Multiple criteria decision making approach with multi-valued neutrosophic linguistic normalized weighted Bonferroni mean Hamacher operator

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Abstract The neutrosophic set and linguistic term set are widely applied in recently years. Motivated by the advantages of them, we combine the multi-valued neutrosophic set and linguistic set and define the concept of the multi-valued neutrosophic linguistic set (MVNLS). Furthermore, Hamacher operation is an extension of the Algebraic and Einstein operation. Additionally, the normalized weighted Bonferroni mean (NWBM) operator can consider the weight of each argument as well as capture the interrelationship of different arguments. Therefore, the combination of NWBM operator and Hamacher operation is more valuable and agile. Firstly, MVNLS and multi-valued neutrosophic linguistic number (MVNLN) are defined, then some new operational rules of MVNLNs on account of Hamacher operations are developed, and the comparison functions for MVNLNs are given. Secondly, multi-valued neutrosophic linguistic normalized weighted Bonferroni mean Hamacher operator (MVNLNWBMH) is proposed, and a number of expected characteristics of new operator are investigated. Meanwhile, some special cases of different parameters p, q and ε are analyzed. Thirdly, the approach utilizing the MVNLNWBMH operator is introduced to manage multiple criteria decision making issue (MCDM) in multi-valued neutrosophic linguistic environment. Ultimately, a practical example is presented and a comparative analysis is carried out, which validate the effectiveness and generalization of the novel approach.

Keywords Multi-valued neutrosophic linguistic, NWBM, Hamacher, Multiple criteria decision making

1 Introduction

In real world, due to the complexity of decision information, the fuzzy theory has attracted widespread attentions and has been developed in various fields. Zaheh [1] firstly proposed the notion of fuzzy sets (FSs). Then, Atanassov [2] introduced the intuitionistic fuzzy sets (IFSs), which overcome the weakness of non-membership degrees. Subsequently, in order to address the hesitation degree of decision-makers, Torra [3] defined hesitant fuzzy sets (HFSs). Fuzzy set theory has gained well promoted, but it still cannot manage the inconsistent and indeterminate information. Under this circumstance, Smarandache [4] proposed Neutrosophic Sets (NSs), whose indeterminacy degree is independent on both true and false membership. NS is an extension of IFS, and makes decision-makers express their preference more accurately, so some achievements on NSs and its extensions have been undertaken. Some various concepts of different NSs are defined. For example, Smarandache [54] and Wang et al. [5] introduced single-valued neutrosophic sets (SVNs) to facilitate its application. Ye [6] pointed out the concept of simplified neutrosopic sets (SNSs).Wang [7] developed the concept of interval neutrosopic sets (INSs). However, under certain conditions, the decision makers likely give different evaluation numbers for expressing their hesitant. Subsequently, the definition of single-valued neutrosophic hesitant fuzzy sets (SVNHFSs) was firstly proposed by Ye [8] in 2014, then Wang [9] also proposed multi-valued neutrosophic sets (MVNSs) in 2015. Actually, the notions of SVNHFSs and MVNSs are equal. For simplicity, we adapt the term of MVNSs in this paper.

Mathematical Problems in Engineering.

On the other hand, the aggregation operators, comparison method for Neutrosophic numbers are also been studied. For SVNSs, Liu [10] employed NWBM operator to solve multiple criteria problem in single-valued neutrosophic environment. Ye [11] gave the definitions of cross-entropy and correlation coefficient. For INSs, Zhang [12] developed some aggregation operators. Liu [13] not only provided the definition of interval neutrosophic hesitant fuzzy sets (INHFSs), but also discussed the generalized hybrid weighted average operator. Broumi and Smarandache [14-16] studied the correlation coefficients, cosine similarity measure and some new operations. Ye [17] proposed similarity measures between interval neutrosophic sets. For MVNSs, Ye [8] developed SVNHFWA and SVNHFWG operators for MCDM problem. Peng [18, 19] extended power aggregation operators and defined some outranking relations under MVNS environment. Ji et al. [20] analyzed a novel TODIM method for MVNSs.

In real life, owing to the ambiguity of decision makers' thinking, people prefer to utilize linguistic variables for describing their assessment value rather than the quantization value. Therefore, linguistic variable has attracted widespread attention in the field of MCDM. The linguistic variable was firstly proposed by Zadeh [21] and applied for the fuzzy reasoning. After that, a series of works on it have been made, Wang [22-24] presented a new approach in view of hesitant fuzzy linguistic information. Meng [25] developed linguistic hesitant fuzzy sets and studied hybrid weighted operator. Tian [26] defined gray linguistic weighted Bonferroni mean operator for MCDM.

In order to indicate the true, indeterminate and false extents concerning a linguistic term, the NSs and linguistic set (LS) are combined. Several neutrosophic linguistic sets and its corresponding operator are defined, for example single valued or simplified neutrosophic linguistic sets and trapezoid linguistic sets [27-30], interval neutrosophic certain or uncertain linguistic sets [31-33].However, due to the hesitancy of people's thinking, the true of a linguistic term may be given several values, and the case is similar to the false and indeterminate extents. The existing literatures don't consider this perspective. Therefore, the multi-valued neutrosophic linguistic set (MVNLS) and multi-valued neutrosophic linguistic number (MVNLN) in this article are proposed in order to better express the information.

Aggregation operator which can fuse multiple arguments into a single comprehensive value is an important tool for MCDM problem. Many researchers have developed some efficient operators [34-41], for instance, the weighted geometric average (WGA) or averaging (WA) operator, prioritized aggregation (PA) operator, Maclaurin symmetric mean and Bonferroni mean (BM) operator. BM operator was originally defined by Bonferroni [42], which has attracted widespread attentions because of its characteristics of capture interrelationship among arguments. Some achievements have been made on it [43-49].In order to aggregate neutrosophic linguistic information, some researches on aggregation operators under neutrosophic linguistic and neutrosophic uncertain linguistic environments are also been applied [27-33,50].Until to now, BM and NWBM fail to accommodate aggregation information for multi-valued neutrosophic linguistic environment. Motivated by this limitation, we will extend the NWBM operator to MVNLS in this article.

T-norms and t-conorms are two functions that satisfy certain conditions respectively. The Archimedean t-conorms and t-norms are well-known, which include algebraic, Einstein and Hamacher, Hamacher operation is an extension of algebraic and Einstein. Generally, the algebraic operators are commonly, there are also a few aggregation operations based on Einstein operations. Due to Hamacher operator is more general, Liu [51, 52] discussed the Hamacher operational rules. So far, there is no research for MVNLS based on Hamacher operations. Since it is better for MVNLS to depict the actual situation, NWBM operator can capture the interrelationship among arguments, and Hamacher operations are more general, it is of great meaning to study the NWBM Hamacher operators under multi-valued neutrosophic linguistic environment for MCDM problems.

The main purposes of the paper are presented in the following:

1. To be better express people's hesitant, combining the MVNS and LS, we give the notions of MVNLS and MVNLN, besides, the score, accuracy and certainty functions are also investigated to compare MVNLNs.

Mathematical Problems in Engineering.

2. Due to the generalization of Hamacher operational rules, we define new operations of MVNLNs based on Hamacher operational rules, and discuss their operational relations.
3. The NWBM considering the interrelationship of different arguments has gained widespread concerns, we extend NWBM operator to MVNLN environment, the MVNLNWBMH operator is defined, moreover, some desirable characteristics are also studied.
4. In order to verify the effectiveness, an example for MCDM problem utilizing MVNLNWBMH operator is illustrated and conduct a comparative analysis. We also analyze the influences of different parameter values for the final outcomes, the results demonstrate the operator proposed is more general and flexible.

The article is arranged in this way. In Section 2, we review a number of notions and operations for MVNS, LS, NWBM operator and Hamacher. In Section 3, we propose the definitions of MVNLS and MVNLN, and develop the operations of MVNLNs on the basis of Hamacher t-conorms and t-norms. Meanwhile, the Algebraic as well as Einstein for MVNLNs are also presented, which are special cases of Hamacher operation. Moreover, the comparison method of MVNLNs is also defined. In Section 4, we propose the MVNLNWBMH operator and investigate its properties. Furthermore, when corresponding parameters are assigned different values, the special examples are also discussed. In Section 5, we establish the MCDM procedure on account of the proposed aggregation operators with MVNLS information. Section 6 presents a concrete example, as well as a comparison analysis is provided to show the practicability utilizing our method. Finally, in Section 7, some results are presented.

2 Preliminaries

Some notions and operation are introduced in this section, which will be useful in the latter analysis.

2.1 Linguistic term sets

Suppose that $S = \left\{s_1, s_2, \cdots, s_\iota\right\}$ is an ordered and finite linguistic set, in which s_j denotes a linguistic variable value and ι is an odd value. When ι is equal to seven, the corresponding linguistic set are provided in the following:

$$S = \left\{s_1, s_2, s_3, s_4, s_5, s_6, s_7\right\} = \left\{\text{extremely poor,very poor,poor,medium,good,very good,extremely good}\right\}.$$

In order to avoid the linguistic information loss, the set above is expanded, that is a contiguous set, $\bar{S} = \left\{s_\alpha \left|\alpha \in R\right.\right\}$.

Definition 1 [53] Let s_i and s_j be any two linguistic variables, the corresponding operations are presented:

(1) $\lambda s_i = s_{\lambda \times i}, \lambda \geq 0$;

(2) $s_i \oplus s_j = s_{i+j}$;

(3) $s_i \otimes s_j = s_{i \times j}$;

(4) $\left(s_i\right)^\lambda = s_{i^\lambda}$.

2.2 Multi-valued neutrosophic sets

Definition 2 [8,9] Suppose that X is a collection of objects, MVNSs A on X is defined by

$$A = \left\{\left\langle x, \tilde{T}_A(x), \tilde{I}_A(x), \tilde{F}_A(x)\right\rangle \left| x \in X\right.\right\},$$

Where $\tilde{T}_A(x) = \left\{\gamma \left|\gamma \in \tilde{T}_A(x)\right.\right\}$, $\tilde{I}_A(x) = \left\{\delta \left|\delta \in \tilde{I}_A(x)\right.\right\}$, $\tilde{F}_A(x) = \left\{\eta \left|\eta \in \tilde{F}_A(x)\right.\right\}$,

$\tilde{T}_A(x)$, $\tilde{I}_A(x)$, and $\tilde{F}_A(x)$ are three collections of crisp numbers belonging to $\left[0, 1\right]$, representing the probable true-membership degree, indeterminacy-membership degree and falsity- membership degree, where x in X belonging to A, respectively, satisfying these conditions $0 \le \gamma, \delta, \eta \le 1$, and $0 \le \sup \tilde{T}_A(x) + \sup \tilde{I}_A(x) + \sup \tilde{F}_A(x) \le 3$. If there is only one element in X, A is indicated by the three tuple $A = \left\langle \tilde{T}_A(x), \tilde{I}_A(x), \tilde{F}_A(x) \right\rangle$, that is known as a multi-valued neutrosophic number (MVNN).Generally, MVNSs is considered as the generalizations of the other sets, such as FSs, IFSs, HFSs, DHFs, and SVNSs.

2.3 Normalized weighted Bonferroni mean

Definition 3 [42] Let $p, q \ge 0$, as well as $a_i (i = 1, 2, \cdots n)$ be a set of nonnegative values, then the BM is defined as

$$BM^{p,q}(a_1, a_2, \cdots, a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1, \\ i \ne j}}^{n} \left(a_i^p a_j^q \right) \right)^{\frac{1}{p+q}}$$

Definition 4 [45]Let $p, q \ge 0$, and $a_i (i = 1, 2, \cdots n)$ be a set of nonnegative values, and the corresponding NWBM can be expressed as below:

$$NWBM^{p,q}(a_1, a_2, \cdots, a_n) = \left(\bigoplus_{\substack{i,j=1, \\ i \ne j}}^{n} \frac{w_i w_j}{1-w_i} \left(a_i^p \otimes a_j^q \right) \right)^{\frac{1}{p+q}}$$

Where $w = (w_1, w_2, \cdots, w_n)$ represents the corresponding weighted vector of $a_i \left(i = 1, 2, \cdots, n \right)$, satisfying $w_i \ge 0$ and $\sum_{i=1}^{n} w_i = 1$. The weight vector can be given by decision-makers in real problem.

Obviously, the NWBM operator possesses a few characteristics such as commutativity, reducibility, monotonicity, boundedness, and idempotency.

2.4 Hamacher operations

We know aggregation operator is given in accordance with different t-norms and t-conorms, there are some exceptional circumstances listed in the following:

(1) Algebraic t-norm and t-conorm

$$a \otimes b = ab, a \oplus b = a + b - ab;$$

(2) Einstein t-norm and t-conorm

$$a \otimes b = \frac{ab}{1 + \left(1 - a\right) \times \left(1 - b\right)}, a \oplus b = \frac{a + b}{1 + ab};$$

(3) Hamacher t-norm and t-conorm

$$a \otimes b = \frac{ab}{\varepsilon + \left(1 - \varepsilon\right)\left(a + b - ab\right)}, a \oplus b = \frac{a + b - ab - (1 - \varepsilon)ab}{1 - (1 - \varepsilon)ab}, \varepsilon > 0$$

Mathematical Problems in Engineering.

In special, when $\varepsilon = 1, \varepsilon = 2$, the Algebraic and Einstein operations are the simplifications of Hamacher t-norm and t-conorm.

3 Multi-valued neutrosophic linguistic set

3.1 MVNLS and its Hamacher operations

Definition 5 Let X be a set of points, an MVNLS A in X is defined as follows:

$$A = \left\{\left\langle x, \left[s_{\theta(x)}, \left(\tilde{T}_A(x), \tilde{I}_A(x), \tilde{F}_A(x)\right)\right]\right\rangle \middle| x \in X\right\},$$

Where $s_{\theta(x)} \in S$, $\tilde{T}_A(x) = \left\{\gamma \middle| \gamma \in \tilde{T}_A(x)\right\}$, $\tilde{I}_A(x) = \left\{\delta \middle| \delta \in \tilde{I}_A(x)\right\}$, $\tilde{F}_A(x) = \left\{\eta \middle| \eta \in \tilde{F}_A(x)\right\}$,

$\tilde{T}_A(x)$, $\tilde{I}_A(x)$, and $\tilde{F}_A(x)$ are three sets of crisp values in $[0, 1]$, denoting three degrees of x in X belonging to $s_{\theta(x)}$, , that are true, indeterminacy and falsity, satisfying these conditions $0 \le \gamma, \delta, \eta \le 1$, and $0 \le \sup \tilde{T}_A(x) + \sup \tilde{I}_A(x) + \sup \tilde{F}_A(x) \le 1$.

Definition 6 Let $A = \left\{\left\langle x, \left[s_{\theta(x)}, \left(\tilde{T}_A(x), \tilde{I}_A(x), \tilde{F}_A(x)\right)\right]\right\rangle \middle| x \in X\right\}$ be an MVNLS, supposing there is only one element in X, then tuple $\left\langle s_{\theta(x)}, \left(\tilde{T}_A(x), \tilde{I}_A(x), \tilde{F}_A(x)\right)\right\rangle$ is depicted as a multi-valued neutrosophic linguistic number (MVNLN). For simplicity, the MVNLN can also be represented as

$$A = \left\{\left\langle s_{\theta(x)}, \left(\tilde{T}_A(x), \tilde{I}_A(x), \tilde{F}_A(x)\right)\right\rangle \middle| x \in X\right\}$$

Definition 7 Let $a_1 = \left\langle s_{\theta(a_1)}, \left(\tilde{T}(a_1), \tilde{I}(a_1), \tilde{F}(a_1)\right)\right\rangle$ and $a_2 = \left\langle s_{\theta(a_2)}, \left(\tilde{T}(a_2), \tilde{I}(a_2), \tilde{F}(a_2)\right)\right\rangle$ be two MVNLNs, and $\lambda > 0$, then the operations of MVNLNs can be defined on the basis of Hamacher operations.

