Ordu University Journal of Science and Technology / Ordu Üniversitesi Bilim ve Teknoloji Dergisi

Ordu Üniv. Bil. Tek. Derg., 2017; 7(2): 330-339
Ordu Univ. J. Sci. Tech., 2017; 7(2): 330-339
e-ISSN: 2146-6459

Research Article / Araştırma Makalesi

On Neutrosophic Continuity

Süleyman ŞENYURT[1*], Gülşah KAYA[2]

[1] Ordu University, Faculty of Arts and Sciences, Department of Mathematics, Ordu, Turkey

(Recived Date / Geliş Tarihi: 06.06.2017; Accepted Date / Kabul Tarihi: 19.10.2017)

Abstract

In this paper, we defined the neutrosophic continuous function, neutrosophic open function, neutrosophic closed function and neutrosophic homeomorphism on neutrosophic topological spaces. Then, we give some characteristics of these functions; neutrosophic closed function is a neutrosophic continuous function, neutrosophic open function is a neutrosophic continuous function.

***Keywords*:** Neutrosophic set, neutrosophic topological space, neutrosophic continuous function, neutrosophic open function, neutrosophic homeomorphism.

Neutrosophic Süreklilik Üzerine

Öz

Bu çalışmada, neutrosophic topolojik uzaylarda neutrosophic sürekli fonksiyon, neutrosophic açık fonksiyon, neutrosophic kapalı fonksiyon ve neutrosophic homeomorfizm tanımlandı. Daha sonra, bu fonksiyonların bazı karakteristik özellikleri hakkında bilgi verildi.

***Anahtar Kelimeler*:** Neutrosophic küme, neutrosophic topolojik uzay, neutrosophic sürekli fonksiyon, neutrosophic açık fonksiyon, neutrosophic homeomorfizm.

1. Introduction

The concept of neutrosophic sets was first introduced by Smarandache (Smarandache, 2005), as a generalization of intuitionistic fuzzy sets (Atanassov, 1986) where we have the degree of membership, the degree of indeterminacy and the degree of non-membership of each element in X. After the introduction of the neutrosophic sets, neutrosophic set operations have been investigated. Topology of neutrosophic sets have been studied intensively by researchers, such as Samarandache (Samarandache, 2002), Lupianez (Lupianez, 2008), (Lupianez, 2009(1)), (Lupianez, 2009(2)) and (Lupianez, 2010).

* *Corresponding Author / Sorumlu Yazar: senyurtsuleyman@hotmail.com*

4. $cl(A^c) = (int(A))^c$,
5. $cl(A) = int(A) \cup fr(A)$.

Definition 2.3 *Let* X *and* Y *be two non empty set,* $f : X \to Y$ *be a function,* $A \in N(X)$ *and* $B \in N(Y)$. *Then, we have* (Salama ve ark., 2014).

1. Image of A under f is defined by

$$f(A) = \{\langle y, f(\mu_A)(y), (1 - f(1-\sigma_A))(y), (1 - f(1-\nu_A))(y)\rangle : y \in Y\}$$

where

$$f(\mu_A)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_A(x), & f^{-1}(y) \neq \emptyset, \\ 0, & f^{-1}(y) = \emptyset, \end{cases}$$

$$(1 - f(1-\sigma_A))(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \sigma_A(x), & f^{-1}(y) \neq \emptyset, \\ 1, & f^{-1}(y) = \emptyset, \end{cases}$$

$$(1 - f(1-\nu_A))(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \nu_A(x), & f^{-1}(y) \neq \emptyset, \\ 1, & f^{-1}(y) = \emptyset. \end{cases}$$

2. Pre-image B under f is defined by

$$f^{-1}(B) = \{\langle x, f^{-1}(\mu_B)(x), f^{-1}(\sigma_B)(x), f^{-1}(\nu_B)(x)\rangle : x \in X\}.$$

Theorem 2.2 *Let* $f : X \to Y$ *be a function,* $A_1, A_2 \in N(X)$ *and* $B_1, B_2 \in N(Y)$. *Then followings are provided:* (Salama ve ark., 2014).

1. If $A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_2)$,
2. $A \subseteq f^{-1}(f(A))$ (If f is an injective function, then equality holds.),
3. If $B_1 \subseteq B_2$, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$,
4. $f(f^{-1}(B)) \subseteq B$ (If f is surjective function, then equality holds.) (If f is a injective function, then equality holds).

