*-1)- *z*^(*n*-1) <= *x*^(*n*-2)+*y*^(*n*-2)- *z*^(*n*-2) there are no positive solutions for this equation, and also proves with the increasing of *x* there are still no positive integer solutions for this equation when *x*^(*n*-1)+*y*^(*n*-1)- *z*^(*n*-1)<= *x*^(*n*-2)+*y*^(*n*-2)- *z*^(*n*-2) is not satisfied.

1. Some Relevant Theorems

There are some theorems for proving or need to be known. ***All symbols in this paper represent positive integers unless they are stated to be not.***

Theorem 1.1. In the equation of

$$\begin{cases} x^n + y^n = z^n \\ \gcd(x, y, z) = 1 \\ n > 2 \end{cases} \tag{1-1}$$

x, y, z meet

$x \neq y$,

$x + y > z$,

and if

$x > y$

then

$z > x > y$.

Proof: Let

$x = y$,

we have

$$2x^n = z^n$$

and

$$\sqrt[n]{2}x = z$$

where $\sqrt[n]{2}$ is not an integer and x, z are all positive integers, so $x \neq y$.

Since

$$(x + y)^n = x^n + C_n^1 x^{n-1} y + ... + C_n^{n-1} x y^{n-1} + y^n > z^n ,$$

so we get

$$x + y > z.$$

Since

$$x^n + y^n = z^n ,$$

so we have

$$z^n > x^n, z^n > y^n$$

and get

$$z > x > y$$

when

$$x > y .$$

Theorem 1.2. In the equation of (1-1), x, y, z meet

$$\gcd(x, y) = \gcd(y, z) = \gcd(x, z) = 1.$$

Proof: Since $x^n + y^n = z^n$, if $\gcd(x, y) > 1$ then we have $\left(x_1^n + y^n{}_1\right) \times [\gcd(x, y)]^n = z^n$ which causes $\gcd(x, y, z) > 1$ since the left side contains the factor of $[\gcd(x, y)]^n$ then the right side must also contains this factor but contradicts against (1-1) in which $\gcd(x, y, z) = 1$, so we have $\gcd(x, y) = 1$. Using the same way we have $\gcd(x, z) = \gcd(y, z) = 1$.

Theorem 1.3. If there is no positive integer solution for

$$x^p + y^p = z^p$$

when $p > 2$ is a prime number then there is also no positive integer solution for

$$\left(x^p\right)^k + \left(y^p\right)^k = \left(z^p\right)^k .$$

Proof: Since $x^p + y^p = z^p$ has no positive integer solution, so there still no positive integer solution for

$$\left(x^k\right)^p + \left(y^k\right)^p = \left(z^k\right)^p$$

which means there is also no positive integer solution for

$$\left(x^p\right)^k + \left(y^p\right)^k = \left(z^p\right)^k .$$

So we only need to prove there is no positive integer solution for equation (1-1) when n is a prime number.

Theorem 1.4. There are no positive integer solutions for equation (1-1) when x or y is a prime number .

Proof: When x is a prime number, since

$$x^n = z^n - y^n = (z-y)\left(z^{n-1} + z^{n-2}y + ... + zy^{n-2} + y^{n-1}\right),$$

so we have

$$\gcd(z-y, x) = x ,$$

which means

$$z - y \geq x ,$$

we have

$$x + y \leq z ,$$

that contradicts against **Theorem 1.1** in which $x + y > z$, so it is with y , which means there are no positive integer solutions for equation (1-1) when x or y is a prime number .

Theorem 1.5. There are no positive integer solutions for equation (1-1) when z is a prime number .

Proof: When z is a prime number, from Theorem 1.12 we only consider the case of $n > 2$ is a prime number, since

$$x^n + y^n = z^n = (x+y)\left(x^{n-1} + ... + y^{n-1}\right),$$

so we have

$$\gcd(x+y, z) = z ,$$

from **Theorem 1.1** we know $x + y > z$, so we get

$$x + y \geq 2z ,$$

that contradicts against **Theorem 1.1** in which $z > x > y \Rightarrow x + y < 2z$, which means there are no positive integer solutions for equation (1-1) when z is a prime number .

Theorem 1.6. There are no positive integer solutions for

$$1^n + y^n = z^n .$$

Proof: Since

$$1 = z^n - y^n = (z-y)\left(z^{n-1} + z^{n-2}y + ... + zy^{n-2} + y^{n-1}\right)$$

where

$$\begin{cases} z - y = 1 \\ \left(z^{n-1} + z^{n-2}y + ... + zy^{n-2} + y^{n-1}\right) = 1 \end{cases}$$

that causes z, y to be non positive integers, so there are no positive integer solutions for

$$1^n + y^n = z^n .$$

Theorem 1.7. There are no positive integer solutions for

$$2^n + y^n = z^n .$$

Proof: Since

$$2^n = z^n - y^n = (z - y)\left(z^{n-1} + z^{n-2}y + ... + zy^{n-2} + y^{n-1}\right),$$

if

$$\begin{cases} z - y = 1 \\ z^{n-1} + z^{n-2}y + ... + zy^{n-2} + y^{n-1} = 2^n \end{cases}$$