(1) $a_1 \oplus a_2$

$$= \left\langle s_{\theta(a_1)+\theta(a_2)}, \left(\bigcup_{\gamma_1 \in \tilde{T}(a_1), \gamma_2 \in \tilde{T}(a_2)} \left\{\frac{\gamma_1 + \gamma_2 - \gamma_1\gamma_2 - (1-\varepsilon)\gamma_1\gamma_2}{1 - (1-\varepsilon)\gamma_1\gamma_2}\right\}, \bigcup_{\delta_1 \in \tilde{I}(a_1), \delta_2 \in \tilde{I}(a_2)} \left\{\frac{\delta_1\delta_2}{\varepsilon + (1-\varepsilon)(\delta_1 + \delta_2 - \delta_1\delta_2)}\right\}, \bigcup_{\eta_1 \in \tilde{F}(a_1), \eta_2 \in \tilde{F}(a_2)} \left\{\frac{\eta_1\eta_2}{\varepsilon + (1-\varepsilon)(\eta_1 + \eta_2 - \eta_1\eta_2)}\right\}\right)\right\rangle;$$

(2) $a_1 \otimes a_2$

$$= \left\langle s_{\theta(a_1)\times\theta(a_2)}, \left(\bigcup_{\gamma_1 \in \tilde{T}(a_1), \gamma_2 \in \tilde{T}(a_2)} \left\{\frac{\gamma_1\gamma_2}{\varepsilon + (1-\varepsilon)(\gamma_1 + \gamma_2 - \gamma_1\gamma_2)}\right\}, \bigcup_{\delta_1 \in \tilde{I}(a_1), \delta_2 \in \tilde{I}(a_2)} \left\{\frac{\delta_1 + \delta_2 - \delta_1\delta_2 - (1-\varepsilon)\delta_1\delta_2}{1 - (1-\varepsilon)\delta_1\delta_2}\right\}, \bigcup_{\eta_1 \in \tilde{F}(a_1), \eta_2 \in \tilde{F}(a_2)} \left\{\frac{\eta_1 + \eta_2 - \eta_1\eta_2 - (1-\varepsilon)\eta_1\eta_2}{1 - (1-\varepsilon)\eta_1\eta_2}\right\}\right)\right\rangle;$$

(3) $\lambda a_1 = \langle s_{\lambda\theta(a_1)},$

$\left(\bigcup_{\gamma_1\in\tilde{T}(a_1)}\left\{\frac{(1+(\varepsilon-1)\gamma_1)^{\lambda}-(1-\gamma_1)^{\lambda}}{(1+(\varepsilon-1)\gamma_1)^{\lambda}+(\varepsilon-1)(1-\gamma_1)^{\lambda}}\right\},\right.$

$\bigcup_{\delta_1\in\tilde{I}(a_1)}\left\{\frac{\varepsilon\delta_1^{\lambda}}{(1+(\varepsilon-1)(1-\delta_1))^{\lambda}+(\varepsilon-1)\delta_1^{\lambda}}\right\},$

$\left.\bigcup_{\eta_1\in\tilde{F}(a_1)}\left\{\frac{\varepsilon\eta_1^{\lambda}}{(1+(\varepsilon-1)(1-\eta_1))^{\lambda}+(\varepsilon-1)\eta_1^{\lambda}}\right\}\right)\rangle;$

(4) $a_1^{\lambda} = \langle s_{\theta^{\lambda}(a_1)},$

$\left(\bigcup_{\gamma_1\in\tilde{T}(a_1)}\left\{\frac{\varepsilon\gamma_1^{\lambda}}{(1+(\varepsilon-1)(1-\gamma_1))^{\lambda}+(\varepsilon-1)\gamma_1^{\lambda}}\right\},\right.$

$\bigcup_{\delta_1\in\tilde{I}(a_1)}\left\{\frac{(1+(\varepsilon-1)\delta_1)^{\lambda}-(1-\delta_1)^{\lambda}}{(1+(\varepsilon-1)\delta_1)^{\lambda}+(\varepsilon-1)(1-\delta_1)^{\lambda}}\right\},$

$\left.\bigcup_{\eta_1\in\tilde{F}(a_1)}\left\{\frac{(1+(\varepsilon-1)\eta_1)^{\lambda}-(1-\eta_1)^{\lambda}}{(1+(\varepsilon-1)\eta_1)^{\lambda}+(\varepsilon-1)(1-\eta_1)^{\lambda}}\right\}\right)\rangle.$

If $\varepsilon = 1$, then the operations based on Hamacher operational rules in Definition 7 will simplified to the Algebraic operational rules in the following:

(5) $a_1 \oplus a_2$

$= \langle s_{\theta(a_1)+\theta(a_2)},$

$\left(\bigcup_{\gamma_1\in\tilde{T}(a_1),\gamma_2\in\tilde{T}(a_2)}\{\gamma_1+\gamma_2-\gamma_1\gamma_2\},\right.$

$\bigcup_{\delta_1\in\tilde{I}(a_1),\delta_2\in\tilde{I}(a_2)}\{\delta_1\delta_2\},$

$\left.\bigcup_{\eta_1\in\tilde{F}(a_1),\eta_2\in\tilde{F}(a_2)}\{\eta_1\eta_2\}\right)\rangle;$

(6) $a_1 \otimes a_2$

$= \langle s_{\theta(a_1)\times\theta(a_2)},$

$\left(\bigcup_{\gamma_1\in\tilde{T}(a_1),\gamma_2\in\tilde{T}(a_2)}\{\gamma_1\gamma_2\},\right.$

$\bigcup_{\delta_1\in\tilde{I}(a_1),\delta_2\in\tilde{I}(a_2)}\{\delta_1+\delta_2-\delta_1\delta_2\},$

$\left.\bigcup_{\eta_1\in\tilde{F}(a_1),\eta_2\in\tilde{F}(a_2)}\{\eta_1+\eta_2-\eta_1\eta_2\}\right)\rangle;$

(7) $\lambda a_1 = \langle s_{\lambda\theta(a_1)},$

$\left(\bigcup_{\gamma_1\in\tilde{T}(a_1)}\{1-(1-\gamma_1)^{\lambda}\},\right.$

$\bigcup_{\delta_1\in\tilde{I}(a_1)}\{\delta_1^{\lambda}\},$

$\left.\bigcup_{\eta_1\in\tilde{F}(a_1)}\{\eta_1^{\lambda}\}\right)\rangle;$

(8) $a_1^{\lambda} = \Big\langle s_{\theta^{\lambda}(a_1)},$
$\Big(\bigcup_{\gamma_1 \in \tilde{T}(a_1)} \{\gamma_1^{\lambda}\},$
$\bigcup_{\delta_1 \in \tilde{I}(a_1)} \{1-(1-\delta_1)^{\lambda}\},$
$\bigcup_{\eta_1 \in \tilde{F}(a_1)} \{1-(1-\eta_1)^{\lambda}\}\Big)\Big\rangle.$

Supposing $\tilde{T}(a_1)$, $\tilde{I}(a_1)$, $\tilde{F}(a_1)$, $\tilde{T}(a_2)$, $\tilde{I}(a_2)$, and $\tilde{F}(a_2)$ contain only one value, then the operations defined above can be reduced to the operations of SVNLNs based on Algebraic operations proposed by Ye [27].

If $\varepsilon = 2$, then the operations based on Hamacher operational rules in Definition 7 will simplified to the Einstein operations of MVNLNs presented below:

(9) $a_1 \oplus a_2$

$$= \Big\langle s_{\theta(a_1)+\theta(a_2)},$$
$$\Big(\bigcup_{\gamma_1 \in \tilde{T}(a_1), \gamma_2 \in \tilde{T}(a_2)} \left\{\frac{\gamma_1+\gamma_2}{1+\gamma_1\gamma_2}\right\},$$
$$\bigcup_{\delta_1 \in \tilde{I}(a_1), \delta_2 \in \tilde{I}(a_2)} \left\{\frac{\delta_1\delta_2}{2-\delta_1-\delta_2+\delta_1\delta_2}\right\},$$
$$\bigcup_{\eta_1 \in \tilde{F}(a_1), \eta_2 \in \tilde{F}(a_2)} \left\{\frac{\eta_1\eta_2}{2-\eta_1-\eta_2+\eta_1\eta_2}\right\}\Big)\Big\rangle;$$

(10) $a_1 \otimes a_2$

$$= \Big\langle s_{\theta(a_1)\times\theta(a_2)},$$
$$\Big(\bigcup_{\gamma_1 \in \tilde{T}(a_1), \gamma_2 \in \tilde{T}(a_2)} \left\{\frac{\gamma_1\gamma_2}{2-\gamma_1-\gamma_2+\gamma_1\gamma_2}\right\},$$
$$\bigcup_{\delta_1 \in \tilde{I}(a_1), \delta_2 \in \tilde{I}(a_2)} \left\{\frac{\delta_1+\delta_2}{1+\delta_1\delta_2}\right\},$$
$$\bigcup_{\eta_1 \in \tilde{F}(a_1), \eta_2 \in \tilde{F}(a_2)} \left\{\frac{\eta_1+\eta_2}{1+\eta_1\eta_2}\right\}\Big)\Big\rangle;$$

(11) $\lambda a_1 = \Big\langle s_{\lambda\theta(a_1)},$

$$\Big(\bigcup_{\gamma_1 \in \tilde{T}(a_1)} \left\{\frac{(1+\gamma_1)^{\lambda}-(1-\gamma_1)^{\lambda}}{(1+\gamma_1)^{\lambda}+(1-\gamma_1)^{\lambda}}\right\},$$
$$\bigcup_{\delta_1 \in \tilde{I}(a_1)} \left\{\frac{2\delta_1^{\lambda}}{(2-\delta_1)^{\lambda}+\delta_1^{\lambda}}\right\},$$
$$\bigcup_{\eta_1 \in \tilde{F}(a_1)} \left\{\frac{2\eta_1^{\lambda}}{(2-\eta_1)^{\lambda}+\eta_1^{\lambda}}\right\}\Big)\Big\rangle;$$

Mathematical Problems in Engineering.

(12) $a_1^{\lambda} = \Big\langle s_{\theta^{\lambda}(a_1)},$

$\Bigg(\bigcup_{\gamma_1 \in \tilde{T}(a_1)} \left\{\frac{2\gamma_1^{\lambda}}{(2-\gamma_1)^{\lambda} + \gamma_1^{\lambda}}\right\},$

$\bigcup_{\delta_1 \in \tilde{I}(a_1)} \left\{\frac{(1+\delta_1)^{\lambda} - (1-\delta_1)^{\lambda}}{(1+\delta_1)^{\lambda} + (1-\delta_1)^{\lambda}}\right\},$

$\bigcup_{\eta_1 \in \tilde{F}(a_1)} \left\{\frac{(1+\eta_1)^{\lambda} - (1-\eta_1)^{\lambda}}{(1+\eta_1)^{\lambda} + (1-\eta_1)^{\lambda}}\right\}\Bigg)\Big\rangle.$

Supposing $\tilde{T}(a_1)$, $\tilde{I}(a_1)$, $\tilde{F}(a_1)$, $\tilde{T}(a_2)$, $\tilde{I}(a_2)$, and $\tilde{F}(a_2)$ contain only one value, then the operations defined above can be reduced to the operations of SVNLNs based on Einstein operations.

Theorem 1 Let $a_1 = \left\langle s_{\theta(a_1)}, \left(\tilde{T}(a_1), \tilde{I}(a_1), \tilde{F}(a_1)\right)\right\rangle$, $a_2 = \left\langle s_{\theta(a_2)}, \left(\tilde{T}(a_2), \tilde{I}(a_2), \tilde{F}(a_2)\right)\right\rangle$, and $a_3 = \left\langle s_{\theta(a_3)}, \left(\tilde{T}(a_3), \tilde{I}(a_3), \tilde{F}(a_3)\right)\right\rangle$ be any three MVNLNs, and $\lambda, \lambda_1, \lambda_2 > 0$, then the properties below are correct:

(1) $a_1 \oplus a_2 = a_2 \oplus a_1$;

(2) $a_1 \otimes a_2 = a_2 \otimes a_1$;

(3) $\lambda\left(a_1 \oplus a_2\right) = \lambda a_1 \oplus \lambda a_2$;

(4) $\lambda_1 a_1 \oplus \lambda_2 a_1 = (\lambda_1 + \lambda_2)a_1$;

(5) $a_1^{\lambda_1} \otimes a_1^{\lambda_2} = a_1^{\lambda_1 + \lambda_2}$;

(6) $a_1^{\lambda} \otimes a_2^{\lambda} = \left(a_1 \otimes a_2\right)^{\lambda}$;

(7) $\left(a_1 \oplus a_2\right) \oplus a_3 = a_1 \oplus \left(a_2 \oplus a_3\right)$;

(8) $\left(a_1 \otimes a_2\right) \otimes a_3 = a_1 \otimes \left(a_2 \otimes a_3\right)$.

Then, equation (4) will be proved as follows:

Proof (4) Since $\lambda_1, \lambda_2 > 0$,

$$\lambda_1 a_1 \oplus \lambda_2 a_1 = \Big\langle s_{\lambda_1\theta(a_1)+\lambda_2\theta(a_1)},$$

$$\Bigg(\bigcup_{\gamma_1 \in \tilde{T}(a_1)} \left\{\frac{\frac{(1+(\varepsilon-1)\gamma_1)^{\lambda_1}-(1-\gamma_1)^{\lambda_1}}{(1+(\varepsilon-1)\gamma_1)^{\lambda_1}+(\varepsilon-1)(1-\gamma_1)^{\lambda_1}} + \frac{(1+(\varepsilon-1)\gamma_1)^{\lambda_2}-(1-\gamma_1)^{\lambda_2}}{(1+(\varepsilon-1)\gamma_1)^{\lambda_2}+(\varepsilon-1)(1-\gamma_1)^{\lambda_2}} - \frac{(1+(\varepsilon-1)\gamma_1)^{\lambda_1}-(1-\gamma_1)^{\lambda_1}}{(1+(\varepsilon-1)\gamma_1)^{\lambda_1}+(\varepsilon-1)(1-\gamma_1)^{\lambda_1}} \cdot \frac{(1+(\varepsilon-1)\gamma_1)^{\lambda_2}-(1-\gamma_1)^{\lambda_2}}{(1+(\varepsilon-1)\gamma_1)^{\lambda_2}+(\varepsilon-1)(1-\gamma_1)^{\lambda_2}} - (1-\varepsilon)\frac{(1+(\varepsilon-1)\gamma_1)^{\lambda_1}-(1-\gamma_1)^{\lambda_1}}{(1+(\varepsilon-1)\gamma_1)^{\lambda_1}+(\varepsilon-1)(1-\gamma_1)^{\lambda_1}} \cdot \frac{(1+(\varepsilon-1)\gamma_1)^{\lambda_2}-(1-\gamma_1)^{\lambda_2}}{(1+(\varepsilon-1)\gamma_1)^{\lambda_2}+(\varepsilon-1)(1-\gamma_1)^{\lambda_2}}}{1-(1-\varepsilon)\frac{(1+(\varepsilon-1)\gamma_1)^{\lambda_1}-(1-\gamma_1)^{\lambda_1}}{(1+(\varepsilon-1)\gamma_1)^{\lambda_1}+(\varepsilon-1)(1-\gamma_1)^{\lambda_1}} \cdot \frac{(1+(\varepsilon-1)\gamma_1)^{\lambda_2}-(1-\gamma_1)^{\lambda_2}}{(1+(\varepsilon-1)\gamma_1)^{\lambda_2}+(\varepsilon-1)(1-\gamma_1)^{\lambda_2}}}\right\},$$

$$\bigcup_{\delta_1 \in \tilde{I}(a_1)} \left\{\frac{\frac{\varepsilon\delta_1^{\lambda_1}}{(1+(\varepsilon-1)(1-\delta_1))^{\lambda_1}+(\varepsilon-1)\delta_1^{\lambda_1}} \cdot \frac{\varepsilon\delta_1^{\lambda_2}}{(1+(\varepsilon-1)(1-\delta_1))^{\lambda_2}+(\varepsilon-1)\delta_1^{\lambda_2}}}{\varepsilon+(1-\varepsilon)\left(\frac{\varepsilon\delta_1^{\lambda_1}}{(1+(\varepsilon-1)(1-\delta_1))^{\lambda_1}+(\varepsilon-1)\delta_1^{\lambda_1}} + \frac{\varepsilon\delta_1^{\lambda_2}}{(1+(\varepsilon-1)(1-\delta_1))^{\lambda_2}+(\varepsilon-1)\delta_1^{\lambda_2}} - \frac{\varepsilon\delta_1^{\lambda_1}}{(1+(\varepsilon-1)(1-\delta_1))^{\lambda_1}+(\varepsilon-1)\delta_1^{\lambda_1}} \cdot \frac{\varepsilon\delta_1^{\lambda_2}}{(1+(\varepsilon-1)(1-\delta_1))^{\lambda_2}+(\varepsilon-1)\delta_1^{\lambda_2}}\right)}\right\},$$

$$\bigcup_{\eta_1 \in \tilde{F}(a_1)} \left\{\frac{\frac{\varepsilon\eta_1^{\lambda_1}}{(1+(\varepsilon-1)(1-\eta_1))^{\lambda_1}+(\varepsilon-1)\eta_1^{\lambda_1}} \cdot \frac{\varepsilon\eta_1^{\lambda_2}}{(1+(\varepsilon-1)(1-\eta_1))^{\lambda_2}+(\varepsilon-1)\eta_1^{\lambda_2}}}{\varepsilon+(1-\varepsilon)\left(\frac{\varepsilon\eta_1^{\lambda_1}}{(1+(\varepsilon-1)(1-\eta_1))^{\lambda_1}+(\varepsilon-1)\eta_1^{\lambda_1}} + \frac{\varepsilon\eta_1^{\lambda_2}}{(1+(\varepsilon-1)(1-\eta_1))^{\lambda_2}+(\varepsilon-1)\eta_1^{\lambda_2}} - \frac{\varepsilon\eta_1^{\lambda_1}}{(1+(\varepsilon-1)(1-\eta_1))^{\lambda_1}+(\varepsilon-1)\eta_1^{\lambda_1}} \cdot \frac{\varepsilon\eta_1^{\lambda_2}}{(1+(\varepsilon-1)(1-\eta_1))^{\lambda_2}+(\varepsilon-1)\eta_1^{\lambda_2}}\right)}\right\}\Bigg)\Big\rangle$$

$$= \Big\langle s_{(\lambda_1+\lambda_2)\theta(a_1)},$$

$$\Bigg(\bigcup_{\gamma_1 \in \tilde{T}(a_1)} \left\{\frac{(1+(\varepsilon-1)\gamma_1)^{\lambda_1+\lambda_2}-(1-\gamma_1)^{\lambda_1+\lambda_2}}{(1+(\varepsilon-1)\gamma_1)^{\lambda_1+\lambda_2}+(\varepsilon-1)(1-\gamma_1)^{\lambda_1+\lambda_2}}\right\},$$

$$\bigcup_{\delta_1 \in \tilde{I}(a_1)} \left\{\frac{\varepsilon\delta_1^{\lambda_1+\lambda_2}}{(1+(\varepsilon-1)(1-\delta_1))^{\lambda_1+\lambda_2}+(\varepsilon-1)\delta_1^{\lambda_1+\lambda_2}}\right\},$$

$$\bigcup_{\eta_1 \in \tilde{F}(a_1)} \left\{\frac{\varepsilon\eta_1^{\lambda_1+\lambda_2}}{(1+(\varepsilon-1)(1-\eta_1))^{\lambda_1+\lambda_2}+(\varepsilon-1)\eta_1^{\lambda_1+\lambda_2}}\right\}\Bigg)\Big\rangle$$

$$= (\lambda_1+\lambda_2)a_1$$

Therefore, equation (4) $\lambda_1 a_1 \oplus \lambda_2 a_1 = (\lambda_1 + \lambda_2)a_1$ can be obtained.