$(\Leftarrow)$: Let $A \in \kappa(\tau)$. Then we have $f^{-1}(A) \in \kappa(\tau)$. $f^{-1}(A)$ set is neutrosophic closed set because of f^{-1} is neutrosophic closed. Thence, f^{-1} function is neutrosophic continuous.

Theorem 3.3 *Let* (X,τ) *and* (Y,σ) *be two neutrosophic topological spaces and* $f: X \to Y$ *be a function. Then,* f *is a neutrosophic continuous function if and only if* $f(\mathrm{cl}(A)) \subseteq \mathrm{cl}(f(A))$ *for all* $A \in \mathrm{N}(X)$.

Proof. $(\Rightarrow)$: Let $A \in \mathrm{N}(X)$ and f be a neutrosophic continuous function.
From Theorem 2.1, we know that

$$A \subseteq \mathrm{cl}(A) \Rightarrow f(A) \subseteq \mathrm{cl}(f(A))$$

Then, apploying 2 Theorem 2.2 , we have

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\mathrm{cl}(f(A))) \text{ and } \mathrm{cl}(A) \subseteq \mathrm{cl}(f^{-1}(\mathrm{cl}(f(A)))).$$

Because f *is a neutrosophic continuous function and* $\mathrm{cl}(f(A))$ is a neutrosophic closed set, we know $\mathrm{cl}(f^{-1}(\mathrm{cl}(f(A))) = f^{-1}(\mathrm{cl}(f(A)))$.
Hence, $f(\mathrm{cl}(A) \subseteq \mathrm{cl}(f(A))$.

$(\Leftarrow)$: Let $f(\mathrm{cl}(A)) \subseteq \mathrm{cl}(f(A))$ for all $A \in \mathrm{N}(X)$. $F \in \kappa(\sigma)$ is given. Then, $\mathrm{cl}(f(f^{-1}(F))) \subseteq \mathrm{cl}(F) = F$ and so $f^{-1}(F) \in \kappa(\tau)$. From Theorem 3.2, f is a neutrosophic continuous function.

Theorem 3.4 *Let* (X,τ) *and* (Y,σ) *be two neutrosophic topological spaces,* $f: X \to Y$ *be a function. Then,* f *is a neutrosophic continuous function if and only if* $\mathrm{cl}(f^{-1}(B)) \subseteq f^{-1}(\mathrm{cl}(B))$ *for all* $B \in \mathrm{N}(Y)$.

Proof. $(\Rightarrow)$: Let $B \in \mathrm{N}(Y)$ and f be a neutrosophic continuous fonction. From Theorem 2.2 and Theorem 2.1, we have $f^{-1}(B) \subseteq f^{-1}(\mathrm{cl}(B))$. Then, $\mathrm{cl}(f^{-1}(B)) \subseteq \mathrm{cl}(f^{-1}(\mathrm{cl}(B)))$. Because we know $\mathrm{cl}(B) \in \kappa(\sigma)$ by Theorem 3.2, $f^{-1}(\mathrm{cl}(B)) \in \kappa(\tau)$. Thus,

$$\mathrm{cl}(f^{-1}(B)) \subseteq \mathrm{cl}(f^{-1}(\mathrm{cl}(B))) = f^{-1}(\mathrm{cl}(B)).$$

$(\Leftarrow)$: Let $\mathrm{cl}(f^{-1}(B)) \subseteq f^{-1}(\mathrm{cl}(B)$ for all $B \in \mathrm{N}(Y)$. $F \in \kappa(\sigma)$ is given. Then,

$$\mathrm{cl}(f^{-1}(F)) \subseteq f^{-1}(\mathrm{cl}(F)) = f^{-1}(F).$$

From Theorem 3.2, f is a neutrosophic continuous function.

Theorem 3.5 *Let* (X,τ) *and* (Y,σ) *be two neutrosophic topological spaces,* $f:(X,\tau) \to (Y,\sigma)$ *be a bijective function.* f *is neutrosophic continuous if and only if*

$int(f(A)) \subseteq f(int(A))$ *for all* $A \in N(X)$.

Proof. $(\Rightarrow)$: Let, $A \in N(X)$ and f be a bijective and neutrosophic continuous function. $f(A) = B$ is given. From Theorem 2.2 and Theorem 2.1 we know that $f^{-1}(int(B)) \subseteq f^{-1}(B)$. Since f is an injective function we know $f^{-1}(B) = A$, so that $f^{-1}(int(B)) \subseteq A$. Therefore, $int(f^{-1}(int(B))) \subseteq int(A)$. Here, $f^{-1}(int(B)) \in \tau$ and $f^{-1}(int(B)) \subseteq int(A)$ then, $f(f^{-1}(int(B)) \subseteq f(int(A))$. Since f is a surjective function we know that $f(f^{-1}(int(B))) = int(B)$. Hence, $int(f(A)) \subseteq f(int(A))$.