then taking the least value for $y = 2, z = 3$, we have

$$3^{n-1} + 2\times 3^{n-2} + ... + 2^{n-1} > 2^n$$

when $n > 2$ that is impossible. If

$$\begin{cases} z - y = 2^i \\ z^{n-1} + z^{n-2}y + ... + zy^{n-2} + y^{n-1} = 2^j \\ i + j = n \\ i \geq 1 \end{cases}$$

then $z > 2$ and taking the least value of $y = 2, z = 3$, we get

$$3^{n-1} + 2\times 3^{n-2} + ... + 2^{n-1} > 2^j$$

with $n > 2$ that is also impossible, so there are no positive integer solutions for

$$2^n + y^n = z^n .$$

Theorem 1.8. There are no positive integer solutions for equation (1-1) when $n \to \infty$ and x, y, z in equation (1-1) meet

$$z < \sqrt[n]{2}x, x > 2, y > 1, z > 3.$$

Proof: Since $x^n + y^n = z^n$, let $x > y$, we get

$$\left(\frac{z}{x}\right)^n - \left(\frac{y}{x}\right)^n = 1,$$

since

$$z > x > y,$$

so we have

$$z < \sqrt[n]{2}x,$$

and

$$\lim_{n\to\infty}\left(\frac{z}{x}\right)^n-\left(\frac{y}{x}\right)^n=\infty>1$$

which means there are no positive integer solutions for equation (1-1) when $n\to\infty$.

According to **Theorem 1.6, 1.7** we have $x>2, y>1, z>3$.

Theorem 1.9. There are no positive integer solutions for equation (1-1) when $x, y, z\le 100$.

Proof: From **Theorem1.8**, we know $z<\sqrt[n]{2}x$, so we have

$$\frac{100}{\sqrt[n]{2}}<x,$$

when $n=3$, we have the smallest values for x, so we get

$$\left(\frac{100}{\sqrt[3]{2}}<x\right)\Rightarrow(79<x),$$

from **Theorem 1.4, 1.5** we know x, y, z are all not prime numbers. There are below combinations of x, y, z when $x, y, z\le 100$:

$$\begin{cases}(x=80\sim 99)^n+(y=4\sim(x-1))^n=(z=81\sim 100)^n\\ x+y>z\\ x^2+y^2>z^2\\ x^j+y^j>z^j\\ j<n\end{cases}.$$

Here we take $7^n+9^n=10^n$ for example to explain how to prove. We plot the graph for this equation as showed in **Figure 1-1**.

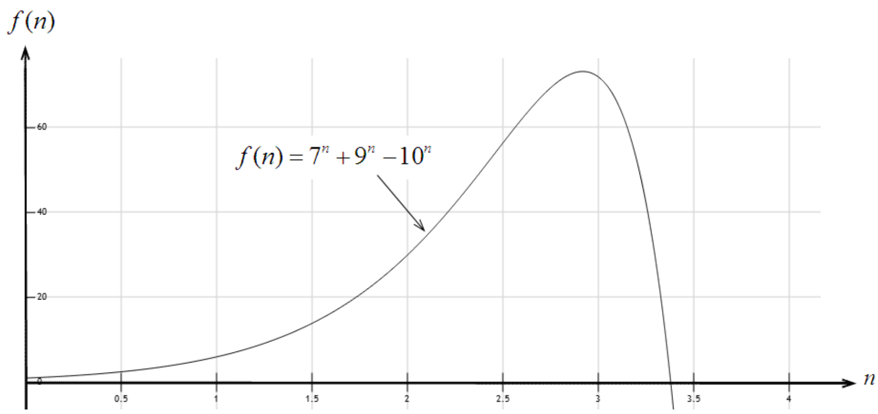

Figure 1-1 *Graph of* $f(n)=7^n+9^n-10^n$

Obviously for equation $f(n) = 7^n + 9^n - 10^n$ in **Figure 1-1**, we have $3 < n < 4$ is not an integer so there are no positive integer solutions, using this method we have the conclusion of there are no positive integer solutions for equation (1-1) when $z \le 100$.

Using the method of which we prove **Theorem 1.6, 1.7** we can prove when $x, y \le 100$, there are no positive integer solutions for equation (1-1).

Theorem 1.10. In the equation of (1-1), x, y, z meet

$$x^{n-i} + y^{n-i} > z^{n-i},$$

$$x^{n+i} + y^{n+i} < z^{n+i},$$

where

$$n > i \ge 1.$$

This theorem holds true when x, y, z are positive real numbers but n must be a positive integer.

Proof: From equation (1-1), since

$$x^n + y^n = z^n,$$

from **Theorem 1.1**, since $z > x > y$, we have

$$x^{n-i} + y^{n-i} > \left[\left(\frac{x}{z}\right)^i x^{n-i} + \left(\frac{y}{z}\right)^i y^{n-i} = z^{n-i}\right],$$

$$x^{n+i} + y^{n+i} < \left(z^i x^{n-i} + z^i y^{n-i} = z^{n+i}\right),$$

so we have

$$x^{n-i} + y^{n-i} > z^{n-i}.$$

$$x^{n+i} + y^{n+i} < z^{n+i}.$$

This theorem means given x, y, z if equation (1-1) has one positive integer solution then this solution is the only one.