Similarly, the other equations in Theorem1 are easily certified in the light of Definition 7.

3.2 Comparison method

The score, accuracy, and certainty functions are important indexes to rank MVNLNs, and its corresponding definition is given below:

Definition 8 Let $a = \left\langle s_{\theta(a)}, \left(\tilde{T}(a), \tilde{I}(a), \tilde{F}(a)\right)\right\rangle$ be an MVNLN, and the score, accuracy, and certainty functions are achieved as below.

(1) $$E(a) = \left(\frac{1}{l_{\tilde{T}(a)} l_{\tilde{I}(a)} l_{\tilde{F}(a)}} \sum_{\gamma\in\tilde{T}(a),\delta\in\tilde{I}(a),\eta\in\tilde{F}(a)} \left(\frac{\gamma + 1 - \delta + 1 - \eta}{3}\right)\right) s_{\theta(a)}$$

$$= s_{\left(\frac{1}{l_{\tilde{T}(a)} l_{\tilde{I}(a)} l_{\tilde{F}(a)}} \sum_{\gamma\in\tilde{T}(a),\delta\in\tilde{I}(a),\eta\in\tilde{F}(a)} \left(\frac{\gamma+1-\delta+1-\eta}{3}\right)\right)\theta(a)}$$

(2) $$H(a) = \left(\frac{1}{l_{\tilde{T}(a)} l_{\tilde{F}(a)}} \sum_{\gamma\in\tilde{T}(a),\eta\in\tilde{F}(a)} (\gamma - \eta)\right) s_{\theta(a)}$$

$$= s_{\left(\frac{1}{l_{\tilde{T}(a)} l_{\tilde{F}(a)}} \sum_{\gamma\in\tilde{T}(a),\eta\in\tilde{F}(a)} (\gamma-\eta)\right)\theta(a)}$$

(3) $$C(a) = \left(\frac{1}{l_{\tilde{T}(a)}} \sum_{\gamma\in\tilde{T}(a)} \gamma\right) s_{\theta(a)}$$

$$= s_{\left(\frac{1}{l_{\tilde{T}(a)}} \sum_{\gamma\in\tilde{T}(a)} \gamma\right)\theta(a)}$$

Where $l_{\tilde{T}(a)}$, $l_{\tilde{I}(a)}$, and $l_{\tilde{F}(a)}$ are the numbers of the values in $\tilde{T}_A(x)$, $\tilde{I}_A(x)$, and $\tilde{F}_A(x)$ respectively. The linguistic variable $s_{\theta(a)}$ is important for an MVNLN. Therefore, the comparison functions defined above in Definition 8 are denoted as the linguistic variable. The bigger the truth degree $\tilde{T}(a)$ concerning the variable $s_{\theta(a)}$ is, meanwhile, the smaller the indeterminacy degree $\tilde{I}(a)$ as well as the false degree $\tilde{F}(a)$ concerning the linguistic variable $s_{\theta(a)}$ are, then the higher the MVNLN is. Relating to the function of score, the greater $\gamma - \delta - \eta$ corresponding to $s_{\theta(a)}$ is, the higher the affirmative statement is. Relating to the function of accuracy, the greater γ minus η is, the certain the statement is. Regarding to the function of certainty, the bigger γ is, the certain the statement is.

Based on Definition 8, the comparison method between MVNLNs are obtained.

Definition 9 Supposing a_1 and a_2 are two MVNLNs, the compared approach is achieved in the following:

(1) Supposing that $E(a_1) > E(a_2)$,then a_1 is greater than a_2, represented as $a_1 \succ a_2$;

(2) Supposing that $E(a_1) = E(a_2)$, and $H(a_1) > H(a_2)$,then a_1 is greater than a_2 , represented as $a_1 \succ a_2$;

(3) Supposing that $E(a_1) = E(a_2)$, $H(a_1) = H(a_2)$, and $C(a_1) > C(a_2)$,then a_1 is greater than a_2 , represented as $a_1 \succ a_2$;

(4) Supposing that $E(a_1) = E(a_2)$, $H(a_1) = H(a_2)$, and $C(a_1) = C(a_2)$,then a_1 equals a_2 , represented as $a_1 \sim a_2$;

Mathematical Problems in Engineering.

4 The multi-valued neutrosophic linguistic normalized weighted Bonferroni mean Hamacher operator

The NWBM operator can not only take into account the advantages of BM and WBM, but also has the property of reducibility and idempotency. However, the NWBM operator has not been applied to the cases where the input arguments are MVNLNs.

Definition 10 Let $a_i(i = 1, 2, \cdots, n)$ be a space of MVNLNs, $a_i = \left\langle s_{\theta(a_i)}, \left(\tilde{T}(a_i), \tilde{I}(a_i), \tilde{F}(a_i)\right)\right\rangle$, $p, q \geq 0$, and $\omega = \left(\omega_1, \omega_2, \cdots, \omega_n\right)$ be the weighted vector for a_i, $\omega_i \in \left[0, 1\right]$ and $\sum_{i=1}^{n} \omega_i = 1$. Then the operator of MVNLNWBMH is achieved as below, the aggregation result is still an MVNLN.

$$MVNLNWBMH(a_1, a_2, \cdots a_n) = \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1 - \omega_i} (a_i^{\,p} \otimes a_j^{\,q}) \right)^{\frac{1}{p+q}}$$

According to the operational laws in Definition 7, the results are derived below:

$$MVNLNWBMH(a_1, a_2, \cdots a_n) = \left\langle \frac{s}{\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\theta^p(a_i)\cdot\theta^q(a_j)\right)\right)^{\frac{1}{p+q}}},\right.$$

$$\left(\bigcup_{\gamma_i\in\tilde{T}(a_i),\gamma_j\in\tilde{T}(a_j)}\left\{\frac{\varepsilon\left(\frac{x-y}{x+(\varepsilon-1)y}\right)^{\frac{1}{p+q}}}{\left(1+(\varepsilon-1)\left(1-\frac{x-y}{x+(\varepsilon-1)y}\right)\right)^{\frac{1}{p+q}}+(\varepsilon-1)\left(\frac{x-y}{x+(\varepsilon-1)y}\right)^{\frac{1}{p+q}}}\right\},\right.$$

$$\bigcup_{\delta_i\in\tilde{I}(a_i),\delta_j\in\tilde{I}(a_j)}\left\{\frac{\left(1+(\varepsilon-1)\frac{h}{g+(\varepsilon-1)h}\right)^{\frac{1}{p+q}}-\left(1-\frac{h}{g+(\varepsilon-1)h}\right)^{\frac{1}{p+q}}}{\left(1+(\varepsilon-1)\frac{h}{g+(\varepsilon-1)h}\right)^{\frac{1}{p+q}}+(\varepsilon-1)\left(1-\frac{h}{g+(\varepsilon-1)h}\right)^{\frac{1}{p+q}}}\right\},$$

$$\left.\left.\bigcup_{\eta_i\in\tilde{F}(a_i),\eta_j\in\tilde{F}(a_j)}\left\{\frac{\left(1+(\varepsilon-1)\frac{u}{v+(\varepsilon-1)u}\right)^{\frac{1}{p+q}}-\left(1-\frac{u}{v+(\varepsilon-1)u}\right)^{\frac{1}{p+q}}}{\left(1+(\varepsilon-1)\frac{u}{v+(\varepsilon-1)u}\right)^{\frac{1}{p+q}}+(\varepsilon-1)\left(1-\frac{u}{v+(\varepsilon-1)u}\right)^{\frac{1}{p+q}}}\right\}\right)\right\rangle$$

$$= \left\langle \frac{s}{\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\theta^p(a_i)\cdot\theta^q(a_j)\right)\right)^{\frac{1}{p+q}}},\right.$$

$$\left(\bigcup_{\gamma_i\in\tilde{T}(a_i),\gamma_j\in\tilde{T}(a_j)}\left\{\frac{\varepsilon(x-y)^{\frac{1}{p+q}}}{\left(x+(\varepsilon^2-1)y\right)^{\frac{1}{p+q}}+(\varepsilon-1)(x-y)^{\frac{1}{p+q}}}\right\},\right.$$

$$\bigcup_{\delta_i\in\tilde{I}(a_i),\delta_j\in\tilde{I}(a_j)}\left\{\frac{\left(g+(\varepsilon^2-1)h\right)^{\frac{1}{p+q}}-(g-h)^{\frac{1}{p+q}}}{\left(g+(\varepsilon^2-1)h\right)^{\frac{1}{p+q}}+(\varepsilon-1)(g-h)^{\frac{1}{p+q}}}\right\},$$

$$\left.\left.\bigcup_{\eta_i\in\tilde{F}(a_i),\eta_j\in\tilde{F}(a_j)}\left\{\frac{\left(v+(\varepsilon^2-1)u\right)^{\frac{1}{p+q}}-(v-u)^{\frac{1}{p+q}}}{\left(v+(\varepsilon^2-1)u\right)^{\frac{1}{p+q}}+(\varepsilon-1)(v-u)^{\frac{1}{p+q}}}\right\}\right)\right\rangle \tag{1}$$

Where $x = \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(\varepsilon-(\varepsilon-1)\gamma_i\right)^p\left(\varepsilon-(\varepsilon-1)\gamma_j\right)^q+(\varepsilon^2-1)\gamma_i^p\gamma_j^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}$,

$$y = \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(\left(\varepsilon - (\varepsilon - 1)\gamma_i\right)^p \left(\varepsilon - (\varepsilon - 1)\gamma_j\right)^q - \gamma_i^{\,p}\gamma_j^{\,q} \right)^{\frac{\omega_i\omega_j}{1-\omega_i}},$$

$$g = \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(\left(1 + (\varepsilon - 1)\delta_i\right)^p \left(1 + (\varepsilon - 1)\delta_j\right)^q + (\varepsilon^2 - 1)\left(1 - \delta_i\right)^p \left(1 - \delta_j\right)^q \right)^{\frac{\omega_i\omega_j}{1-\omega_i}},$$

$$h = \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(\left(1 + (\varepsilon - 1)\delta_i\right)^p \left(1 + (\varepsilon - 1)\delta_j\right)^q - \left(1 - \delta_i\right)^p \left(1 - \delta_j\right)^q \right)^{\frac{\omega_i\omega_j}{1-\omega_i}},$$

$$v = \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(\left(1 + (\varepsilon - 1)\eta_i\right)^p \left(1 + (\varepsilon - 1)\eta_j\right)^q + (\varepsilon^2 - 1)\left(1 - \eta_i\right)^p \left(1 - \eta_j\right)^q \right)^{\frac{\omega_i\omega_j}{1-\omega_i}}.$$

$$u = \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(\left(1 + (\varepsilon - 1)\eta_i\right)^p \left(1 + (\varepsilon - 1)\eta_j\right)^q - \left(1 - \eta_i\right)^p \left(1 - \eta_j\right)^q \right)^{\frac{\omega_i\omega_j}{1-\omega_i}},$$

Proof. According to the operational rules for MVNLNs, the results below can be gained

$$a_i^{\,p} = \left\langle s_{\theta^p(a_i)}, \left(\bigcup_{\gamma_i \in \tilde{T}(a_i)} \left\{ \frac{\varepsilon \cdot \gamma_i^{\,p}}{\left(1 + (\varepsilon - 1)(1 - \gamma_i)\right)^p + (\varepsilon - 1)\gamma_i^{\,p}} \right\}, \right.\right.$$
$$\bigcup_{\delta_i \in \tilde{I}(a_i)} \left\{ \frac{\left(1 + (\varepsilon - 1)\delta_i\right)^p - \left(1 - \delta_i\right)^p}{\left(1 + (\varepsilon - 1)\delta_i\right)^p + (\varepsilon - 1)\left(1 - \delta_i\right)^p} \right\},$$
$$\left.\left. \bigcup_{\eta_i \in \tilde{F}(a_i)} \left\{ \frac{\left(1 + (\varepsilon - 1)\eta_i\right)^p - \left(1 - \eta_i\right)^p}{\left(1 + (\varepsilon - 1)\eta_i\right)^p + (\varepsilon - 1)\left(1 - \eta_i\right)^p} \right\} \right) \right\rangle$$

$$a_j^{\,q} = \left\langle s_{\theta^q(a_j)}, \left(\bigcup_{\gamma_j \in \tilde{T}(a_j)} \left\{ \frac{\varepsilon \cdot \gamma_j^{\,q}}{\left(1 + (\varepsilon - 1)(1 - \gamma_j)\right)^q + (\varepsilon - 1)\gamma_j^{\,q}} \right\}, \right.\right.$$
$$\bigcup_{\delta_j \in \tilde{I}(a_j)} \left\{ \frac{\left(1 + (\varepsilon - 1)\delta_j\right)^q - \left(1 - \delta_j\right)^q}{\left(1 + (\varepsilon - 1)\delta_j\right)^q + (\varepsilon - 1)\left(1 - \delta_j\right)^q} \right\},$$
$$\left.\left. \bigcup_{\eta_j \in \tilde{F}(a_j)} \left\{ \frac{\left(1 + (\varepsilon - 1)\eta_j\right)^q - \left(1 - \eta_j\right)^q}{\left(1 + (\varepsilon - 1)\eta_j\right)^q + (\varepsilon - 1)\left(1 - \eta_j\right)^q} \right\} \right) \right\rangle$$

$$a_i^{\,p} \otimes a_j^{\,q} = \left\langle s_{\theta^p(a_i)\theta^q(a_j)}, \left(\bigcup_{\gamma_i \in \tilde{T}(a_i), \gamma_j \in \tilde{T}(a_j)} \left\{ \frac{\varepsilon \gamma_i^{\,p} \gamma_j^{\,q}}{\left(\varepsilon - (\varepsilon - 1)\gamma_i\right)^p \cdot \left(\varepsilon - (\varepsilon - 1)\gamma_j\right)^q + (\varepsilon - 1)\gamma_i^{\,p}\gamma_j^{\,q}} \right\}, \right. \right.$$

$$\bigcup_{\delta_i \in \tilde{I}(a_i), \delta_j \in \tilde{I}(a_j)} \left\{ \frac{\left(1 + (\varepsilon - 1)\delta_i\right)^p \cdot \left(1 + (\varepsilon - 1)\delta_j\right)^q - (1 - \delta_i)^p (1 - \delta_j)^q}{\left(1 + (\varepsilon - 1)\delta_i\right)^p \cdot \left(1 + (\varepsilon - 1)\delta_j\right)^q + (\varepsilon - 1)(1 - \delta_i)^p (1 - \delta_j)^q} \right\},$$

$$\left. \left. \bigcup_{\eta_i \in \tilde{F}(a_i), \eta_j \in \tilde{F}(a_j)} \left\{ \frac{\left(1 + (\varepsilon - 1)\eta_i\right)^p \cdot \left(1 + (\varepsilon - 1)\eta_j\right)^q - (1 - \eta_i)^p (1 - \eta_j)^q}{\left(1 + (\varepsilon - 1)\eta_i\right)^p \cdot \left(1 + (\varepsilon - 1)\eta_j\right)^q + (\varepsilon - 1)(1 - \eta_i)^p (1 - \eta_j)^q} \right\} \right) \right\rangle$$