$(\Leftarrow)$: Let $int(f(A)) \subseteq f(int(A))$ for all $A \in N(X)$. Because $V \in \sigma$, f is surjective we know that $V = int(V) = int(f(f^{-1}(V))) \subseteq (f(int(f^{-1}(V)))$. $f^{-1}(V) \subseteq int(f^{-1}(V))$ then $f^{-1}(V) \in \tau$ by injective f. Hence, f is a neutrosophic continuous function.

Theorem 3.6 *Let* (X,τ) *and* (Y,σ) *be two neutrosophic topological spaces and* $f: X \to Y$ *be a function. Then,* f *is a neutrosophic continuous function if and only if* $f^{-1}(int(B)) \subseteq int(f^{-1}(B))$ *for all* $B \in N(Y)$.

Proof. $(\Rightarrow)$: Let $B \in N(Y)$ and f be a neutrosophic continuous function.
$int(B) \subseteq B \Rightarrow f^{-1}(int(B)) \subseteq f^{-1}(B) \Rightarrow int(f^{-1}(int(B)) \subseteq int(f^{-1}(B))$
Since $int(B) \in \sigma$ and $f^{-1}(int(B)) \in \tau$. So that, $int(f^{-1}(int(B))) = f^{-1}(int(B)) \subseteq int(f^{-1}(B))$.

$(\Leftarrow)$: Let $f^{-1}(int(B)) \subseteq int(f^{-1}(B))$ for all $B \in N(Y)$ and $G \in \tau$. Then, $f^{-1}(G) \subseteq int(f^{-1}(G))$ and $f^{-1}(G) = int(f^{-1}(G))$ so that $f^{-1}(G) \in \tau$. Hence f is a neutrosophic continuous function.

Theorem 3.7 *Let* (X,τ) *and* (Y,σ) *be two neutrosophic topological spaces and* $f: X \to Y$ *be a bijective function. Then* f *is a neutrosophic continuous function if and only if* $f(fr(A)) \subseteq fr(f(A))$ *for all* $A \in N(X)$.

Proof. $(\Rightarrow)$: Let f is a bijective and neutrosophic continuous function and $A \in N(X)$. From Definition 2.2, we know that $fr(A) = cl(A) \cap (int(A))^c$. Therefore, from Theorem 3.2, $f(int(A)) \subseteq int(f(A))$ and from Theorem 3.5 we find $f(cl(A)) \subseteq cl(f(A))$. Hence,

$$f(fr(A)) = f(cl(A) \cap (int(A))^c)$$

$$\subseteq f(\mathrm{c}l(A)) \cap f((\mathrm{i}nt(A))^c)$$
$$= f(\mathrm{c}l(A)) \cap f((\mathrm{i}nt(A)))^c$$
$$= \mathrm{f}r(f(A))$$

$(\Leftarrow)$: Let $f(\mathrm{f}r(A)) \subseteq \mathrm{f}r(f(A))$ for all $A \in \mathcal{N}(X)$.

$$f(\mathrm{c}l(A)) = f(A) \cup (\mathrm{f}r(A))$$
$$= f(A) \cup f((\mathrm{f}r(A))$$
$$\subseteq f(A) \cup \mathrm{f}r(f(A))$$
$$= \mathrm{c}l(f(A))$$

By Theorem 3.3 we find f is a neutrosophic continuous function.

Theorem 3.8 *Let* (X,τ) *and* (Y,σ) *be two neutrosophic topological spaces and* $f: X \to Y$ *be a bijective function. Then,* f *is a neutrosophic continuous function if and only if* $\mathrm{f}r(f^{-1}(B)) \subseteq f^{-1}(\mathrm{f}r(B))$ *for all* $B \in \mathcal{N}(Y)$.