Theorem 1.11. There are no positive integer solutions for equation (1-1) when

$$\frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} \le 1.$$

And in order to have positive integer solutions for equation (1-1),

$$\frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} > 40$$

must be satisfied.

Proof: In equation (1-1), let

$$\begin{cases} a = x^{n-2} \\ b = y^{n-2} \\ c = z^{n-2} \end{cases},$$

we have

$$\begin{cases} ax^2 + by^2 = cz^2 \\ a^{\frac{n-1}{n-2}}x + b^{\frac{n-1}{n-2}}y = c^{\frac{n-1}{n-2}}z \end{cases}.$$

Since we reduce the order of equation so the method is called "Order reducing method for equations". Let $x > y$ and

$$\begin{cases} y = x - f \\ z = x + e \end{cases},$$

we have

$$\begin{cases} ax^2 + b(x-f)^2 = c(x+e)^2 \\ a^{\frac{n-1}{n-2}}x + b^{\frac{n-1}{n-2}}(x-f) = c^{\frac{n-1}{n-2}}(x+e) \end{cases}$$

and

$$\begin{cases} (a+b-c)x^2 - 2(bf+ce)x + \left(bf^2 - ce^2\right) = 0 \\ a^{\frac{n-1}{n-2}}x + b^{\frac{n-1}{n-2}}(x-f) - c^{\frac{n-1}{n-2}}(x+e) = 0 \end{cases},$$

the roots are

$$x = \frac{(bf+ce) \pm \sqrt{(bf+ce)^2 - (a+b-c)\left(bf^2 - ce^2\right)}}{x^{n-2} + y^{n-2} - z^{n-2}}, \qquad (1\text{-}2)$$

and

$$x = \frac{c^{\frac{n-1}{n-2}}e + b^{\frac{n-1}{n-2}}f}{a^{\frac{n-1}{n-2}} + b^{\frac{n-1}{n-2}} - c^{\frac{n-1}{n-2}}} = \frac{bfy + cez}{x^{n-1} + y^{n-1} - z^{n-1}}. \qquad (1\text{-}3)$$

There are two cases for bf^2, ce^2 when $bf^2 \ge ce^2$ and $bf^2 < ce^2$.

Case A: If $bf^2 \ge ce^2$, from (1-2) when

$$x = \frac{(bf+ce) + \sqrt{(bf+ce)^2 - (a+b-c)\left(bf^2 - ce^2\right)}}{x^{n-2} + y^{n-2} - z^{n-2}},$$

from **Theorem 1.10** we know $a + b - c = x^{n-2} + y^{n-2} - z^{n-2} > 0$, so we have

$$x \le \frac{2(bf + ce)}{x^{n-2} + y^{n-2} - z^{n-2}},$$

also from **Theorem 1.10** we have $x^{n-1} + y^{n-1} - z^{n-1} > 0$, compare to (1-3) we get

$$\frac{bfy + cez}{x^{n-1} + y^{n-1} - z^{n-1}} \le \frac{2(bf + ce)}{x^{n-2} + y^{n-2} - z^{n-2}}.$$

When $\frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} \le 1$, we have

$$bfy + cez \le 2(bf + ce)$$

that is impossible since from **Theorem 1.8** we know $y \ge 2$ and $z > 3$.

When

$$x = \frac{(bf + ce) - \sqrt{(bf + ce)^2 - (a + b - c)(bf^2 - ce^2)}}{x^{n-2} + y^{n-2} - z^{n-2}}.$$

we have

$$x \le \frac{bf + ce}{x^{n-2} + y^{n-2} - z^{n-2}},$$

compare to (1-3) we get

$$\frac{bfy + cez}{x^{n-1} + y^{n-1} - z^{n-1}} \le \frac{bf + ce}{x^{n-2} + y^{n-2} - z^{n-2}}.$$

When $\frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} \le 1$, we have

$$bfy + cez \le bf + ce$$

that is impossible since from **Theorem 1.8** we have already known $y \ge 2$ and $z > 3$.

Case B: If $bf^2 < ce^2$, from (1-2) when

$$x = \frac{(bf + ce) + \sqrt{(bf + ce)^2 + (a + b - c)(ce^2 - bf^2)}}{x^{n-2} + y^{n-2} - z^{n-2}},$$

we can prove $(bf + ce)^2 > (a + b - c)(ce^2 - bf^2)$ since if not we have

$$(bf + ce)^2 \le (a + b - c)(ce^2 - bf^2)$$

and

$$\left[(2b+a)-c\right]bf^2+2bfce+\left[2c-(a+b)\right]ce^2\le 0$$

that is impossible since $a+b-c>0$ and $c>a, c>b, 2c-(a+b)>0.$ So we have

$$x<\frac{(bf+ce)\left(1+\sqrt{2}\right)}{x^{n-2}+y^{n-2}-z^{n-2}}$$

compare to (2-4) we get

$$\frac{bfy+cez}{x^{n-1}+y^{n-1}-z^{n-1}}<\frac{(bf+ce)\left(1+\sqrt{2}\right)}{x^{n-2}+y^{n-2}-z^{n-2}}.$$

When $\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}}\le 1$, we have

$$bfy+cez<(bf+ce)\left(1+\sqrt{2}\right)<2.5(bf+ce)$$

and

$$bf(x-f)+ce(x+e)<2.5(bf+ce)$$

that leads to

$$x<\left[\frac{2.5(bf+ce)+bf^2-ce^2}{bf+ce}=2.5-\frac{ce^2-bf^2}{bf+ce}\right]<2.5$$

where possible values for x are 1, 2 but according to **Theorem 1.6**, **1.7** we know there are no positive integer solutions.