Firstly, we need to testify the mathematical formula below.

$$\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1 - \omega_i} (a_i^{\,p} \otimes a_j^{\,q})$$

$$= \left\langle s_{\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} \cdot \theta^p(a_i) \cdot \theta^q(a_j)}, \left(\bigcup_{\gamma_i \in \tilde{T}(a_i), \gamma_j \in \tilde{T}(a_j)} \left\{ \frac{\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(\varepsilon - (\varepsilon - 1)\gamma_i\right)^p \left(\varepsilon - (\varepsilon - 1)\gamma_j\right)^q + (\varepsilon^2 - 1)\gamma_i^{\,p}\gamma_j^{\,q} \right)^{\frac{\omega_i\omega_j}{1-\omega_i}} - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(\varepsilon - (\varepsilon - 1)\gamma_i\right)^p \left(\varepsilon - (\varepsilon - 1)\gamma_j\right)^q - \gamma_i^{\,p}\gamma_j^{\,q} \right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}{\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(\varepsilon - (\varepsilon - 1)\gamma_i\right)^p \left(\varepsilon - (\varepsilon - 1)\gamma_j\right)^q + (\varepsilon^2 - 1)\gamma_i^{\,p}\gamma_j^{\,q} \right)^{\frac{\omega_i\omega_j}{1-\omega_i}} + (\varepsilon - 1) \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(\varepsilon - (\varepsilon - 1)\gamma_i\right)^p \left(\varepsilon - (\varepsilon - 1)\gamma_j\right)^q - \gamma_i^{\,p}\gamma_j^{\,q} \right)^{\frac{\omega_i\omega_j}{1-\omega_i}}} \right\}, \right. \right.$$

$$\bigcup_{\delta_i \in \tilde{I}(a_i), \delta_j \in \tilde{I}(a_j)} \left\{ \frac{\varepsilon \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(1 + (\varepsilon - 1)\delta_i\right)^p \left(1 + (\varepsilon - 1)\delta_j\right)^q - (1 - \delta_i)^p (1 - \delta_j)^q \right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}{\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(1 + (\varepsilon - 1)\delta_i\right)^p \left(1 + (\varepsilon - 1)\delta_j\right)^q + (\varepsilon^2 - 1)(1 - \delta_i)^p (1 - \delta_j)^q \right)^{\frac{\omega_i\omega_j}{1-\omega_i}} + (\varepsilon - 1) \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(1 + (\varepsilon - 1)\delta_i\right)^p \left(1 + (\varepsilon - 1)\delta_j\right)^q - (1 - \delta_i)^p (1 - \delta_j)^q \right)^{\frac{\omega_i\omega_j}{1-\omega_i}}} \right\},$$

$$\left. \left. \bigcup_{\eta_i \in \tilde{F}(a_i), \eta_j \in \tilde{F}(a_j)} \left\{ \frac{\varepsilon \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(1 + (\varepsilon - 1)\eta_i\right)^p \left(1 + (\varepsilon - 1)\eta_j\right)^q - (1 - \eta_i)^p (1 - \eta_j)^q \right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}{\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(1 + (\varepsilon - 1)\eta_i\right)^p \left(1 + (\varepsilon - 1)\eta_j\right)^q + (\varepsilon^2 - 1)(1 - \eta_i)^p (1 - \eta_j)^q \right)^{\frac{\omega_i\omega_j}{1-\omega_i}} + (\varepsilon - 1) \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(1 + (\varepsilon - 1)\eta_i\right)^p \left(1 + (\varepsilon - 1)\eta_j\right)^q - (1 - \eta_i)^p (1 - \eta_j)^q \right)^{\frac{\omega_i\omega_j}{1-\omega_i}}} \right\} \right) \right\rangle. \tag{2}$$

The mathematical induction on n is adopt to prove Eq.(2)

(1) Supposing $n = 2$, the equation below is obtained.

$$\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{2} \frac{\omega_i \omega_j}{1 - \omega_i} (a_i^{\,p} \otimes a_j^{\,q}) = \frac{\omega_1 \omega_2}{1 - \omega_1} (a_1^{\,p} \otimes a_2^{\,q}) \oplus \frac{\omega_2 \omega_1}{1 - \omega_2} (a_2^{\,p} \otimes a_1^{\,q})$$

$$= \left\langle s_{\frac{\omega_1\omega_2}{1-\omega_1} \cdot \theta^p(a_1) \cdot \theta^q(a_2) + \frac{\omega_2\omega_1}{1-\omega_2} \cdot \theta^p(a_2) \cdot \theta^q(a_1)}, \left(\bigcup_{\gamma_i \in \tilde{T}(a_i), \gamma_j \in \tilde{T}(a_j)} \left\{ \frac{x_1 x_2 - y_1 y_2}{x_1 x_2 + (\varepsilon - 1) y_1 y_2} \right\}, \right. \right.$$

$$\bigcup_{\delta_i \in \tilde{I}(a_i), \delta_j \in \tilde{I}(a_j)} \left\{ \frac{\varepsilon m_1 m_2}{z_1 z_2 + (\varepsilon - 1) m_1 m_2} \right\},$$

$$\left. \left. \bigcup_{\eta_i \in \tilde{F}(a_i), \eta_j \in \tilde{F}(a_j)} \left\{ \frac{\varepsilon v_1 v_2}{u_1 u_2 + (\varepsilon - 1) v_1 v_2} \right\} \right) \right\rangle$$

Where $x_1 = \left(1 + (\varepsilon - 1) \cdot \frac{\varepsilon \gamma_1^{\,p} \gamma_2^{\,q}}{\left(\varepsilon - (\varepsilon - 1)\gamma_1\right)^p \cdot \left(\varepsilon - (\varepsilon - 1)\gamma_2\right)^q + (\varepsilon - 1)\gamma_1^{\,p}\gamma_2^{\,q}} \right)^{\frac{\omega_1\omega_2}{1-\omega_1}}$,

$$x_2 = \left(1 + \left(\varepsilon - 1\right) \cdot \frac{\varepsilon \gamma_2{}^p \gamma_1{}^q}{\left(\varepsilon - \left(\varepsilon - 1\right)\gamma_2\right)^p \cdot \left(\varepsilon - \left(\varepsilon - 1\right)\gamma_1\right)^q + \left(\varepsilon - 1\right)\gamma_2{}^p \gamma_1{}^q}\right)^{\frac{\omega_2\omega_1}{1-\omega_2}},$$

$$y_1 = \left(1 - \frac{\varepsilon \gamma_1{}^p \gamma_2{}^q}{\left(\varepsilon - \left(\varepsilon - 1\right)\gamma_1\right)^p \cdot \left(\varepsilon - \left(\varepsilon - 1\right)\gamma_2\right)^q + \left(\varepsilon - 1\right)\gamma_1{}^p \gamma_2{}^q}\right)^{\frac{\omega_1\omega_2}{1-\omega_1}},$$

$$y_2 = \left(1 - \frac{\varepsilon \gamma_2{}^p \gamma_1{}^q}{\left(\varepsilon - \left(\varepsilon - 1\right)\gamma_2\right)^p \cdot \left(\varepsilon - \left(\varepsilon - 1\right)\gamma_1\right)^q + \left(\varepsilon - 1\right)\gamma_2{}^p \gamma_1{}^q}\right)^{\frac{\omega_2\omega_1}{1-\omega_2}},$$

$$m_1 = \left(\frac{\left(1 + \left(\varepsilon - 1\right)\delta_1\right)^p \cdot \left(1 + \left(\varepsilon - 1\right)\delta_2\right)^q - \left(1 - \delta_1\right)^p \left(1 - \delta_2\right)^q}{\left(1 + \left(\varepsilon - 1\right)\delta_1\right)^p \cdot \left(1 + \left(\varepsilon - 1\right)\delta_2\right)^q + \left(\varepsilon - 1\right)\left(1 - \delta_1\right)^p \left(1 - \delta_2\right)^q}\right)^{\frac{\omega_1\omega_2}{1-\omega_1}},$$

$$m_2 = \left(\frac{\left(1 + \left(\varepsilon - 1\right)\delta_2\right)^p \cdot \left(1 + \left(\varepsilon - 1\right)\delta_1\right)^q - \left(1 - \delta_2\right)^p \left(1 - \delta_1\right)^q}{\left(1 + \left(\varepsilon - 1\right)\delta_2\right)^p \cdot \left(1 + \left(\varepsilon - 1\right)\delta_1\right)^q + \left(\varepsilon - 1\right)\left(1 - \delta_2\right)^p \left(1 - \delta_1\right)^q}\right)^{\frac{\omega_2\omega_1}{1-\omega_2}},$$

$$z_1 = \left(1 + \left(\varepsilon - 1\right)\left(1 - \frac{\left(1 + \left(\varepsilon - 1\right)\delta_1\right)^p \cdot \left(1 + \left(\varepsilon - 1\right)\delta_2\right)^q - \left(1 - \delta_1\right)^p \left(1 - \delta_2\right)^q}{\left(1 + \left(\varepsilon - 1\right)\delta_1\right)^p \cdot \left(1 + \left(\varepsilon - 1\right)\delta_2\right)^q + \left(\varepsilon - 1\right)\left(1 - \delta_1\right)^p \left(1 - \delta_2\right)^q}\right)\right)^{\frac{\omega_1\omega_2}{1-\omega_1}},$$

$$z_2 = \left(1 + \left(\varepsilon - 1\right)\left(1 - \frac{\left(1 + \left(\varepsilon - 1\right)\delta_2\right)^p \cdot \left(1 + \left(\varepsilon - 1\right)\delta_1\right)^q - \left(1 - \delta_2\right)^p \left(1 - \delta_1\right)^q}{\left(1 + \left(\varepsilon - 1\right)\delta_2\right)^p \cdot \left(1 + \left(\varepsilon - 1\right)\delta_1\right)^q + \left(\varepsilon - 1\right)\left(1 - \delta_2\right)^p \left(1 - \delta_1\right)^q}\right)\right)^{\frac{\omega_2\omega_1}{1-\omega_2}},$$

$$v_1 = \left(\frac{\left(1 + \left(\varepsilon - 1\right)\eta_1\right)^p \cdot \left(1 + \left(\varepsilon - 1\right)\eta_2\right)^q - \left(1 - \eta_1\right)^p \left(1 - \eta_2\right)^q}{\left(1 + \left(\varepsilon - 1\right)\eta_1\right)^p \cdot \left(1 + \left(\varepsilon - 1\right)\eta_2\right)^q + \left(\varepsilon - 1\right)\left(1 - \eta_1\right)^p \left(1 - \eta_2\right)^q}\right)^{\frac{\omega_1\omega_2}{1-\omega_1}},$$

$$v_2 = \left(\frac{\left(1 + \left(\varepsilon - 1\right)\eta_2\right)^p \cdot \left(1 + \left(\varepsilon - 1\right)\eta_1\right)^q - \left(1 - \eta_2\right)^p \left(1 - \eta_1\right)^q}{\left(1 + \left(\varepsilon - 1\right)\eta_2\right)^p \cdot \left(1 + \left(\varepsilon - 1\right)\eta_1\right)^q + \left(\varepsilon - 1\right)\left(1 - \eta_2\right)^p \left(1 - \eta_1\right)^q}\right)^{\frac{\omega_2\omega_1}{1-\omega_2}},$$

$$u_1 = \left(1 + \left(\varepsilon - 1\right)\left(1 - \frac{\left(1 + \left(\varepsilon - 1\right)\eta_1\right)^p \cdot \left(1 + \left(\varepsilon - 1\right)\eta_2\right)^q - \left(1 - \eta_1\right)^p \left(1 - \eta_2\right)^q}{\left(1 + \left(\varepsilon - 1\right)\eta_1\right)^p \cdot \left(1 + \left(\varepsilon - 1\right)\eta_2\right)^q + \left(\varepsilon - 1\right)\left(1 - \eta_1\right)^p \left(1 - \eta_2\right)^q}\right)\right)^{\frac{\omega_1\omega_2}{1-\omega_1}},$$

$$u_2 = \left(1 + \left(\varepsilon - 1\right)\left(1 - \frac{\left(1 + \left(\varepsilon - 1\right)\eta_2\right)^p \cdot \left(1 + \left(\varepsilon - 1\right)\eta_1\right)^q - \left(1 - \eta_2\right)^p \left(1 - \eta_1\right)^q}{\left(1 + \left(\varepsilon - 1\right)\eta_2\right)^p \cdot \left(1 + \left(\varepsilon - 1\right)\eta_1\right)^q + \left(\varepsilon - 1\right)\left(1 - \eta_2\right)^p \left(1 - \eta_1\right)^q}\right)\right)^{\frac{\omega_2\omega_1}{1-\omega_2}},$$

Then,

$$\bigoplus_{\substack{i,j=1\\ i\neq j}}^{2}\frac{\omega_i\omega_j}{1-\omega_i}(a_i^{\,p}\otimes a_j^{\,q})=\frac{\omega_1\omega_2}{1-\omega_1}(a_1^{\,p}\otimes a_2^{\,q})\oplus\frac{\omega_2\omega_1}{1-\omega_2}(a_2^{\,p}\otimes a_1^{\,q})$$

$$=\left\langle s_{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{2}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\theta^{p}(a_i)\cdot\theta^{q}(a_j)\right)},\left(\bigcup_{\gamma_i\in\tilde{T}(a_i),\gamma_j\in\tilde{T}(a_j)}\left\{\frac{\prod\limits_{\substack{i,j=1\\ i\neq j}}^{2}\left(\left(\varepsilon-(\varepsilon-1)\gamma_i\right)^{p}\left(\varepsilon-(\varepsilon-1)\gamma_j\right)^{q}+\left(\varepsilon^{2}-1\right)\gamma_i^{\,p}\gamma_j^{\,q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}-\prod\limits_{\substack{i,j=1\\ i\neq j}}^{2}\left(\left(\varepsilon-(\varepsilon-1)\gamma_i\right)^{p}\left(\varepsilon-(\varepsilon-1)\gamma_j\right)^{q}-\gamma_i^{\,p}\gamma_j^{\,q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}{\prod\limits_{\substack{i,j=1\\ i\neq j}}^{2}\left(\left(\varepsilon-(\varepsilon-1)\gamma_i\right)^{p}\left(\varepsilon-(\varepsilon-1)\gamma_j\right)^{q}+\left(\varepsilon^{2}-1\right)\gamma_i^{\,p}\gamma_j^{\,q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}+(\varepsilon-1)\prod\limits_{\substack{i,j=1\\ i\neq j}}^{2}\left(\left(\varepsilon-(\varepsilon-1)\gamma_i\right)^{p}\left(\varepsilon-(\varepsilon-1)\gamma_j\right)^{q}-\gamma_i^{\,p}\gamma_j^{\,q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}\right\},\right.\right.$$

$$\bigcup_{\delta_i\in\tilde{I}(a_i),\delta_j\in\tilde{I}(a_j)}\left\{\frac{\varepsilon\prod\limits_{\substack{i,j=1\\ i\neq j}}^{2}\left(\left(1+(\varepsilon-1)\delta_i\right)^{p}\left(1+(\varepsilon-1)\delta_j\right)^{q}-\left(1-\delta_i\right)^{p}\left(1-\delta_j\right)^{q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}{\prod\limits_{\substack{i,j=1\\ i\neq j}}^{2}\left(\left(1+(\varepsilon-1)\delta_i\right)^{p}\left(1+(\varepsilon-1)\delta_j\right)^{q}+(\varepsilon^{2}-1)\left(1-\delta_i\right)^{p}\left(1-\delta_j\right)^{q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}+(\varepsilon-1)\prod\limits_{\substack{i,j=1\\ i\neq j}}^{2}\left(\left(1+(\varepsilon-1)\delta_i\right)^{p}\left(1+(\varepsilon-1)\delta_j\right)^{q}-\left(1-\delta_i\right)^{p}\left(1-\delta_j\right)^{q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}\right\},$$

$$\left.\left.\bigcup_{\eta_i\in\tilde{F}(a_i),\eta_j\in\tilde{F}(a_j)}\left\{\frac{\varepsilon\prod\limits_{\substack{i,j=1\\ i\neq j}}^{2}\left(\left(1+(\varepsilon-1)\eta_i\right)^{p}\left(1+(\varepsilon-1)\eta_j\right)^{q}-\left(1-\eta_i\right)^{p}\left(1-\eta_j\right)^{q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}{\prod\limits_{\substack{i,j=1\\ i\neq j}}^{2}\left(\left(1+(\varepsilon-1)\eta_i\right)^{p}\left(1+(\varepsilon-1)\eta_j\right)^{q}+(\varepsilon^{2}-1)\left(1-\eta_i\right)^{p}\left(1-\eta_j\right)^{q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}+(\varepsilon-1)\prod\limits_{\substack{i,j=1\\ i\neq j}}^{2}\left(\left(1+(\varepsilon-1)\eta_i\right)^{p}\left(1+(\varepsilon-1)\eta_j\right)^{q}-\left(1-\eta_i\right)^{p}\left(1-\eta_j\right)^{q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}\right\}\right)\right\rangle.$$

We can achieve Eq. (2) is right when $n = k$.