Proof. $(\Rightarrow)$: Let f is a bijective and neutrosophic continuous function and $B \in \mathcal{N}(Y)$. By Theorem 3.4 and Theorem 3.6, we know that $\mathrm{c}l(f^{-1}(B)) \subseteq f^{-1}(\mathrm{c}l(B))$ and $f^{-1}(\mathrm{i}nt(B)) \subseteq \mathrm{i}nt(f^{-1}(B))$.

$$f^{-1}(\mathrm{f}r(B)) = f^{-1}(\mathrm{c}l(B) \cup (\mathrm{i}nt(B))^c)$$
$$= f^{-1}(\mathrm{c}l(B)) \cup f^{-1}((\mathrm{i}nt(B))^c)$$
$$= f^{-1}(\mathrm{c}l(B)) \cap f(^{-1}(\mathrm{i}nt(B)))^c$$

From Theorem 3.3 and Theorem 3.6 we know $\mathrm{c}l(f^{-1}(B)) \subseteq f^{-1}(\mathrm{c}l(B))$ and $(\mathrm{i}nt(f^{-1}(B)))^c \subseteq (f^{-1}(\mathrm{i}nt(B)))^c$; hence, $f^{-1}(B) \subseteq f^{-1}(\mathrm{f}r(B))$.

$(\Leftarrow)$: Let $\mathrm{f}r(f^{-1}(B)) \subseteq f^{-1}(\mathrm{f}r(B))$ for all $B \in \mathcal{N}(Y)$. Then,

$$\mathrm{f}r(f^{-1}(B)) \cup f^{-1}(B) \subseteq f^{-1}(\mathrm{f}r(B)) \cup f^{-1}(B)$$

Hence,

$$\mathrm{c}l(f^{-1}(B)) \subseteq f^{-1}(\mathrm{f}r(B) \cup B)$$
$$= f^{-1}(\mathrm{c}l(B))$$

From Theorem 3.4, f is a neutrosophic contiuous function.

Definition 3.2 *Let* (X,τ) *and* (Y,σ) *be two neutrosophic topological spaces and*

$f:(X,\tau)\to(Y,\sigma)$ *be a function.*

1. If $f(U)\in\sigma$ for all $U\in\tau$, then f is called a neutrosophic open function.
2. If $f(F)\in\kappa(\sigma)$ for all $F\in\kappa(\tau)$, then f is called a neutrosophic closed function.

Example 3.3 *Let* $X=\{x_1,x_2\}$, $Y=\{y_1,y_2\}$, $A\in \mathrm{N}(X)$, $B\in \mathrm{N}(Y)$ *and* $f:X\to Y$ *be a function such that*

$A=\{\langle x_1,0.5,0.4,0.3\rangle,\langle x_2,0.7,0.8,0.2\rangle\}$,

$B=\{\langle y_1,0.1,0.7,0.6\rangle,\langle y_2,0.8,0.9,0.5\rangle\}$,

and

$f(x_1)=y_2$ and $f(x_2)=y_1$.

Then, $\tau=\{\tilde{X},\tilde{\varnothing},A\}$ and $\sigma=\{\tilde{Y},\tilde{\varnothing},B\}$ are two neutrosophic topological spaces. Therefore, f is a neutrosophic open function. But f is not a neutrosophic closed function.

Theorem 3.9 *Let* (X,τ) *and* (Y,σ) *be two neutrosophic topological spaces and* $f:X\to Y$ *be a neutrosophic continuous function. Then,* f *is a neutrosophic open function if and only if* $f(\mathrm{i}nt(A))$ ô $\mathrm{i}nt(f(A))$ *for all* $A\in N(X)$.

Proof. $(\Rightarrow)$: Let $f:(X,\tau)\to(Y,\sigma)$ be a neutrosophic continuous function and $A\in N(X)$

$\mathrm{i}nt(A)$ ô A

$f(\mathrm{i}nt(A))$ ô $f(A)$

$\mathrm{i}nt(f(\mathrm{i}nt(A)))$ ô $\mathrm{i}nt(f(A))$

$f(\mathrm{i}nt(A))$ ô $\mathrm{i}nt(f(A))$.

$(\Leftarrow)$: Let $f(\mathrm{i}nt(A))$ ô $\mathrm{i}nt(f(A))$ for all $A\in N(X)$. If $G\in X$ neutrosophic subset is a neutrosophic open function, $f(G)$ is a subset of $f(\mathrm{i}nt(G))$. So, $f(G)$ is a neutrosophic open function.

Theorem 3.10 *Let* (X,τ) *and* (Y,σ) *be two neutrosophic topological spaces an* $f:X\to Y$ *be a bijective function. Then,* f^{-1} *is a neutrosophic continuous function if and only if* f^{-1} *is a neutrosophic open function.*

Proof. ($\Rightarrow$): Let $U\in\sigma$ and f be a neutrosophic continuous function. Then, we have $f^{-1}(G)=g(G)$. Hence, f^{-1} is a neutrosohic open function.