When

$$x=\frac{(bf+ce)-\sqrt{(bf+ce)^2+(a+b-c)\left(ce^2-bf^2\right)}}{x^{n-2}+y^{n-2}-z^{n-2}}$$

is not possible since $x\le 0$.

So we have the conclusion of there are no positive integer solutions for equation (1-1) when $\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}}\le 1.$

Obviously we have

$$bfy+cez<2.5\left(\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}}\right)(bf+ce),$$

from **Theorem 1.9** we know $x, y, z\le 100$ there are no positive integer solutions for equation

(1-1), so we have

$$\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}}>40,$$

which must be satisfied to have positive integer solutions for equation (1-1).

Theorem 1.12. Function $f(x)=A^X$ and $g(x)=A^X+B^X$ are all monotonically increasing "Convex functions", where A, B are all positive real numbers and X is a real number.

Proof: Since monotonically increasing "Convex functions" meets

$$f'(x)=\frac{df(x)}{dx}>0,$$

$$f''(x)=\frac{d^2 f(x)}{dx^2}>0,$$

for $f(x)=A^X$ and $g(x)=A^X+B^X$, we have

$$f'(x)=A^X \ln A>0,$$
$$f''(x)=A^X \ln^2 A>0,$$

$$g'(x)=A^X \ln A+B^X \ln B>0,$$
$$g''(x)=A^X \ln^2 A+B^X \ln^2 B>0,$$

so $f(x)=A^X$ and $g(x)=A^X+B^X$ are all monotonically increasing "Convex functions".

This theorem means that functions $g(n)=x^n+y^n$, $f(n)=z^n$ are all monotonically increasing "Convex functions".

2. Proving Method

From **Theorem 1.11** we know in order to have positive integer solutions for this equation, $\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}}>1$ must be satisfied. We give the graph of this equation as showed in

Figure 2-1 when $\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}}>1$, where $AB // CD'$.

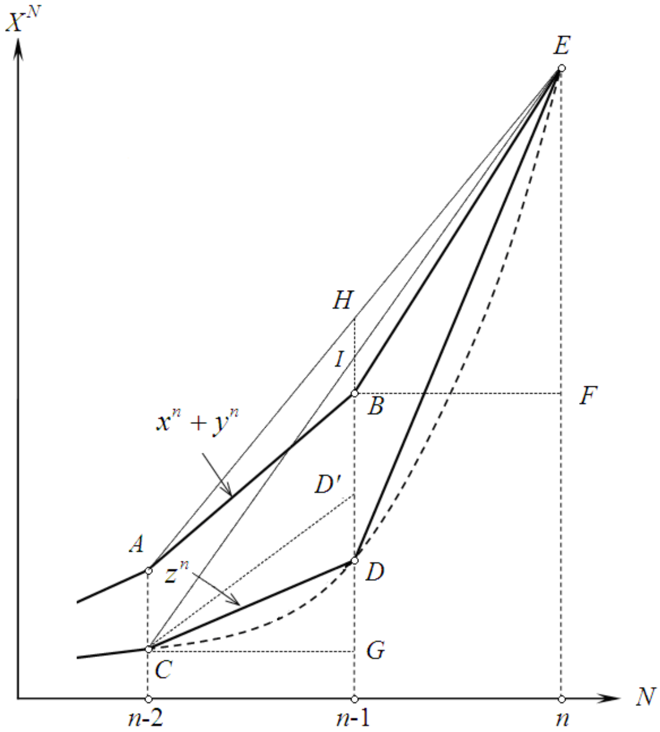

Figure 2-1 *Graph of* $x^n + y^n = z^n$ *when* $\dfrac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}} > 1$

1. In **Figure 2-1** we have

$$\angle CDE = 360^0 - \arctan\left(\frac{z^n - z^{n-1}}{1}\right) - \arctan\left(\frac{1}{z^{n-1} - z^{n-2}}\right) - 90^0,$$

and

$$BD = x^{n-1} + y^{n-1} - z^{n-1},$$
$$AC = x^{n-2} + y^{n-2} - z^{n-2}.$$

When $\dfrac{BD}{AC} > 1$ we have

$$\angle ABE - \angle CDE = \angle D'CD + \angle BED > 0,$$

which means

$$\angle ABE > \angle CDE.$$

It is also very clear that if $\angle ABE \le \angle CDE$ then $\dfrac{BD}{AC} < 1$.