(2) Supposing $n = k$, the equation below is right, then

$$\bigoplus_{\substack{i,j=1\\ i\neq j}}^{k}\frac{\omega_i\omega_j}{1-\omega_i}(a_i^{\,p}\otimes a_j^{\,q})$$

$$=\left\langle s_{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{k}\frac{\omega_i\omega_j}{1-\omega_i}\cdot\theta^{p}(a_i)\cdot\theta^{q}(a_j)},\left(\bigcup_{\gamma_i\in\tilde{T}(a_i),\gamma_j\in\tilde{T}(a_j)}\left\{\frac{\prod\limits_{\substack{i,j=1\\ i\neq j}}^{k}\left(\left(\varepsilon-(\varepsilon-1)\gamma_i\right)^{p}\left(\varepsilon-(\varepsilon-1)\gamma_j\right)^{q}+\left(\varepsilon^{2}-1\right)\gamma_i^{\,p}\gamma_j^{\,q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}-\prod\limits_{\substack{i,j=1\\ i\neq j}}^{k}\left(\left(\varepsilon-(\varepsilon-1)\gamma_i\right)^{p}\left(\varepsilon-(\varepsilon-1)\gamma_j\right)^{q}-\gamma_i^{\,p}\gamma_j^{\,q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}{\prod\limits_{\substack{i,j=1\\ i\neq j}}^{k}\left(\left(\varepsilon-(\varepsilon-1)\gamma_i\right)^{p}\left(\varepsilon-(\varepsilon-1)\gamma_j\right)^{q}+\left(\varepsilon^{2}-1\right)\gamma_i^{\,p}\gamma_j^{\,q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}+(\varepsilon-1)\prod\limits_{\substack{i,j=1\\ i\neq j}}^{k}\left(\left(\varepsilon-(\varepsilon-1)\gamma_i\right)^{p}\left(\varepsilon-(\varepsilon-1)\gamma_j\right)^{q}-\gamma_i^{\,p}\gamma_j^{\,q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}\right\},\right.\right.$$

$$\bigcup_{\delta_i\in\tilde{I}(a_i),\delta_j\in\tilde{I}(a_j)}\left\{\frac{\varepsilon\prod\limits_{\substack{i,j=1\\ i\neq j}}^{k}\left(\left(1+(\varepsilon-1)\delta_i\right)^{p}\left(1+(\varepsilon-1)\delta_j\right)^{q}-\left(1-\delta_i\right)^{p}\left(1-\delta_j\right)^{q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}{\prod\limits_{\substack{i,j=1\\ i\neq j}}^{k}\left(\left(1+(\varepsilon-1)\delta_i\right)^{p}\left(1+(\varepsilon-1)\delta_j\right)^{q}+(\varepsilon^{2}-1)\left(1-\delta_i\right)^{p}\left(1-\delta_j\right)^{q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}+(\varepsilon-1)\prod\limits_{\substack{i,j=1\\ i\neq j}}^{k}\left(\left(1+(\varepsilon-1)\delta_i\right)^{p}\left(1+(\varepsilon-1)\delta_j\right)^{q}-\left(1-\delta_i\right)^{p}\left(1-\delta_j\right)^{q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}\right\},$$

$$\left.\left.\bigcup_{\eta_i\in\tilde{F}(a_i),\eta_j\in\tilde{F}(a_j)}\left\{\frac{\varepsilon\prod\limits_{\substack{i,j=1\\ i\neq j}}^{k}\left(\left(1+(\varepsilon-1)\eta_i\right)^{p}\left(1+(\varepsilon-1)\eta_j\right)^{q}-\left(1-\eta_i\right)^{p}\left(1-\eta_j\right)^{q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}{\prod\limits_{\substack{i,j=1\\ i\neq j}}^{k}\left(\left(1+(\varepsilon-1)\eta_i\right)^{p}\left(1+(\varepsilon-1)\eta_j\right)^{q}+(\varepsilon^{2}-1)\left(1-\eta_i\right)^{p}\left(1-\eta_j\right)^{q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}+(\varepsilon-1)\prod\limits_{\substack{i,j=1\\ i\neq j}}^{k}\left(\left(1+(\varepsilon-1)\eta_i\right)^{p}\left(1+(\varepsilon-1)\eta_j\right)^{q}-\left(1-\eta_i\right)^{p}\left(1-\eta_j\right)^{q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}\right\}\right)\right\rangle.$$

If $n = k + 1$, we need to calculate the equation below.

$$\bigoplus_{\substack{i,j=1\\ i\neq j}}^{k+1}\frac{\omega_i\omega_j}{1-\omega_i}(a_i^{\,p}\otimes a_j^{\,q})=\bigoplus_{\substack{i,j=1\\ i\neq j}}^{k}\frac{\omega_i\omega_j}{1-\omega_i}(a_i^{\,p}\otimes a_j^{\,q})\oplus\bigoplus_{i=1}^{k}\frac{\omega_i\omega_{k+1}}{1-\omega_i}(a_i^{\,p}\otimes a_{k+1}^{\,q})\oplus\bigoplus_{j=1}^{k}\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}(a_{k+1}^{\,p}\otimes a_j^{\,q})$$

The mathematical induction on k are used to testify the equations below.

$$\bigoplus_{i=1}^{k}\frac{\omega_i\omega_{k+1}}{1-\omega_i}(a_i^{\,p}\otimes a_{k+1}^{\,q})=\left\langle s_{\sum\limits_{i=1}^{k}\left(\frac{\omega_i\omega_{k+1}}{1-\omega_i}\cdot\theta^{p}(a_i)\cdot\theta^{q}(a_{k+1})\right)},\right.$$

$$\left(\bigcup_{\gamma_i\in\tilde{T}(a_i),\gamma_{k+1}\in\tilde{T}(a_{k+1})}\left\{\frac{\prod\limits_{i=1}^{k}\left(\left(\varepsilon-(\varepsilon-1)\gamma_i\right)^{p}\left(\varepsilon-(\varepsilon-1)\gamma_{k+1}\right)^{q}+\left(\varepsilon^{2}-1\right)\gamma_i^{\,p}\gamma_{k+1}^{\,q}\right)^{\frac{\omega_i\omega_{k+1}}{1-\omega_i}}-\prod\limits_{i=1}^{k}\left(\left(\varepsilon-(\varepsilon-1)\gamma_i\right)^{p}\left(\varepsilon-(\varepsilon-1)\gamma_{k+1}\right)^{q}-\gamma_i^{\,p}\gamma_{k+1}^{\,q}\right)^{\frac{\omega_i\omega_{k+1}}{1-\omega_i}}}{\prod\limits_{i=1}^{k}\left(\left(\varepsilon-(\varepsilon-1)\gamma_i\right)^{p}\left(\varepsilon-(\varepsilon-1)\gamma_{k+1}\right)^{q}+\left(\varepsilon^{2}-1\right)\gamma_i^{\,p}\gamma_{k+1}^{\,q}\right)^{\frac{\omega_i\omega_{k+1}}{1-\omega_i}}+(\varepsilon-1)\prod\limits_{i=1}^{k}\left(\left(\varepsilon-(\varepsilon-1)\gamma_i\right)^{p}\left(\varepsilon-(\varepsilon-1)\gamma_{k+1}\right)^{q}-\gamma_i^{\,p}\gamma_{k+1}^{\,q}\right)^{\frac{\omega_i\omega_{k+1}}{1-\omega_i}}}\right\},\right.$$

$$\bigcup_{\delta_i\in\tilde{I}(a_i),\delta_{k+1}\in\tilde{I}(a_{k+1})}\left\{\frac{\varepsilon\prod\limits_{i=1}^{k}\left(\left(1+(\varepsilon-1)\delta_i\right)^{p}\left(1+(\varepsilon-1)\delta_{k+1}\right)^{q}-\left(1-\delta_i\right)^{p}\left(1-\delta_{k+1}\right)^{q}\right)^{\frac{\omega_i\omega_{k+1}}{1-\omega_i}}}{\prod\limits_{i=1}^{k}\left(\left(1+(\varepsilon-1)\delta_i\right)^{p}\left(1+(\varepsilon-1)\delta_{k+1}\right)^{q}+(\varepsilon^{2}-1)\left(1-\delta_i\right)^{p}\left(1-\delta_{k+1}\right)^{q}\right)^{\frac{\omega_i\omega_{k+1}}{1-\omega_i}}+(\varepsilon-1)\prod\limits_{i=1}^{k}\left(\left(1+(\varepsilon-1)\delta_i\right)^{p}\left(1+(\varepsilon-1)\delta_{k+1}\right)^{q}-\left(1-\delta_i\right)^{p}\left(1-\delta_{k+1}\right)^{q}\right)^{\frac{\omega_i\omega_{k+1}}{1-\omega_i}}}\right\},$$

$$\left.\left.\bigcup_{\eta_i\in\tilde{F}(a_i),\eta_{k+1}\in\tilde{F}(a_{k+1})}\left\{\frac{\varepsilon\prod\limits_{i=1}^{k}\left(\left(1+(\varepsilon-1)\eta_i\right)^{p}\left(1+(\varepsilon-1)\eta_{k+1}\right)^{q}-\left(1-\eta_i\right)^{p}\left(1-\eta_{k+1}\right)^{q}\right)^{\frac{\omega_i\omega_{k+1}}{1-\omega_i}}}{\prod\limits_{i=1}^{k}\left(\left(1+(\varepsilon-1)\eta_i\right)^{p}\left(1+(\varepsilon-1)\eta_{k+1}\right)^{q}+(\varepsilon^{2}-1)\left(1-\eta_i\right)^{p}\left(1-\eta_{k+1}\right)^{q}\right)^{\frac{\omega_i\omega_{k+1}}{1-\omega_i}}+(\varepsilon-1)\prod\limits_{i=1}^{k}\left(\left(1+(\varepsilon-1)\eta_i\right)^{p}\left(1+(\varepsilon-1)\eta_{k+1}\right)^{q}-\left(1-\eta_i\right)^{p}\left(1-\eta_{k+1}\right)^{q}\right)^{\frac{\omega_i\omega_{k+1}}{1-\omega_i}}}\right\}\right)\right\rangle$$

$$\mathop{\oplus}_{j=1}^{k}\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}({a_{k+1}}^p\otimes {a_j}^q)=\left\langle s_{\sum_{j=1}^{k}\left(\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}\cdot\theta^p(a_{k+1})\cdot\theta^q(a_j)\right)},\right.$$

$$\left(\bigcup_{\gamma_{k+1}\in\tilde{T}(a_{k+1}),\gamma_j\in\tilde{T}(a_j)}\left\{\frac{\prod_{j=1}^{k}\left(\left(\varepsilon-(\varepsilon-1)\gamma_{k+1}\right)^p\left(\varepsilon-(\varepsilon-1)\gamma_j\right)^q+\left(\varepsilon^2-1\right){\gamma_{k+1}}^p{\gamma_j}^q\right)^{\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}}-\prod_{j=1}^{k}\left(\left(\varepsilon-(\varepsilon-1)\gamma_{k+1}\right)^p\left(\varepsilon-(\varepsilon-1)\gamma_j\right)^q-{\gamma_{k+1}}^p{\gamma_j}^q\right)^{\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}}}{\prod_{j=1}^{k}\left(\left(\varepsilon-(\varepsilon-1)\gamma_{k+1}\right)^p\left(\varepsilon-(\varepsilon-1)\gamma_j\right)^q+\left(\varepsilon^2-1\right){\gamma_{k+1}}^p{\gamma_j}^q\right)^{\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}}+(\varepsilon-1)\prod_{j=1}^{k}\left(\left(\varepsilon-(\varepsilon-1)\gamma_{k+1}\right)^p\left(\varepsilon-(\varepsilon-1)\gamma_j\right)^q-{\gamma_{k+1}}^p{\gamma_j}^q\right)^{\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}}}\right\},\right.$$

$$\bigcup_{\delta_{k+1}\in\tilde{I}(a_{k+1}),\delta_j\in\tilde{I}(a_j)}\left\{\frac{\varepsilon\prod_{j=1}^{k}\left(\left(1+(\varepsilon-1)\delta_{k+1}\right)^p\left(1+(\varepsilon-1)\delta_j\right)^q-\left(1-\delta_{k+1}\right)^p\left(1-\delta_j\right)^q\right)^{\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}}}{\prod_{j=1}^{k}\left(\left(1+(\varepsilon-1)\delta_{k+1}\right)^p\left(1+(\varepsilon-1)\delta_j\right)^q+(\varepsilon^2-1)\left(1-\delta_{k+1}\right)^p\left(1-\delta_j\right)^q\right)^{\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}}+(\varepsilon-1)\prod_{j=1}^{k}\left(\left(1+(\varepsilon-1)\delta_{k+1}\right)^p\left(1+(\varepsilon-1)\delta_j\right)^q-\left(1-\delta_{k+1}\right)^p\left(1-\delta_j\right)^q\right)^{\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}}}\right\},$$

$$\left.\left.\bigcup_{\eta_{k+1}\in\tilde{F}(a_{k+1}),\eta_j\in\tilde{F}(a_j)}\left\{\frac{\varepsilon\prod_{j=1}^{k}\left(\left(1+(\varepsilon-1)\eta_{k+1}\right)^p\left(1+(\varepsilon-1)\eta_j\right)^q-\left(1-\eta_{k+1}\right)^p\left(1-\eta_j\right)^q\right)^{\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}}}{\prod_{j=1}^{k}\left(\left(1+(\varepsilon-1)\eta_{k+1}\right)^p\left(1+(\varepsilon-1)\eta_j\right)^q+(\varepsilon^2-1)\left(1-\eta_{k+1}\right)^p\left(1-\eta_j\right)^q\right)^{\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}}+(\varepsilon-1)\prod_{j=1}^{k}\left(\left(1+(\varepsilon-1)\eta_{k+1}\right)^p\left(1+(\varepsilon-1)\eta_j\right)^q-\left(1-\eta_{k+1}\right)^p\left(1-\eta_j\right)^q\right)^{\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}}}\right\}\right)\right\rangle$$

Therefore,

$$\mathop{\oplus}_{\substack{i,j=1\\ i\neq j}}^{k+1}\frac{\omega_i\omega_j}{1-\omega_i}({a_i}^p\otimes {a_j}^q)=\mathop{\oplus}_{\substack{i,j=1\\ i\neq j}}^{k}\frac{\omega_i\omega_j}{1-\omega_i}({a_i}^p\otimes {a_j}^q)\oplus\mathop{\oplus}_{i=1}^{k}\frac{\omega_i\omega_{k+1}}{1-\omega_i}({a_i}^p\otimes {a_{k+1}}^q)\oplus\mathop{\oplus}_{j=1}^{k}\frac{\omega_{k+1}\omega_j}{1-\omega_{k+1}}({a_{k+1}}^p\otimes {a_j}^q)$$

$$=\left\langle s_{\sum_{\substack{i,j=1\\ i\neq j}}^{k+1}\frac{\omega_i\omega_j}{1-\omega_i}\cdot\theta^p(a_i)\cdot\theta^q(a_j)},\left(\bigcup_{\gamma_i\in\tilde{T}(a_i),\gamma_j\in\tilde{T}(a_j)}\left\{\frac{\prod_{\substack{i,j=1\\ i\neq j}}^{k+1}\left(\left(\varepsilon-(\varepsilon-1)\gamma_i\right)^p\left(\varepsilon-(\varepsilon-1)\gamma_j\right)^q+\left(\varepsilon^2-1\right){\gamma_i}^p{\gamma_j}^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}-\prod_{\substack{i,j=1\\ i\neq j}}^{k+1}\left(\left(\varepsilon-(\varepsilon-1)\gamma_i\right)^p\left(\varepsilon-(\varepsilon-1)\gamma_j\right)^q-{\gamma_i}^p{\gamma_j}^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}{\prod_{\substack{i,j=1\\ i\neq j}}^{k+1}\left(\left(\varepsilon-(\varepsilon-1)\gamma_i\right)^p\left(\varepsilon-(\varepsilon-1)\gamma_j\right)^q+\left(\varepsilon^2-1\right){\gamma_i}^p{\gamma_j}^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}+(\varepsilon-1)\prod_{\substack{i,j=1\\ i\neq j}}^{k+1}\left(\left(\varepsilon-(\varepsilon-1)\gamma_i\right)^p\left(\varepsilon-(\varepsilon-1)\gamma_j\right)^q-{\gamma_i}^p{\gamma_j}^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}\right\},\right.\right.$$

$$\bigcup_{\delta_i\in\tilde{I}(a_i),\delta_j\in\tilde{I}(a_j)}\left\{\frac{\varepsilon\prod_{\substack{i,j=1\\ i\neq j}}^{k+1}\left(\left(1+(\varepsilon-1)\delta_i\right)^p\left(1+(\varepsilon-1)\delta_j\right)^q-\left(1-\delta_i\right)^p\left(1-\delta_j\right)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}{\prod_{\substack{i,j=1\\ i\neq j}}^{k+1}\left(\left(1+(\varepsilon-1)\delta_i\right)^p\left(1+(\varepsilon-1)\delta_j\right)^q+(\varepsilon^2-1)\left(1-\delta_i\right)^p\left(1-\delta_j\right)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}+(\varepsilon-1)\prod_{\substack{i,j=1\\ i\neq j}}^{k+1}\left(\left(1+(\varepsilon-1)\delta_i\right)^p\left(1+(\varepsilon-1)\delta_j\right)^q-\left(1-\delta_i\right)^p\left(1-\delta_j\right)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}\right\},$$

$$\left.\left.\bigcup_{\eta_i\in\tilde{F}(a_i),\eta_j\in\tilde{F}(a_j)}\left\{\frac{\varepsilon\prod_{\substack{i,j=1\\ i\neq j}}^{k+1}\left(\left(1+(\varepsilon-1)\eta_i\right)^p\left(1+(\varepsilon-1)\eta_j\right)^q-\left(1-\eta_i\right)^p\left(1-\eta_j\right)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}{\prod_{\substack{i,j=1\\ i\neq j}}^{k+1}\left(\left(1+(\varepsilon-1)\eta_i\right)^p\left(1+(\varepsilon-1)\eta_j\right)^q+(\varepsilon^2-1)\left(1-\eta_i\right)^p\left(1-\eta_j\right)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}+(\varepsilon-1)\prod_{\substack{i,j=1\\ i\neq j}}^{k+1}\left(\left(1+(\varepsilon-1)\eta_i\right)^p\left(1+(\varepsilon-1)\eta_j\right)^q-\left(1-\eta_i\right)^p\left(1-\eta_j\right)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}}\right\}\right)\right\rangle.$$

That is, If $n=k+1$, Eq. (2) is right. Therefore, for all n, Eq. (2) is right.