$(\Leftarrow)$: Let f^{-1} be a neutrosophic open function. Then, we have $f^{-1}(U)\in\tau$ for all $U\in\sigma$. So, f^{-1} is a neutrosophic continuous function.

Theorem 3.11 Let (X,τ) *and* (Y,σ) *be two neutrosophic topological spaces an* $f:X\to Y$ *be a bijective function. Then,* f^{-1} *is a neutrosophic continuous function if and only if* f^{-1} *is a neutrosophic closed function.*

Proof. ($\Rightarrow$): Let $A\in\kappa(\sigma)$. Then, $f^{-1}(A)\in\kappa(\tau)$ is a neutrosophic continuous function, so $f^{-1}(A)$ is a neutrosophic closed function. From Theorem 3.2, we have f^{-1} is a neutrosophic closed function.
($\Leftarrow$): Let $A\in\kappa(\sigma)$. Then $f^{-1}(A)\in\kappa(\tau)$. f^{-1} is a neutrosophic closed function, so $f^{-1}(A)$ is a neutrosophic closed function. From Theorem 3.2 f *is* a neutrosophic continuous function.

Definition 3.3 *Let* (X,τ) *and* (Y,σ) *be two neutrosophic topological spaces and* $f:X\to Y$ *be a function. If following conditions hold, then* f *is called neutrosophic homeomorphism*

1. f is a bijective function,
2. f is a neutrosophic continuous function,
3. f^{-1} is a neutrosophic continuous function.

Example 3.5 *Let* $X=\{x_1,x_2\}$, $Y=\{y_1,y_2\}$ *and* $A\in\mathrm{N}(X)$ *and* $B\in\mathrm{N}(Y)$ *such that*

$A=\{\langle x_1,0.5,0.4,0.3\rangle,\langle x_2,0.7,0.8,0.2\rangle\}$,

$B=\{\langle y_1,0.1,0.7,0.6\rangle,\langle y_2,0.8,0.9,0.5\rangle\}$.

Then, $\tau=\{\tilde{X},\tilde{\emptyset},A\}$ and $\sigma=\{\tilde{Y},\tilde{\emptyset},B\}$ are two neurosophic topology over X and Y, respectively. Moreover, let $f:X\to Y$ be a function such that $f(x_1)=y_1$ and $f(x_2)=y_2$. It can be seen clearly that f is a neutrosophic homeomorphism.

Theorem 3.12 *Let* (X,τ) *and* (Y,σ) *be two neutrosophic topological spaces and* $f:X\to Y$ *be a bijective function. Then f is a neutrosophic homeomorphism if and only if f is a neutrosophic continuous and neutrosophic closed function.*

Proof. Let *f* is a neutrosophic homeomorphism. From Definition 3.3 we know f is a neutrosophic continuous function. Then, from Theorem 3.10 we have f^{-1} is a neutrosophic closed function. So, $\left(f^{-1}\right)^{-1}=f$ is a neutrosophic closed function.

References

1. Atanassov, K., (1986). *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 20, 87–96
2. Karataş, S. and Kuru, C., (2016). *Neutrosophic topology*, Neutrosophic Sets and Systems, 13, 90–96
3. Lupiáñez, F. G., (2008). *On neutrosophic topology*, The International Journal of Systems and Cybernetics, 37(6), 797–800
4. Lupiáñez, F. G., (2009). *Interval neutrosophic sets and topology*, The International Journal of Systems and Cybernetics, 38(3-4), 621–624,
5. Lupiáñez, F. G., (2009). *On various neutrosophic topologies*, The International Journal of Systems and Cybernetics, 38(6), 1009–1013,
6. Lupiáñez, F. G., (2010). *On neutrosophic paraconsistent topology*, The International Journal of Systems and Cybernetics, 39(4), 598–601
7. Salama, A.A., Smarandache, F. and Valeri, K., (2014). *Neutrosophic Closed Set and Neutrosophic Continuous Functıons,* Neutrosophic Sets and Systems, 4, 4–8,
8. Smarandache, F., (2005). *Neutrosophic set - a generalization of the intuitionistic fuzzy set*, International Journal of Pure and Applied Mathematics, 24(3), 287–297
9. Smarandache, F., (2002). *Neutrosophy and neutrosophic logic, first international conference on neutrosophy, neutrosophic logic, set, probabilty, and statistics,* University of New Mexico, Gallup, NM 87301. USA