From **Theorem 1.9** we know if $z \le 100$ then there are no positive integer solutions for equation (1-1), when $n = 3$ (*which is the worst case*) we have

$$\angle CDE = 270^0 - \arctan\left(\frac{z^n - z^{n-1}}{1}\right) - \arctan\left(\frac{1}{z^{n-1} - z^{n-2}}\right)$$

$$= 270^0 - \arctan(100^3 - 100^2) - \arctan\left(\frac{1}{100^2 - 100}\right) > 179.99^0,$$

and

$$\angle ABE > \angle CDE > 179.99^0,$$

which means $\angle ABE, \angle CDE \to 180^0$, so ABE, CDE are almost lines with $z > 100, n \geq 3$, that leads to $\frac{BD}{AC} \to \frac{1}{2} < 1$, which contradicts against $BD > AC$. So when z, n is large enough then $\frac{BD}{AC} = \frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} < 1$, from **Theorem 1.11** we know there are no positive integer solutions for equation (1-1).

2. For function

$$f(z) = \angle CDE = 270^0 - \arctan\left(\frac{z^n - z^{n-1}}{1}\right) - \arctan\left(\frac{1}{z^{n-1} - z^{n-2}}\right)$$

$$= \frac{3}{2}\pi - \arctan\left(\frac{z^n - z^{n-1}}{1}\right) - \arctan\left(\frac{1}{z^{n-1} - z^{n-2}}\right),$$

we give the function plot for it in **Figure 2-2**.

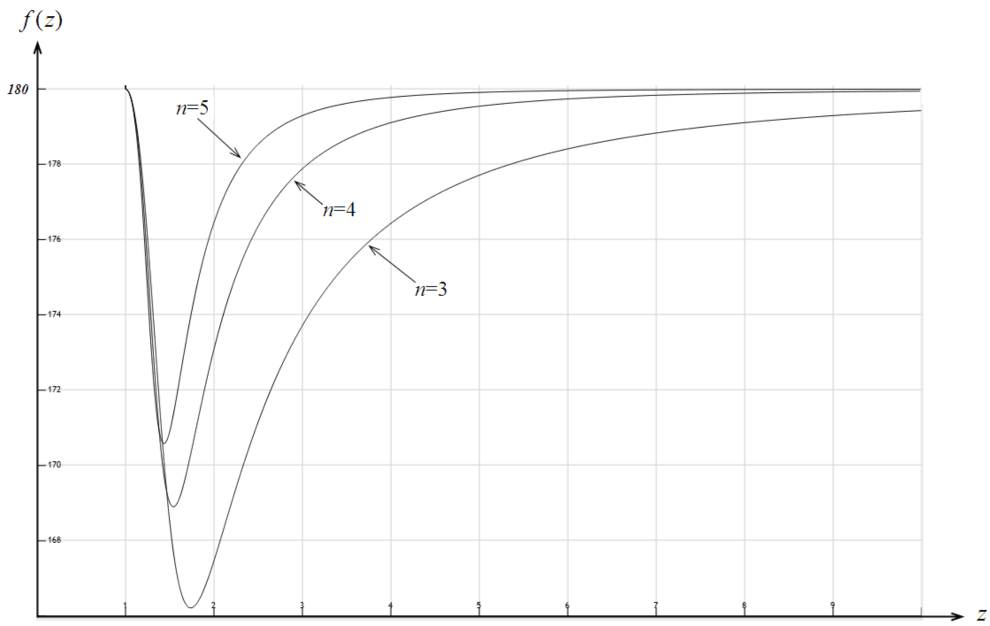

Figure 2-2 *Graph of* $f(z) = \angle CDE = 270^0 - \arctan\left(\frac{z^n - z^{n-1}}{1}\right) - \arctan\left(\frac{1}{z^{n-1} - z^{n-2}}\right)$

where we take $\pi = 3.1415926535897932$

Obviously $f(z) = \angle CDE$ is a "Monotonically increasing function" when $z \geq 3$, and with the increasing of z the value of $f(z) = \angle CDE$ is close to 180^0. It is very clear that $\angle ABE - \angle CDE$ is decreasing with the increasing of z, since

$$\left(\angle ABE - \angle CDE = \angle D'CD + \angle BED\right) < 180^0 - \angle CDE,$$

where $\angle CDE$ is increasing. When $n = 3$ since $\angle CDE > 179.99^0$, so we have

$$\left(\angle D'CD + \angle BED\right) < 180^0 - \angle CDE < 180^0 - 179.99^0 < 0.01^0,$$

which means

$$\angle BED, \angle D'CD < 0.01^0,$$

and when z or n is large enough, we have

$$\angle ABE - \angle CDE = \left(\angle BED + \angle D'CD\right) \to 0,$$

which means $BD < AC$ that contradicts against $BD > AC$. So when z or n is large enough then $\dfrac{BD}{AC} = \dfrac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} < 1$, from **Theorem 1.11** we know there are no positive integer solutions for equation (1-1).