Then, Eq. (1) is right.

In the following, the properties of MVNLNWBMH operator will be proved.

(1) Reducibility. Let $\omega=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)$. Then $MVNLNWBMH(a_1,a_2,\cdots a_n)=MVNLBMH(a_1,a_2,\cdots a_n)$.

Proof. Since $\omega=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)$, then according to the operations in Definition 10, the result below can be obtained.

Mathematical Problems in Engineering.

$$MVNLNWBMH(a_1, a_2, \cdots a_n) = \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} (a_i^{\,p} \otimes a_j^{\,q}) \right)^{\frac{1}{p+q}}$$

$$= \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\frac{1}{n^2}}{1-\frac{1}{n}} (a_i^{\,p} \otimes a_j^{\,q}) \right)^{\frac{1}{p+q}}$$

$$= \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{1}{n(n-1)} (a_i^{\,p} \otimes a_j^{\,q}) \right)^{\frac{1}{p+q}}$$

$$= \left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} (a_i^{\,p} \otimes a_j^{\,q}) \right)^{\frac{1}{p+q}}$$

$$= MVNLBMH(a_1, a_2, \cdots a_n).$$

(2) Idempotency. Let $a_i = a\left(i = 1, 2, \cdots, n\right)$. Then $MVNLNWBMH(a_1, a_2, \cdots a_n) = a$.

Proof. For each i, owing to $a_i = a$, the formula below is obtained on the basis of Eq. (5) in Theorem 1.

$$MVNLNWBMH(a_1, a_2, \cdots a_n) = \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} (a_i^{\,p} \otimes a_j^{\,q}) \right)^{\frac{1}{p+q}}$$

$$= \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} (a^{p} \otimes a^{q}) \right)^{\frac{1}{p+q}}$$

$$= \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} a^{p+q} \right)^{\frac{1}{p+q}}$$

$$= a \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} \right)^{\frac{1}{p+q}}$$

$$= a.$$

(3) Commutativity. Let $\left(\tilde{a}_1, \tilde{a}_2, \cdots \tilde{a}_n\right)$ be any permutation of $\left(a_1, a_2, \cdots a_n\right)$. Then

$$MVNLNWBMH\left(\tilde{a}_1, \tilde{a}_2, \cdots \tilde{a}_n\right) = MVNLNWBMH(a_1, a_2, \cdots a_n).$$

Proof. Owing to $\left(\tilde{a}_1, \tilde{a}_2, \cdots \tilde{a}_n\right)$ is permutation of $(a_1, a_2, \cdots a_n)$, then the equation below can be obtained.

$$MVNLNWBMH(a_1, a_2, \cdots a_n) = \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} (a_i^{\ p} \otimes a_j^{\ q}) \right)^{\frac{1}{p+q}}$$

$$= \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} (\tilde{a}_i^{\ p} \otimes \tilde{a}_j^{\ q}) \right)^{\frac{1}{p+q}}$$

(4) Monotonicity. Suppose $a_i = \left\langle s_{\theta(a_i)}, \left(\tilde{T}(a_i), \tilde{I}(a_i), \tilde{F}(a_i)\right)\right\rangle$ $(i = 1, 2, \cdots, n)$ and $b_i = \left\langle s_{\theta(b_i)}, \left(\tilde{T}(b_i), \tilde{I}(b_i), \tilde{F}(b_i)\right)\right\rangle$ $(i = 1, 2, \cdots, n)$ are two sets of MVNLNs, when $s_{\theta(a_i)} \geq s_{\theta(b_i)}$, $\tilde{T}(a_i) \geq \tilde{T}(b_i)$, $\tilde{I}(a_i) \leq \tilde{I}(b_i)$ and $\tilde{F}(a_i) \leq \tilde{F}(b_i)$ for each i, then $MVNLNWBMH(a_1, a_2, \cdots a_n) \geq MVNLNWBMH(b_1, b_2, \cdots b_n)$.

Proof. (I) Linguistic term part

Owing to $p, q \geq 0$, and $s_{\theta(a_i)} \geq s_{\theta(b_i)}$ for each i, the result below is gained.

$\theta^p(a_i) \geq \theta^p(b_i)$ and $\theta^q(a_i) \geq \theta^q(b_i)$

$$\Rightarrow \theta^p(a_i)\theta^q(a_i) \geq \theta^p(b_i)\theta^q(b_i)$$

$$\Rightarrow \frac{\omega_i \omega_j}{1-\omega_i} \theta^p(a_i)\theta^q(a_i) \geq \frac{\omega_i \omega_j}{1-\omega_i} \theta^p(b_i)\theta^q(b_i)$$

$$\Rightarrow \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} \theta^p(a_i)\theta^q(a_i) \geq \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} \theta^p(b_i)\theta^q(b_i)$$

$$\Rightarrow \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} \theta^p(a_i)\theta^q(a_i) \right)^{\frac{1}{p+q}} \geq \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} \theta^p(b_i)\theta^q(b_i) \right)^{\frac{1}{p+q}}$$

(II) True, indeterminacy and falsity membership parts

Owing to $\tilde{T}(a_i) \geq \tilde{T}(b_i)$, $\tilde{I}(a_i) \leq \tilde{I}(b_i)$ and $\tilde{F}(a_i) \leq \tilde{F}(b_i)$ for each i, $p, q \geq 0$, then the following results can be proved easily.

$$\frac{\varepsilon \left(x_{a_i} - y_{a_i}\right)^{\frac{1}{p+q}}}{\left(x_{a_i} + \left(\varepsilon^2 - 1\right) y_{a_i}\right)^{\frac{1}{p+q}} + \left(\varepsilon - 1\right)\left(x_{a_i} - y_{a_i}\right)^{\frac{1}{p+q}}} \geq \frac{\varepsilon \left(x_{b_i} - y_{b_i}\right)^{\frac{1}{p+q}}}{\left(x_{b_i} + \left(\varepsilon^2 - 1\right) y_{b_i}\right)^{\frac{1}{p+q}} + \left(\varepsilon - 1\right)\left(x_{b_i} - y_{b_i}\right)^{\frac{1}{p+q}}};$$

$$\frac{\left(g_{a_i} + \left(\varepsilon^2 - 1\right) h_{a_i}\right)^{\frac{1}{p+q}} - \left(g_{a_i} - h_{a_i}\right)^{\frac{1}{p+q}}}{\left(g_{a_i} + \left(\varepsilon^2 - 1\right) h_{a_i}\right)^{\frac{1}{p+q}} + \left(\varepsilon - 1\right)\left(g_{a_i} - h_{a_i}\right)^{\frac{1}{p+q}}} \leq \frac{\left(g_{b_i} + \left(\varepsilon^2 - 1\right) h_{b_i}\right)^{\frac{1}{p+q}} - \left(g_{b_i} - h_{b_i}\right)^{\frac{1}{p+q}}}{\left(g_{b_i} + \left(\varepsilon^2 - 1\right) h_{b_i}\right)^{\frac{1}{p+q}} + \left(\varepsilon - 1\right)\left(g_{b_i} - h_{b_i}\right)^{\frac{1}{p+q}}};$$

$$\frac{\left(v_{a_i} + \left(\varepsilon^2 - 1\right) u_{a_i}\right)^{\frac{1}{p+q}} - \left(v_{a_i} - u_{a_i}\right)^{\frac{1}{p+q}}}{\left(v_{a_i} + \left(\varepsilon^2 - 1\right) u_{a_i}\right)^{\frac{1}{p+q}} + \left(\varepsilon - 1\right)\left(v_{a_i} - u_{a_i}\right)^{\frac{1}{p+q}}} \leq \frac{\left(v_{b_i} + \left(\varepsilon^2 - 1\right) u_{b_i}\right)^{\frac{1}{p+q}} - \left(v_{b_i} - u_{b_i}\right)^{\frac{1}{p+q}}}{\left(v_{b_i} + \left(\varepsilon^2 - 1\right) u_{b_i}\right)^{\frac{1}{p+q}} + \left(\varepsilon - 1\right)\left(v_{b_i} - u_{b_i}\right)^{\frac{1}{p+q}}}.$$

Where the corresponding x, y, g, h, u and v are defined in Definition 10.

(III) Comparing $MVNLNWBMH(a_1, a_2, \cdots a_n)$ with $MVNLNWBMH(b_1, b_2, \cdots b_n)$

Suppose $a = \left\langle s_{\theta(a)}, \left(\tilde{T}(a), \tilde{I}(a), \tilde{F}(a)\right)\right\rangle = MVNLNWBMH(a_1, a_2, \cdots a_n)$ and

$b = \left\langle s_{\theta(b)}, \left(\tilde{T}(b), \tilde{I}(b), \tilde{F}(b)\right)\right\rangle = MVNLNWBMH(b_1, b_2, \cdots b_n)$.

Because $s_{\theta(a)} \geq s_{\theta(b)}$, $\tilde{T}(a) \geq \tilde{T}(b)$, $\tilde{I}(a) \leq \tilde{I}(b)$ and $\tilde{F}(a) \leq \tilde{F}(b)$, thus $a \geq b$.

Then $MVNLNWBMH(a_1, a_2, \cdots a_n) \geq MVNLNWBMH(b_1, b_2, \cdots b_n)$.

In the following, a few special examples for MVNLNWBMH operator regarding different values ε, p and q will be explored.

(1) If $q = 0$, then the MVNLNWBMH operator defined by equation (1) will be reduced to the generalized multi-valued neutrosophic linguistic Hamacher weighted average (GMVNLHWA) operator shown as below.

$$GMVNLHWA(a_1, a_2, \cdots a_n) = \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i}(a_i^{\ p} \otimes a_j^{\ q})\right)^{\frac{1}{p+q}}$$

$$= \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\omega_i \omega_j}{1-\omega_i} a_i^{\ p}\right)^{\frac{1}{p}}$$

$$= \left(\bigoplus_{i=1}^{n} \frac{\omega_i\left(1-\omega_i\right)}{1-\omega_i} a_i^{\ p}\right)^{\frac{1}{p}}$$

$$= \left(\bigoplus_{i=1}^{n} \omega_i a_i^{\ p}\right)^{\frac{1}{p}}$$

$$= \left\langle s_{\left(\sum_{i=1}^{n}\left(\omega_i \cdot \theta^{p}(a_i)\right)\right)^{\frac{1}{p}}}, \right.$$

$$\left(\bigcup_{\gamma_i \in \tilde{T}(a_i), \gamma_j \in \tilde{T}(a_j)} \left\{ \frac{\varepsilon\left(x' - y'\right)^{\frac{1}{p}}}{\left(x' + \left(\varepsilon^2 - 1\right) y'\right)^{\frac{1}{p}} + \left(\varepsilon - 1\right)\left(x' - y'\right)^{\frac{1}{p}}} \right\}, \right.$$

$$\bigcup_{\delta_i \in \tilde{I}(a_i), \delta_j \in \tilde{I}(a_j)} \left\{ \frac{\left(g' + \left(\varepsilon^2 - 1\right) h'\right)^{\frac{1}{p}} - \left(g' - h'\right)^{\frac{1}{p}}}{\left(g' + \left(\varepsilon^2 - 1\right) h'\right)^{\frac{1}{p}} + \left(\varepsilon - 1\right)\left(g' - h'\right)^{\frac{1}{p}}} \right\},$$

$$\left.\left. \bigcup_{\eta_i \in \tilde{F}(a_i), \eta_j \in \tilde{F}(a_j)} \left\{ \frac{\left(v' + \left(\varepsilon^2 - 1\right) u'\right)^{\frac{1}{p}} - \left(v' - u'\right)^{\frac{1}{p}}}{\left(v' + \left(\varepsilon^2 - 1\right) u'\right)^{\frac{1}{p}} + \left(\varepsilon - 1\right)\left(v' - u'\right)^{\frac{1}{p}}} \right\} \right) \right\rangle$$

Where $x' = \prod_{i=1}^{n}\left(\left(1 + \left(\varepsilon - 1\right)\left(1 - \gamma_i\right)\right)^{p} + \left(\varepsilon^2 - 1\right)\gamma_i^{\ p}\right)^{\omega_i}$,

$$y' = \prod_{i=1}^{n}\left(\left(\varepsilon - \left(\varepsilon - 1\right)\gamma_i\right)^{p} - \gamma_i^{\ p}\right)^{\omega_i},$$

$$h' = \prod_{i=1}^{n}\left(\left(1 + \left(\varepsilon - 1\right)\delta_i\right)^{p} - \left(1 - \delta_i\right)^{p}\right)^{\omega_i},$$

$$g' = \prod_{i=1}^{n}\left(\left(1+(\varepsilon-1)\delta_i\right)^p + \left(\varepsilon^2-1\right)\left(1-\delta_i\right)^p\right)^{\omega_i},$$

$$u' = \prod_{i=1}^{n}\left(\left(1+(\varepsilon-1)\eta_i\right)^p - \left(1-\eta_i\right)^p\right)^{\omega_i},$$

$$v' = \prod_{i=1}^{n}\left(\left(1+(\varepsilon-1)\eta_i\right)^p + \left(\varepsilon^2-1\right)\left(1-\eta_i\right)^p\right)^{\omega_i}$$

(2) When $p = 1, q = 0$, then the operator of MVNLNWBMH defined by equation (1) will be reduced to the multi-valued neutrosophic linguistic Hamacher weighted arithmetic average (MVNLHWAA) operator shown as below.

$$MVNLHWAA(a_1, a_2, \cdots a_n) = \bigoplus_{i=1}^{n}\omega_i a_i$$

$$= \left\langle s_{\sum_{i=1}^{n}\omega_i\theta(a_i)},\right.$$

$$\left(\bigcup_{\gamma_i\in\tilde{T}(a_i)}\left\{\frac{\prod_{i=1}^{n}\left(1+(\varepsilon-1)\gamma_i\right)^{\omega_i} - \prod_{i=1}^{n}\left(1-\gamma_i\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\varepsilon-1)\gamma_i\right)^{\omega_i} + (\varepsilon-1)\prod_{i=1}^{n}\left(1-\gamma_i\right)^{\omega_i}}\right\},\right.$$

$$\bigcup_{\delta_i\in\tilde{I}(a_i)}\left\{\frac{\varepsilon\prod_{i=1}^{n}\delta_i^{\omega_i}}{(\varepsilon-1)\prod_{i=1}^{n}\delta_i^{\omega_i} + \prod_{i=1}^{n}\left(\varepsilon-(\varepsilon-1)\delta_i\right)^{\omega_i}}\right\},$$

$$\left.\left.\bigcup_{\eta_i\in\tilde{F}(a_i)}\left\{\frac{\varepsilon\prod_{i=1}^{n}\eta_i^{\omega_i}}{(\varepsilon-1)\prod_{i=1}^{n}\eta_i^{\omega_i} + \prod_{i=1}^{n}\left(\varepsilon-(\varepsilon-1)\eta_i\right)^{\omega_i}}\right\}\right)\right\rangle$$

If $\varepsilon = 1$, the MVNLHWAA operator will be reduced to MVNLWAA operator shown as below.