3. In **Figure 2-1** we have

$$\angle ABE = \frac{3}{2}\pi - \arctan\left(\frac{x^n + y^n - x^{n-1} - y^{n-1}}{1}\right) - \arctan\left(\frac{1}{x^{n-1} + y^{n-1} - x^{n-2} - y^{n-2}}\right),$$

$$\angle CDE = \frac{3}{2}\pi - \arctan\left(\frac{z^n - z^{n-1}}{1}\right) - \arctan\left(\frac{1}{z^{n-1} - z^{n-2}}\right),$$

so

$$\angle ABE - \angle CDE = \begin{bmatrix} \arctan\left(\dfrac{z^n - z^{n-1}}{1}\right) + \arctan\left(\dfrac{1}{z^{n-1} - z^{n-2}}\right) - \arctan\left(\dfrac{x^n + y^n - x^{n-1} - y^{n-1}}{1}\right) \\ - \arctan\left(\dfrac{1}{x^{n-1} + y^{n-1} - x^{n-2} - y^{n-2}}\right) \end{bmatrix}.$$

From (1-1) we have

$$z = \left(x^n + y^n\right)^{\frac{1}{n}},$$

we get

$$\angle ABE - \angle CDE = \left[\begin{array}{l} \arctan\left(\frac{x^n + y^n - (x^n + y^n)^{\frac{n-1}{n}}}{1} \right) + \arctan\left(\frac{1}{(x^n + y^n)^{\frac{n-1}{n}} - (x^n + y^n)^{\frac{n-2}{n}}} \right) \\ - \arctan\left(\frac{x^n + y^n - x^{n-1} - y^{n-1}}{1} \right) - \arctan\left(\frac{1}{x^{n-1} + y^{n-1} - x^{n-2} - y^{n-2}} \right) \end{array} \right].$$

We give the plot of $f(x, y) = \angle ABE - \angle CDE$ using Excel VBA program that showed below:

```
n = 3
For x = 1 To 10 ^ 5 Step 1
    For y = 1 To x - 1 Step 1
        z = (x ^ n + y ^ n) ^ (1 / n)
        t1 = z ^ n - z ^ (n - 1)
        t2 = 1 / (z ^ (n - 1) - z ^ (n - 2))
        t3 = (x ^ n + y ^ n) - (x ^ (n - 1) + y ^ (n - 1))
        t4 = 1 / ((x ^ (n - 1) + y ^ (n - 1)) - x ^ (n - 2) - y ^ (n - 2))
        CDE = Application.Atan2(t1, 1) - Application.Atan2(t2, 1)
        ABE = Application.Atan2(t3, 2) - Application.Atan2(t4, 2)
        R = CDE - ABE
         Cells(i, 1) = "z=" & z
         Cells(i, 2) = "x=" & x
         Cells(i, 3) = "y=" & y
         Cells(i, 4) = R
        i = i + 1
       If i > 65536 Then End
    Next y
Next x
```

Figure 2-3 shows the results, obviously $f(x, y) = \angle ABE - \angle CDE, n = 3$ is decreasing.

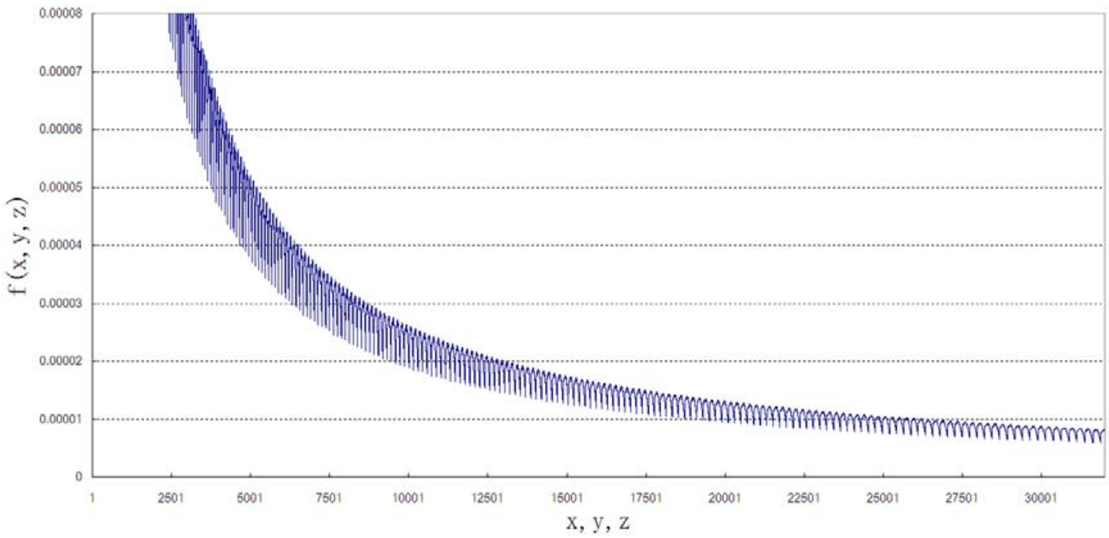

Figure 2-3 *Graph of* $f(x,y)=\angle ABE-\angle CDE, n=3$

4. In **Figure 2-1** we have

$$BD^2=BE^2+DE^2-2BE\times DE\times\cos(\angle BED)$$
$$=\left[\begin{array}{l}(z^n-z^{n-1})^2+1+\\(x^n+y^n-x^{n-1}-y^{n-1})^2+1\\-2\sqrt{(z^n-z^{n-1})^2+1}\times\sqrt{(x^n+y^n-x^{n-1}-y^{n-1})^2+1}\times\\\cos\left(\arctan\left(\frac{1}{x^n+y^n-x^{n-1}-y^{n-1}}\right)-\arctan\left(\frac{1}{z^n-z^{n-1}}\right)\right)\end{array}\right],$$