$$MVNLWAA(a_1, a_2, \cdots a_n) = \left\langle s_{\sum_{i=1}^{n}\omega_i\theta(a_i)},\right.$$

$$\left(\bigcup_{\gamma_i\in\tilde{T}(a_i)}\left\{1-\prod_{i=1}^{n}\left(1-\gamma_i\right)^{\omega_i}\right\},\right.$$

$$\bigcup_{\delta_i\in\tilde{I}(a_i)}\left\{\prod_{i=1}^{n}\delta_i^{\omega_i}\right\},$$

$$\left.\left.\bigcup_{\eta_i\in\tilde{F}(a_i)}\left\{\prod_{i=1}^{n}\eta_i^{\omega_i}\right\}\right)\right\rangle$$

If $\varepsilon = 2$, the MVNLHWAA operator will be reduced to the multi-valued neutrosophic linguistic Einstein weighted arithmetic average (MVNLEWAA) operator shown in the following.

$$MVNLWAA(a_1, a_2, \cdots a_n) = \left\langle s_{\sum_{i=1}^{n} \omega_i \theta(a_i)}, \right.$$

$$\left(\bigcup_{\gamma_i \in \tilde{T}(a_i)} \left\{ \frac{\prod_{i=1}^{n} (1+\gamma_i)^{\omega_i} - \prod_{i=1}^{n} (1-\gamma_i)^{\omega_i}}{\prod_{i=1}^{n} (1+\gamma_i)^{\omega_i} + \prod_{i=1}^{n} (1-\gamma_i)^{\omega_i}} \right\}, \right.$$

$$\bigcup_{\delta_i \in \tilde{I}(a_i)} \left\{ \frac{2\prod_{i=1}^{n} \delta_i^{\omega_i}}{\prod_{i=1}^{n} (2-\delta_i)^{\omega_i} + \prod_{i=1}^{n} \delta_i^{\omega_i}} \right\},$$

$$\left. \left. \bigcup_{\eta_i \in \tilde{F}(a_i)} \left\{ \frac{2\prod_{i=1}^{n} \eta_i^{\omega_i}}{\prod_{i=1}^{n} (2-\eta_i)^{\omega_i} + \prod_{i=1}^{n} \eta_i^{\omega_i}} \right\} \right) \right\rangle$$

(3) If $p \to 0, q = 0$, then the MVNLNWBMH operator defined by equation (1) will be reduced to the multi-valued neutrosophic linguistic Hamacher weighted geometric average (MVNLHWGA) operator shown as below.

$$MVNLHWGA(a_1, a_2, \cdots a_n) = \bigotimes_{i=1}^{n} a_i^{\omega_i}$$

$$= \left\langle s_{\prod_{i=1}^{n} \theta^{\omega_i}(a_i)}, \right.$$

$$\left(\bigcup_{\gamma_i \in \tilde{T}(a_i)} \left\{ \frac{\varepsilon \prod_{i=1}^{n} \gamma_i^{\omega_i}}{(\varepsilon - 1)\prod_{i=1}^{n} \gamma_i^{\omega_i} + \prod_{i=1}^{n} \left(\varepsilon - (\varepsilon - 1)\gamma_i\right)^{\omega_i}} \right\}, \right.$$

$$\bigcup_{\delta_i \in \tilde{I}(a_i)} \left\{ \frac{\prod_{i=1}^{n} \left(1 + (\varepsilon - 1)\delta_i\right)^{\omega_i} - \prod_{i=1}^{n} (1-\delta_i)^{\omega_i}}{\prod_{i=1}^{n} \left(1 + (\varepsilon - 1)\delta_i\right)^{\omega_i} + (\varepsilon - 1)\prod_{i=1}^{n} (1-\delta_i)^{\omega_i}} \right\},$$

$$\left. \left. \bigcup_{\eta_i \in \tilde{F}(a_i)} \left\{ \frac{\prod_{i=1}^{n} \left(1 + (\varepsilon - 1)\eta_i\right)^{\omega_i} - \prod_{i=1}^{n} (1-\eta_i)^{\omega_i}}{\prod_{i=1}^{n} \left(1 + (\varepsilon - 1)\eta_i\right)^{\omega_i} + (\varepsilon - 1)\prod_{i=1}^{n} (1-\eta_i)^{\omega_i}} \right\} \right) \right\rangle$$

If $\varepsilon = 1$, the MVNLHWGA operator will be reduced to MVNLWGA operator shown as below.

$$MVNLHWGA(a_1, a_2, \cdots a_n) = \Big\langle s_{\prod_{i=1}^{n} \theta^{\omega_i}(a_i)},$$

$$\Big(\bigcup_{\gamma_i \in \tilde{T}(a_i)} \Big\{ \prod_{i=1}^{n} \gamma_i^{\omega_i} \Big\},$$

$$\bigcup_{\delta_i \in \tilde{I}(a_i)} \Big\{ 1 - \prod_{i=1}^{n} (1 - \delta_i)^{\omega_i} \Big\},$$

$$\bigcup_{\eta_i \in \tilde{F}(a_i)} \Big\{ 1 - \prod_{i=1}^{n} (1 - \eta_i)^{\omega_i} \Big\} \Big) \Big\rangle$$

If $\varepsilon = 2$, the MVNLHWGA operator will be reduced to the multi-valued neutrosophic linguistic Einstein weighted geometric average (MVNLEWGA) operator shown as below.

$$MVNLHWGA(a_1, a_2, \cdots a_n) = \Big\langle s_{\prod_{i=1}^{n} \theta^{\omega_i}(a_i)},$$

$$\Bigg(\bigcup_{\gamma_i \in \tilde{T}(a_i)} \left\{ \frac{2\prod_{i=1}^{n} \gamma_i^{\omega_i}}{\prod_{i=1}^{n} (2 - \gamma_i)^{\omega_i} + \prod_{i=1}^{n} \gamma_i^{\omega_i}} \right\},$$

$$\bigcup_{\delta_i \in \tilde{I}(a_i)} \left\{ \frac{\prod_{i=1}^{n} (1 + \delta_i)^{\omega_i} - \prod_{i=1}^{n} (1 - \delta_i)^{\omega_i}}{\prod_{i=1}^{n} (1 + \delta_i)^{\omega_i} + \prod_{i=1}^{n} (1 - \delta_i)^{\omega_i}} \right\},$$

$$\bigcup_{\eta_i \in \tilde{F}(a_i)} \left\{ \frac{\prod_{i=1}^{n} (1 + \eta_i)^{\omega_i} - \prod_{i=1}^{n} (1 - \eta_i)^{\omega_i}}{\prod_{i=1}^{n} (1 + \eta_i)^{\omega_i} + \prod_{i=1}^{n} (1 - \eta_i)^{\omega_i}} \right\} \Bigg) \Big\rangle$$

(4) If $\varepsilon = 1$, then the MVNLNWBMH operator defined by equation (1) will be reduced to the MVNLNWBM operator shown as below.

Mathematical Problems in Engineering.

$$MVNLNWBM(a_1, a_2, \cdots a_n) = \left\langle s_{\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\theta^p(a_i)\cdot\theta^q(a_j)\right)\right)^{\frac{1}{p+q}}},\right.$$

$$\left(\bigcup_{\gamma_i\in\tilde{T}(a_i),\gamma_j\in\tilde{T}(a_j)}\left\{\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\gamma_i^{\,p}\gamma_j^{\,q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}}\right\},\right.$$

$$\bigcup_{\delta_i\in\tilde{I}(a_i),\delta_j\in\tilde{I}(a_j)}\left\{1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\delta_i\right)^p\left(1-\delta_j\right)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}}\right\},$$

$$\left.\left.\bigcup_{\eta_i\in\tilde{F}(a_i),\eta_j\in\tilde{F}(a_j)}\left\{1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\eta_i\right)^p\left(1-\eta_j\right)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}}\right\}\right)\right\rangle$$

(5) If $\varepsilon = 2$, then the MVNLNWBMH operator defined by equation (1) will be reduced to the MVNLNWBME, that is the simplification of multi-valued neutrosophic linguistic NWBM Einstein.

$$MVNLNWBMH(a_1, a_2, \cdots a_n) = \left\langle s_{\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\theta^p(a_i)\cdot\theta^q(a_j)\right)\right)^{\frac{1}{p+q}}},\right.$$

$$\left(\bigcup_{\gamma_i\in\tilde{T}(a_i),\gamma_j\in\tilde{T}(a_j)}\left\{\frac{2\left(x-y\right)^{\frac{1}{p+q}}}{\left(x+3y\right)^{\frac{1}{p+q}}+\left(x-y\right)^{\frac{1}{p+q}}}\right\},\right.$$

$$\bigcup_{\delta_i\in\tilde{I}(a_i),\delta_j\in\tilde{I}(a_j)}\left\{\frac{\left(g+3h\right)^{\frac{1}{p+q}}-\left(g-h\right)^{\frac{1}{p+q}}}{\left(g+3h\right)^{\frac{1}{p+q}}+\left(g-h\right)^{\frac{1}{p+q}}}\right\},$$

$$\left.\left.\bigcup_{\eta_i\in\tilde{F}(a_i),\eta_j\in\tilde{F}(a_j)}\left\{\frac{\left(v+3u\right)^{\frac{1}{p+q}}-\left(v-u\right)^{\frac{1}{p+q}}}{\left(v+3u\right)^{\frac{1}{p+q}}+\left(v-u\right)^{\frac{1}{p+q}}}\right\}\right)\right\rangle$$

Where $x = \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(2-\gamma_i\right)^p\left(2-\gamma_j\right)^q+3\gamma_i^{\,p}\gamma_j^{\,q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}$,

$$y = \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(2-\gamma_i\right)^p\left(2-\gamma_j\right)^q-\gamma_i^{\,p}\gamma_j^{\,q}\right)^{\frac{\omega_i\omega_j}{1-\omega_i}},$$

$$h = \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(\left(1+\delta_i\right)^p\left(1+\delta_j\right)^q-\left(1-\delta_i\right)^p\left(1-\delta_j\right)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}},$$

$$g = \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(1+\delta_i\right)^p \left(1+\delta_j\right)^q + 3\left(1-\delta_i\right)^p \left(1-\delta_j\right)^q \right)^{\frac{\omega_i \omega_j}{1-\omega_i}},$$

$$u = \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(1+\eta_i\right)^p \left(1+\eta_j\right)^q - \left(1-\eta_i\right)^p \left(1-\eta_j\right)^q \right)^{\frac{\omega_i \omega_j}{1-\omega_i}},$$

$$v = \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(1+\eta_i\right)^p \left(1+\eta_j\right)^q + 3\left(1-\eta_i\right)^p \left(1-\eta_j\right)^q \right)^{\frac{\omega_i \omega_j}{1-\omega_i}}.$$

From the above analysis, we can obtain the MVNLNWBMH operator is more generalized.

5 The multiple criteria decision making approach based on the MVNLNWBMH operator

The proposed MVNLNWBMH operator is presented to cope with MCDM problem under multi-valued neutrosophic linguistic environment in this subsection.

Suppose that $A = \{A_1, A_2, \cdots, A_m\}$ represent m alternatives, and $C = \{C_1, C_2, \cdots, C_n\}$ represent n criteria. Let $\omega = \{\omega_1, \omega_2, \cdots, \omega_n\}$ be the corresponding weights of criteria, where $\omega_j \geq 0 (j = 1, 2, \cdots, n)$, and $\sum_{j=1}^{n} \omega_j = 1$ .The evaluation value of the criteria $C_j\ (j = 1, 2, \cdots, n)$ regarding the alternative $A_i\ (i = 1, 2, \cdots, m)$ is provided by experts. Each value is represented by MVNLNNs. Suppose that $R = \left[a_{ij}\right]_{m \times n}$ is the multi-valued neutrosophic linguistic decision matrix, $a_{ij} = \left\langle s_{\theta(a_{ij})}, \left(\tilde{T}(a_{ij}), \tilde{I}(a_{ij}), \tilde{F}(a_{ij})\right) \right\rangle$ is the evaluation information which represents the assessment value of alternative $A_i (i = 1, 2, \cdots, m)$ on criteria $C_j (j = 1, 2, \cdots, n)$ with respect to the linguistic value $s_{\theta(a_{ij})}$, where $\tilde{T}(a_{ij})$ indicates the satisfaction degree, $\tilde{I}(a_{ij})$ indicates the indeterminacy degree and $\tilde{F}(a_{ij})$ indicates the dissatisfaction degree.

Then, the main method for ranking and selecting the best alternative is presented in the following.

Step1. The decision matrix is normalized.

Generally, criteria in MCDM problems consist of two types: maximum type and minimum type, the minimum type should be transformed into the maximum type for eliminating the influence of distinguished types. Suppose that $R = \left[a_{ij}\right]_{m \times n}$ is the original decision matrix, which can be normalized as follows:

$$b_{ij} = \begin{cases} a_{ij}, & \text{for maximizing criteria} \\ \left\langle s_{\iota-\theta(a_{ij})}, \left(\tilde{T}(a_{ij}), \tilde{I}(a_{ij}), \tilde{F}(a_{ij})\right) \right\rangle, & \text{for maximizing criteria} \end{cases}$$

Thus, the normalized matrix $B = \left[b_{ij}\right]_{m \times n}$ is gained.

Step2. The comprehensive value of each alternative is calculated.

The comprehensive value represented by $a_i (i = 1, 2, \cdots, m)$ can be obtained by utilizing the MVNLNWBMH operator in Definition 10, which can aggregate the overall value for each alternative with respect to all criteria.

Step3. The compared values of three functions are calculated.

According to the equations given in Definition 8, the score value denoted by $\mathrm{E}(a_i)$, the accuracy value denoted by $\mathrm{H}(a_i)$ and the certainty value denoted by $\mathrm{C}(a_i)$ can be obtained.

Step4. The alternatives are selected.

Based on Definition 9, all alternatives $A_i (i = 1, 2, \cdots, m)$ can be ranked on the basis of $\mathrm{E}(a_i)$, $\mathrm{H}(a_i)$ and $\mathrm{C}(a_i)$, and the best alternative(s) can be selected.

6 An numberical example

In order to validate the effectiveness and practical of the novel approach, an investment project is adapted from Ye.

An investment company wants to expand its business. Four alternatives will be chosen, A_1 represents auto corporation, A_2 represents food corporation, A_3 represents computer company corporation), A_4 represents weapon corporation. Each alternative is evaluated under three criteria, C_1 denotes risk, C_2 denotes growth, C_3 denotes the impact of environment, where C_3 is the minimizing criteria. The corresponding weighted vector is $\omega = \{0.35, 0.25, 0.4\}$. In real situation, the decision maker may hesitant and give several possible value for the satisfaction, indeterminacy and dissatisfaction regarding the alternative A_i corresponding to the criteria C_j under the linguistic term set S. Therefore, the assessment value is given in the form of MVNLNs, and the linguistic term set is employed as $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\} = \{\text{extremely poor,very poor,poor,medium,good,very good,extremely good}\}$.

The multi-valued neutrosophic linguistic decision matrix $R = [a_{ij}]_{4\times3}$ is shown as follows.

$$R = [a_{ij}]_{4\times3} = \begin{bmatrix} \langle s_5, (\{0.3, 0.4, 0.5\}, \{0.1\}, \{0.3, 0.4\}) \rangle & \langle s_6, (\{0.5, 0.6\}, \{0.2, 0.3\}, \{0.3, 0.4\}) \rangle & \langle s_5, (\{0.2, 0.3\}, \{0.1, 0.2\}, \{0.5, 0.6\}) \rangle \\ \langle s_6, (\{0.6, 0.7\}, \{0.1, 0.2\}, \{0.2, 0.3\}) \rangle & \langle s_5, (\{0.6, 0.7\}, \{0.1\}, \{0.3\}) \rangle & \langle s_5, (\{0.6, 0.7\}, \{0.1, 0.2\}, \{0.1, 0.2\}) \rangle \\ \langle s_6, (\{0.5, 0.6\}, \{0.4\}, \{0.2, 0.3\}) \rangle & \langle s_5, (\{0.6\}, \{0.3\}, \{0.4\}) \rangle & \langle s_4, (\{0.5, 0.6\}, \{0.1\}, \{0.3\}) \rangle \\ \langle s_4, (\{0.7, 0.8\}, \{0.1\}, \{0.1, 0.2\}) \rangle & \langle s_4, (\{0.6, 0.7\}, \{0.1\}, \{0.2\}) \rangle & \langle s_6, (\{0.3, 0.5\}, \{0.2\}, \{0.1, 0.2, 0.3\}) \rangle \end{bmatrix}$$

6.1 The procedure using the proposed aggregation operator

Step1. The decision matrix is normalized.