and

$$AC^2=AE^2+CE^2-2AE\times CE\times\cos(\angle AEC)$$
$$=\left[\begin{array}{l}(z^n-z^{n-2})^2+4+\\(x^n+y^n-x^{n-2}-y^{n-2})^2+4\\-2\sqrt{(z^n-z^{n-2})^2+4}\times\sqrt{(x^n+y^n-x^{n-2}-y^{n-2})^2+4}\times\\\cos\left(\arctan\left(\frac{2}{x^n+y^n-x^{n-2}-y^{n-2}}\right)-\arctan\left(\frac{2}{z^n-z^{n-2}}\right)\right)\end{array}\right],$$

from (1-1) we have

$$y=(z^n-x^n)^{\frac{1}{n}}.$$

We give the plot of $f(z,x)=\frac{BD}{AC}$ using Excel VBA program that showed below:

```
n = 3
For z = 10 ^ 7 To 10 ^ 9 Step 1
   For x = z / (2 ^ (1 / n)) To z - 1 Step 1
      y = (z ^ n - x ^ n) ^ (1 / n)
      t1 = z ^ n - z ^ (n - 1)
      t2 = x ^ n + y ^ n - x ^ (n - 1) - y ^ (n - 1)
      t3 = z ^ n - z ^ (n - 2)
      t4 = x ^ n + y ^ n - x ^ (n - 2) - y ^ (n - 2)
      BD = (t1 ^ 2 + t2 ^ 2 + 2 - 2 * Sqr((t1 ^ 2 + 1) * (t2 ^ 2 + 1)) * Cos(Application.Atan2(t2,
1) - Application.Atan2(t1, 1)))
      AC = (t3 ^ 2 + t4 ^ 2 + 8 - 2 * Sqr((t3 ^ 2 + 4) * (t4 ^ 2 + 4)) * Cos(Application.Atan2(t4,
2) - Application.Atan2(t3, 2)))
      R = (BD / AC) ^ 0.5
      Cells(j, 1) = "z=" & z
      Cells(j, 2) = "x=" & x
      Cells(j, 3) = "y=" & y
```

```
        Cells(j, 4) = R
        j = j + 1
        If j > 65536 Then End
    Next x
Next z
```

We give the plot of $f(z,x)=\dfrac{BD}{AC}, n=3$ when $z=10^7, x=\dfrac{z}{\sqrt[n]{2}}\sim z$, $n=3$, it is showed in **Figure 2-4**.

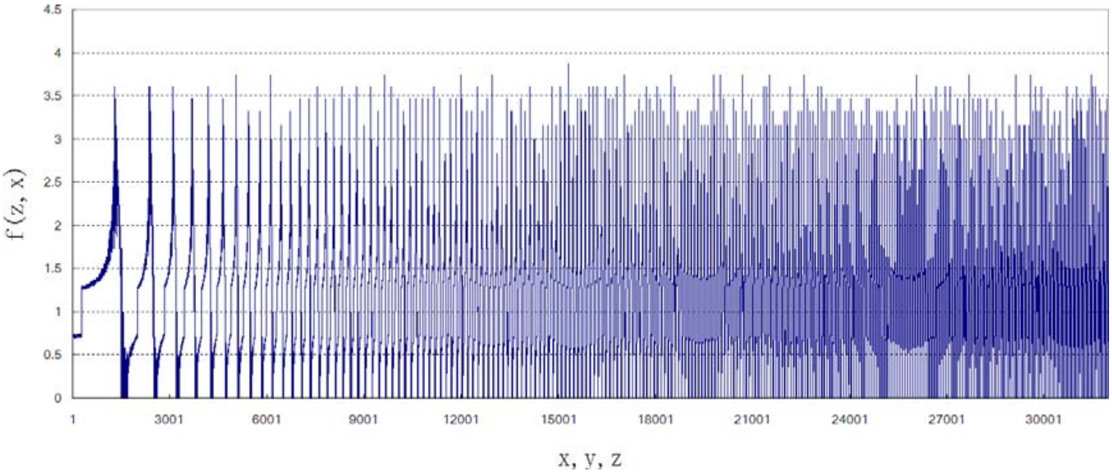

Figure 2-4 *Graph of* $f(z,x)=\dfrac{BD}{AC}, n=3$

With the increasing of z, n the value of $f(z,x)=\dfrac{BD}{AC}$ will be smaller, and we are sure of when $z, n\to\infty$ or get larger, the conclusion holds. In fact even $z=10^6$, we can still have a result of $f(z,x)=\dfrac{BD}{AC}<40$.

5. In **Figure 2-1** let S_{CDE}, S_{ABE} be the areas of triangles $\Delta CDE, \Delta ABE$, we have

$$S_{CDE}=\frac{CD\times DE\times\sin(\angle CDE)}{2}$$

$$=\frac{\left[\begin{array}{l}\sqrt{\left(z^n-z^{n-1}\right)^2+1}\times\sqrt{\left(z^{n-1}-z^{n-2}\right)^2+1}\times\\ \sin\left(\dfrac{3}{2}\pi-\arctan\left(\dfrac{z^n-z^{n-1}}{1}\right)-\arctan\left(\dfrac{1}{z^{n-1}-z^{n-2}}\right)\right)\end{array}\right]}{2}$$

$$= \frac{-\left[\begin{array}{l}\sqrt{\left(z^n - z^{n-1}\right)^2 + 1} \times \sqrt{\left(z^{n-1} - z^{n-2}\right)^2 + 1} \times \\ \cos\left(\arctan\left(\frac{z^n - z^{n-1}}{1}\right) + \arctan\left(\frac{1}{z^{n-1} - z^{n-2}}\right)\right)\end{array}\right]}{2}$$

$$= \frac{DI}{2} + \frac{DI}{2} = DI,$$

$$S_{ABE} = \frac{AB \times BE \times \sin(\angle ABE)}{2}$$

$$= \frac{\left[\begin{array}{l}\sqrt{\left(x^n + y^n - x^{n-1} - y^{n-1}\right)^2 + 1} \times \sqrt{\left(x^{n-1} + y^{n-1} - x^{n-2} - y^{n-2}\right)^2 + 1} \times \\ \sin\left(\frac{3}{2}\pi - \arctan\left(\frac{x^n + y^n - x^{n-1} - y^{n-1}}{1}\right) - \arctan\left(\frac{1}{x^{n-1} + y^{n-1} - x^{n-2} - y^{n-2}}\right)\right)\end{array}\right]}{2}$$

$$= \frac{-\left[\begin{array}{l}\sqrt{\left(x^n + y^n - x^{n-1} - y^{n-1}\right)^2 + 1} \times \sqrt{\left(x^{n-1} + y^{n-1} - x^{n-2} - y^{n-2}\right)^2 + 1} \times \\ \cos\left(\arctan\left(\frac{x^n + y^n - x^{n-1} - y^{n-1}}{1}\right) + \arctan\left(\frac{1}{x^{n-1} + y^{n-1} - x^{n-2} - y^{n-2}}\right)\right)\end{array}\right]}{2}$$

$$= \frac{BH}{2} + \frac{BH}{2} = BH,$$

from (1-1) we have

$$y = \left(z^n - x^n\right)^{\frac{1}{n}}.$$

We give the plot of $f(z, x) = \frac{BH}{DI}$ using Excel VBA program that showed below:

```
n = 7
For z = 3 To 10 ^ 7 Step 1
    For x = z / (2 ^ (1 / n)) To z - 1
        y = (z ^ n - x ^ n) ^ (1 / n)
        If y >= x Then y = x - 1
        t11 = z ^ n - z ^ (n - 1)
        t12 = z ^ (n - 1) - z ^ (n - 2)
        CDE = Application.Atan2(1, t11) + Application.Atan2(t12, 1)
        scde = -Sqr(t11 ^ 2 + 1) * Sqr(t12 ^ 2 + 1) * Cos(CDE) / 2
        t21 = x ^ n + y ^ n - x ^ (n - 1) - y ^ (n - 1)
        t22 = x ^ (n - 1) + y ^ (n - 1) - x ^ (n - 2) - y ^ (n - 2)
        ABE = Application.Atan2(1, t21) + Application.Atan2(t22, 1)
        sabe = -Sqr(t21 ^ 2 + 1) * Sqr(t22 ^ 2 + 1) * Cos(ABE) / 2
        R = scde / sabe
```

```
        Cells(i, j) = R
        i = i + 1
        If i = 65535 Then j = j + 1: i = 0
        If j = 10 Then End
    Next x
Next z
```

The result of this program shows that when $n \ge 7$, the values of S_{CDE}, S_{ABE} are all negative that contradicts against **Theorem 1.12** (since $\angle CDE, \angle ABE < 180^0$, so S_{CDE}, S_{ABE} must be positive values), which means there are no positive integer solutions for equation (1-1). In fact the results of this program include the possible positive integer solutions, so if there is a contradiction then (1-1) can not have positive integer solutions. Obviously the larger of z^n then ABE, CDE are almost lines, but for positive integers that could lead to negative values of S_{CDE}, S_{ABE}. For $f(z,x) = \frac{BH}{DI}$, the program shows that $f(z,x) = \frac{BH}{DI} \to 1$, which means $\frac{BD}{AC} \to \frac{1}{2}$ when $n \ge 3$.

3. Conclusion

In this paper we first prove there are no positive integer solutions for equation (1-1) when $\frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} \le 1$, and then prove with the increasing of *x* the conclusion still holds when $\frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} > 1$ under the assumption of $z < 10^6, n = 3$. And when $n \ge 7$ there are no positive integer solutions for equation (1-1).