Because C_3 is the minimizing criteria, which should be converted to the maximizing criteria, then the normalized decision matrix $B = [b_{ij}]_{m\times n}$ · can be obtained as follows:

$$B = [b_{ij}]_{4\times3} = \begin{bmatrix} \langle s_5, (\{0.3, 0.4, 0.5\}, \{0.1\}, \{0.3, 0.4\}) \rangle & \langle s_6, (\{0.5, 0.6\}, \{0.2, 0.3\}, \{0.3, 0.4\}) \rangle & \langle s_2, (\{0.2, 0.3\}, \{0.1, 0.2\}, \{0.5, 0.6\}) \rangle \\ \langle s_6, (\{0.6, 0.7\}, \{0.1, 0.2\}, \{0.2, 0.3\}) \rangle & \langle s_5, (\{0.6, 0.7\}, \{0.1\}, \{0.3\}) \rangle & \langle s_2, (\{0.6, 0.7\}, \{0.1, 0.2\}, \{0.1, 0.2\}) \rangle \\ \langle s_6, (\{0.5, 0.6\}, \{0.4\}, \{0.2, 0.3\}) \rangle & \langle s_5, (\{0.6\}, \{0.3\}, \{0.4\}) \rangle & \langle s_3, (\{0.5, 0.6\}, \{0.1\}, \{0.3\}) \rangle \\ \langle s_4, (\{0.7, 0.8\}, \{0.1\}, \{0.1, 0.2\}) \rangle & \langle s_4, (\{0.6, 0.7\}, \{0.1\}, \{0.2\}) \rangle & \langle s_1, (\{0.3, 0.5\}, \{0.2\}, \{0.1, 0.2, 0.3\}) \rangle \end{bmatrix}$$

Step2. The comprehensive value of each alternative is calculated.

Derive the comprehensive value $a_i (i = 1, 2, \cdots, m)$ of each alternative $A_i (i = 1, 2, \cdots, m)$ by using the MVNLNWBMH operator presented in Definition 10. Here let $p = q = 1, \varepsilon = 1$. The MVNLNWBMH operator is shown as below:

$$MVNLNWBM(a_1, a_2, \cdots a_n) = \left\langle s_{\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\frac{\omega_i\omega_j}{1-\omega_i}\cdot\theta(a_i)\cdot\theta(a_j)\right)\right)^{\frac{1}{2}}},\right.$$

$$\left(\bigcup_{\gamma_i\in\tilde{T}(a_i),\gamma_j\in\tilde{T}(a_j)}\left\{\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\gamma_i\gamma_j\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{2}}\right\},\right.$$

$$\bigcup_{\delta_i\in\tilde{I}(a_i),\delta_j\in\tilde{I}(a_j)}\left\{1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\delta_i\right)\left(1-\delta_j\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{2}}\right\},$$

$$\left.\left.\bigcup_{\eta_i\in\tilde{F}(a_i),\eta_j\in\tilde{F}(a_j)}\left\{1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\eta_i\right)\left(1-\eta_j\right)\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{2}}\right\}\right)\right\rangle$$

And we have

$a_1 = \langle s_{3.9529},$

$(\{0.3088, 0.3515, 0.3309, 0.3752, 0.3429, 0.3869, 0.3671, 0.4118, 0.3746, 0.4199, 0.4009, 0.4463\},$

$\{0.1255, 0.1645, 0.1482, 0.19\},$

$\{0.3768, 0.4171, 0.4049, 0.446, 0.4109, 0.4511, 0.4382, 0.4783\})\rangle;$

$a_2 = \langle s_{3.9904},$

$(\{0.6, 0.6368, 0.6275, 0.6648, 0.6345, 0.6727, 0.6622, 0.7\},$

$\{0.1, 0.1363, 0.1335, 0.1714\},$

$\{0.1882, 0.2268, 0.2261, 0.2629\})\rangle;$

$a_3 = \langle s_{4.4850},$

$(\{0.5272, 0.5644, 0.5618, 0.6\},$

$\{0.263\},$

$\{0.2918, 0.3273\})\rangle;$

Step3. The compared values of three functions are calculated.

By using equations in Definition 8, we can obtain $E(a_i)$, $H(a_i)$ and $C(a_i), (i = 1, 2, 3, 4)$ as follows:

$E(a_1) = s_{2.3605}, H(a_1) = s_{-0.2036}, C(a_1) = s_{1.4879};$

$E(a_2) = s_{3.0440}, H(a_2) = s_{1.6912}, C(a_2) = s_{2.5930};$

$E(a_3) = s_{2.9763}, H(a_3) = s_{1.1382}, C(a_3) = s_{2.5267};$

$E(a_4) = s_{2.0002}, H(a_4) = s_{1.0641}, C(a_4) = s_{1.5461};$

Step4. The alternatives are selected.

By using the compared approach in Definition 9, $E(a_2) \succ E(a_3) \succ E(a_1) \succ E(a_4)$ can be obtained, so the final raking of alternatives is $A_2 \succ A_3 \succ A_1 \succ A_4$. Apparently, A_2 is the best one, and A_4 is the worst one. We don't need to compare the other functions because the values of score function differ.

6.2 Comparison analysis

We take different value into consideration in step 2 to select the alternative for discussing the impact of different values p, q and ε. The comparisons are presented in Table 1 and Table 2.

Table 1 Ranking of alternatives utilizing different p, q and $\varepsilon = 1$

$\varepsilon = 1, p, q$	Score function $\mathrm{E}(a_i), (i = 1, 2, 3, 4)$	Ranking
$p \to 0, q = 0$	$E(a_1) = s_{2.1214}, E(a_2) = s_{2.8131}, E(a_3) = s_{2.8732}, E(a_4) = s_{1.7094}$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$p = 0.01, q = 0$	$E(a_1) = s_{2.1907}, E(a_2) = s_{2.8460}, E(a_3) = s_{2.9575}, E(a_4) = s_{1.7629}$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$p = 0.1, q = 0$	$E(a_1) = s_{2.2165}, E(a_2) = s_{2.8790}, E(a_3) = s_{2.9713}, E(a_4) = s_{1.8005}$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$p = 1, q = 0$	$E(a_1) = s_{2.4697}, E(a_2) = s_{3.1970}, E(a_3) = s_{3.1084}, E(a_4) = s_{2.1573}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p = 2, q = 0$	$E(a_1) = s_{2.7154}, E(a_2) = s_{3.4882}, E(a_3) = s_{3.2507}, E(a_4) = s_{2.4519}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p = 5, q = 0$	$E(a_1) = s_{3.1768}, E(a_2) = s_{3.9655}, E(a_3) = s_{3.5749}, E(a_4) = s_{2.8445}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p = 10, q = 0$	$E(a_1) = s_{3.5203}, E(a_2) = s_{4.2619}, E(a_3) = s_{3.8673}, E(a_4) = s_{3.0415}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p = 0, q = 1$	$E(a_1) = s_{2.6253}, E(a_2) = s_{3.3082}, E(a_3) = s_{3.1563}, E(a_4) = s_{2.2904}$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$p = 0.01, q = 1$	$E(a_1) = s_{2.6151}, E(a_2) = s_{3.2980}, E(a_3) = s_{3.1492}, E(a_4) = s_{2.2778}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p = 0.1, q = 1$	$E(a_1) = s_{2.5377}, E(a_2) = s_{3.2202}, E(a_3) = s_{3.0964}, E(a_4) = s_{2.1839}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p = 1, q = 1$	$E(a_1) = s_{2.3605}, E(a_2) = s_{3.0440}, E(a_3) = s_{2.9763}, E(a_4) = s_{2.0002}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p = 2, q = 1$	$E(a_1) = s_{2.4824}, E(a_2) = s_{3.1918}, E(a_3) = s_{3.0464}, E(a_4) = s_{2.1664}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p = 5, q = 1$	$E(a_1) = s_{2.9208}, E(a_2) = s_{3.6644}, E(a_3) = s_{3.3532}, E(a_4) = s_{2.5870}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p = 10, q = 1$	$E(a_1) = s_{3.3239}, E(a_2) = s_{4.0427}, E(a_3) = s_{3.6946}, E(a_4) = s_{2.8680}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p = 0, q = 2$	$E(a_1) = s_{2.8597}, E(a_2) = s_{3.5712}, E(a_3) = s_{3.2871}, E(a_4) = s_{2.5589}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p = 0.01, q = 2$	$E(a_1) = s_{2.8519}, E(a_2) = s_{3.5630}, E(a_3) = s_{3.2820}, E(a_4) = s_{2.5494}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p = 0.1, q = 2$	$E(a_1) = s_{2.7889}, E(a_2) = s_{3.4959}, E(a_3) = s_{3.2400}, E(a_4) = s_{2.4731}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p = 1, q = 2$	$E(a_1) = s_{2.5228}, E(a_2) = s_{3.2087}, E(a_3) = s_{3.0542}, E(a_4) = s_{2.1890}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p = 2, q = 2$	$E(a_1) = s_{2.5363}, E(a_2) = s_{3.2305}, E(a_3) = s_{3.0440}, E(a_4) = s_{2.2430}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p = 5, q = 2$	$E(a_1) = s_{2.8535}, E(a_2) = s_{3.5768}, E(a_3) = s_{3.2558}, E(a_4) = s_{2.5581}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p = 10, q = 2$	$E(a_1) = s_{3.2306}, E(a_2) = s_{3.9375}, E(a_3) = s_{3.5836}, E(a_4) = s_{2.8212}$	$A_2 \succ A_3 \succ A_1 \succ A_4$

Table 2 Ranking of alternatives utilizing different p, q and $\varepsilon = 2$

$\varepsilon = 2, p, q$	Score function $\mathrm{E}(a_i), (i = 1, 2, 3, 4)$	Ranking
$p \to 0, q = 0$	$E(a_1) = s_{2.1344}, E(a_2) = s_{2.8137}, E(a_3) = s_{2.8863}, E(a_4) = s_{1.7187}$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$p = 0.01, q = 0$	$E(a_1) = s_{2.1944}, E(a_2) = s_{2.8478}, E(a_3) = s_{2.9631}, E(a_4) = s_{1.7661}$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$p = 0.1, q = 0$	$E(a_1) = s_{2.2171}, E(a_2) = s_{2.8803}, E(a_3) = s_{2.9752}, E(a_4) = s_{1.8021}$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$p = 1, q = 0$	$E(a_1) = s_{2.4553}, E(a_2) = s_{3.1938}, E(a_3) = s_{3.0982}, E(a_4) = s_{2.1483}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p = 2, q = 0$	$E(a_1) = s_{2.7142}, E(a_2) = s_{3.4825}, E(a_3) = s_{3.2361}, E(a_4) = s_{2.4453}$	$A_2 \succ A_3 \succ A_1 \succ A_4$

$p=5, q=0$	$E(a_1)=s_{3.2212}, E(a_2)=s_{3.9697}, E(a_3)=s_{3.5908}, E(a_4)=s_{2.8608}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p=10, q=0$	$E(a_1)=s_{3.5770}, E(a_2)=s_{4.2923}, E(a_3)=s_{3.9217}, E(a_4)=s_{3.0737}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p=0, q=1$	$E(a_1)=s_{2.6103}, E(a_2)=s_{3.3048}, E(a_3)=s_{3.1457}, E(a_4)=s_{2.2818}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p=0.01, q=1$	$E(a_1)=s_{2.6011}, E(a_2)=s_{3.2948}, E(a_3)=s_{3.1394}, E(a_4)=s_{2.2700}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p=0.1, q=1$	$E(a_1)=s_{2.5312}, E(a_2)=s_{3.2188}, E(a_3)=s_{3.0925}, E(a_4)=s_{2.1809}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p=1, q=1$	$E(a_1)=s_{2.3702}, E(a_2)=s_{3.0464}, E(a_3)=s_{2.9849}, E(a_4)=s_{2.0055}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p=2, q=1$	$E(a_1)=s_{2.4994}, E(a_2)=s_{3.1940}, E(a_3)=s_{3.0556}, E(a_4)=s_{2.1751}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p=5, q=1$	$E(a_1)=s_{2.9637}, E(a_2)=s_{3.6734}, E(a_3)=s_{3.3833}, E(a_4)=s_{2.6091}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p=10, q=1$	$E(a_1)=s_{3.3787}, E(a_2)=s_{4.0718}, E(a_3)=s_{3.7516}, E(a_4)=s_{2.9010}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p=0, q=2$	$E(a_1)=s_{2.8577}, E(a_2)=s_{3.5652}, E(a_3)=s_{3.2720}, E(a_4)=s_{2.5525}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p=0.01, q=2$	$E(a_1)=s_{2.8506}, E(a_2)=s_{3.5572}, E(a_3)=s_{3.2675}, E(a_4)=s_{2.5435}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p=0.1, q=2$	$E(a_1)=s_{2.7919}, E(a_2)=s_{3.4917}, E(a_3)=s_{3.2304}, E(a_4)=s_{2.4707}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p=1, q=2$	$E(a_1)=s_{2.5398}, E(a_2)=s_{3.2108}, E(a_3)=s_{3.0635}, E(a_4)=s_{2.1976}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p=2, q=2$	$E(a_1)=s_{2.5586}, E(a_2)=s_{3.2343}, E(a_3)=s_{3.0581}, E(a_4)=s_{2.2559}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p=5, q=2$	$E(a_1)=s_{2.8950}, E(a_2)=s_{3.5871}, E(a_3)=s_{3.2878}, E(a_4)=s_{2.5824}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$p=10, q=2$	$E(a_1)=s_{3.2843}, E(a_2)=s_{3.9650}, E(a_3)=s_{3.6392}, E(a_4)=s_{2.8549}$	$A_2 \succ A_3 \succ A_1 \succ A_4$

In table 1, we take the parameter value $\varepsilon = 1$, which is based on Algebraic operation, and the MVNLNWBM operator is applied. In table 2, we take the parameter value $\varepsilon = 2$, which is based on Einstein operation, and the MVNLNWBME operator is applied. As we can see from table 1 and table 2, the ordering of alternatives taking different parameters p, q and ε may be different, because the different parameters will cause different score function value. However, A_2 or A_3 is always the best selection, and A_4 is always the worst selection. Whether $\varepsilon = 1$ or $\varepsilon = 2$, the same ranking results is obtained with regard to the same parameter value p and the same parameter value q except for one situation in which $p = 0, q = 1$ and $\varepsilon = 1$ .Specially, if $p \to 0, q = 0$, MVNLNWBMH will reduce to MVNLHWGA operator. If $p \to 0, q = 0$ and $\varepsilon = 1$, MVNLHWGA will reduce to MVNLWGA operator. When $p = 1, q = 0$, MVNLNWBMH will reduce to MVNLHWAA operator. If $p = 1, q = 0$ and $\varepsilon = 1$, MVNLHWAA will reduce to MVNLWAA operator. If $\varepsilon = 1$ in Table 1, the ranking results on the basis of the MVNLWGA and MVNLWAA operators differ, which due to the two operators emphasis on different major points, and the same situation happens in Table 2.When the parameters $q = 0$ and $p \to 0, p = 0.01, p = 0.1$, respectively, the rankings are identical in two Tables, the ranking order is always $A_3 \succ A_2 \succ A_1 \succ A_4$. When p and q are assign the other values in two Tables, the ranking order is changed, and the result is $A_2 \succ A_3 \succ A_1 \succ A_4$ .That is, the best selection is from A_3 to A_2 except for one situation where $p = 0, q = 1$ in Table 1.

For illustrating the effective and flexible of the novel approach, the method in literature is adopted in multi-valued neutrosophic linguistic environment in this paper, and the same ranking orders are obtained in ref [8] where the SVNHFWA and SVNHFWG operators are adopted to fuse single-valued neutrosophic hesitant fuzzy information. When $\varepsilon = 1, p \to 0, q = 0$ and $\varepsilon = 1, p = 0, q = 0$, the two operators are special cases of MVNLNWBMH operator .Therefore, novel operator in this paper has better flexibility and generalization. In actual cases, the decision makers can assign different parameter values ε, p and q. . Generally, for convenience, we can set $\varepsilon = p = 1$, which can not only simplify the calculation, but also consider the interrelationship of multiple values.

7 Conclusions

In this paper, the MVNLS are proposed by combining the MVNS and LS, which not only describe linguistic terms, but also give the quantitative value of three membership degrees concerning the linguistic variables, which has better flexibility to express the decision information. Moreover, NWBM is a useful operator which has the trait of taking into account the interrelationship of different arguments, and overcome drawbacks of non-reducibility and non-idempotency. Hamacher operations are the extension of Algebraic and Einstein operations, which is more general. Considering these advantages, we have developed Hamacher operational laws for MVNLNs and extended the NWBM to fuse MVNL information. Thus, the MVNLNWBMH operator is proposed, which is appropriate to deal with MVNL information. Some desirable properties of the novel operator are discussed in detail, and some special cases are analyzed. Furthermore, the comparison method for MVNLNs is also studied, and the rankings of alternatives affecting by different parameters p, q and ε are also compared. For verifying the novel approach, we successfully applied the approach to an example. The results show the novel approach has the following advantages. The MVNLNWBMH operator is more flexible and more general, and which can capture the interrelationship among arguments and express decision information more practical, the decision makers can assign appropriate values according to the real situation. In future, we will explore to apply the operator to the different domains, for instance, fault diagnosis, machine learning and medical diagnosis.

As a future possible research, we will extend our research by using the refined neutrosophic set [61], i.e. the truth value T is refined into types of sub-truths such as T1, T2, etc., similarly indeterminacy I is refined into types of sub-indeterminacies I1, I2, etc., and the sub-falsehood F is split into F1, F2, etc.

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Compliance with ethical standards

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