Generalizations of neutrosophic subalgebras in BCK/BCI-algebras based on neutrosophic points

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Abstract: More general form of neutrosophic points is considered, and generalizations of results in the papers [Y.B. Jun, Neutrosophic subalgebras of several types in BCK/BCI-algebras, Ann. Fuzzy Math. Inform. 14 (2017), no. 1, 75–86] and [A. Borumand Saeid and Y.B. Jun, Neutrosophic subalgebras of BCK/BCI-algebras based on neutrosophic points, Ann. Fuzzy Math. Inform. 14 (2017), 87–97] are discussed. The concepts of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra, $(q_{(k_T,k_I,k_F)}, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra and $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra are introduced, and several properties are investigated. Characterizations of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra are discussed.

Keywords: $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra; $(q_{(k_T,k_I,k_F)}, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra; $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra

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1 Introduction

As a more general platform which extends the notions of the classical set, fuzzy set, interval valued fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set, Smarandache developed the concept of neutrosophic set which consists of three member functions, so called truth membership function, indeterminacy membership function and falsity membership function (see [1, 2, 3]). The neutrosophic set model is an important tool for dealing with real scientific and engineering applications because it can handle not only incomplete information but also the inconsistent information and indeterminate information which exist commonly in real situations. For further particulars on neutrosophic set theory, we refer the readers to the site

http://fs.gallup.unm.edu/FlorentinSmarandache.htm

Guo et al [4] extended the rough set model on two different universes in intuitionistic fuzzy approximation spaces to a single-valued neutrosophic environment. Jun [5] introduced the notion of (Φ, Ψ)-neutrosophic subalgebra of a BCK/BCI-algebra X for $\Phi, \Psi \in \{\in, q, \in \vee q\}$, and investigated related properties. He provided characterizations of an $(\in, \in)$-neutrosophic subalgebra and an $(\in, \in \vee q)$-neutrosophic subalgebra. Given special sets, so called neutrosophic $\in$-subsets, neutrosophic q-subsets and neutrosophic $\in \vee q$-subsets, he provided conditions for the neutrosophic $\in$-subsets, neutrosophic q-subsets and neutrosophic $\in \vee q$-subsets to be subalgebras. He considerd conditions for a neutrosophic set to be a $(q, \in \vee q)$-neutrosophic subalgebra. Saeid and Jun [6] gave relations between an $(\in, \in \vee q)$-neutrosophic subalgebra and a $(q, \in \vee q)$-neutrosophic subalgebra. They discussed characterization of an $(\in, \in \vee q)$-neutrosophic subalgebra by using neutrosophic $\in$-subsets, and provided conditions for an $(\in, \in \vee q)$-neutrosophic subalgebra to be a $(q, \in \vee q)$-neutrosophic subalgebra. They investigated properties on neutrosophic q-subsets and neutrosophic $\in \vee q$-subsets.

The aim of this article is to provide an algebraic tool of neutrosophic set theory which can be used in applied sciences, for example, decision making problems, medical sciences etc. We consider a general form of neutrosophic points, and then we discuss generalizations of the papers [6] and [5]. As a generalization of $(\in, \in \vee q)$-neutrosophic subalgebras, we introduce the notions of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra, and $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra in BCK/BCI-algebras, and investigate several properties. We consider characterizations of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra. We discuss relations between $(\in, \in)$-neutrosophic subalgebra, $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra and $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra.

2 Preliminaries

By a *BCI-algebra* we mean a set X with a binary operation $*$ and the special element 0 satisfying the axioms:

(a1) $((x * y) * (x * z)) * (z * y) = 0$,

(a2) $(x * (x * y)) * y = 0$,

(a3) $x * x = 0$,

(a4) $x * y = y * x = 0 \Rightarrow x = y$,

for all $x, y, z \in X$. If a BCI-algebra X satisfies the axiom

(a5) $0 * x = 0$ for all $x \in X$,

then we say that X is a *BCK-algebra*. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

The collection of all BCK-algebras and all BCI-algebras are denoted by $\mathcal{B}_K(X)$ and $\mathcal{B}_I(X)$, respectively. Also $\mathcal{B}(X) := \mathcal{B}_K(X) \cup \mathcal{B}_I(X)$.

We refer the reader to the books [7] and [8] for further information regarding BCK/BCI-algebras.

Let X be a non-empty set. A neutrosophic set (NS) in X (see [2]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0,1]$ is a truth membership function, $A_I : X \to [0,1]$ is an indeterminate membership function, and $A_F : X \to [0,1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we consider the following sets (see [5]):

$$\begin{aligned}
&T_\in(A;\alpha) := \{x \in X \mid A_T(x) \geq \alpha\},\\
&I_\in(A;\beta) := \{x \in X \mid A_I(x) \geq \beta\},\\
&F_\in(A;\gamma) := \{x \in X \mid A_F(x) \leq \gamma\},\\
&T_q(A;\alpha) := \{x \in X \mid A_T(x) + \alpha > 1\},\\
&I_q(A;\beta) := \{x \in X \mid A_I(x) + \beta > 1\},\\
&F_q(A;\gamma) := \{x \in X \mid A_F(x) + \gamma < 1\},\\
&T_{\in\vee\, q}(A;\alpha) := \{x \in X \mid A_T(x) \geq \alpha \text{ or } A_T(x) + \alpha > 1\},\\
&I_{\in\vee\, q}(A;\beta) := \{x \in X \mid A_I(x) \geq \beta \text{ or } A_I(x) + \beta > 1\},\\
&F_{\in\vee\, q}(A;\gamma) := \{x \in X \mid A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma < 1\}.
\end{aligned}$$

We say $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are *neutrosophic* $\in$*-subsets*; $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are *neutrosophic* q*-subsets*; and $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are *neutrosophic* $\in \vee\, q$*-subsets*. It is clear that

$$T_{\in\vee\, q}(A;\alpha) = T_\in(A;\alpha) \cup T_q(A;\alpha), \tag{2.1}$$

$$I_{\in\vee\, q}(A;\beta) = I_\in(A;\beta) \cup I_q(A;\beta), \tag{2.2}$$

$$F_{\in\vee\, q}(A;\gamma) = F_\in(A;\gamma) \cup F_q(A;\gamma). \tag{2.3}$$

Given $\Phi, \Psi \in \{\in, q, \in \vee\, q\}$, a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is called a $(\Phi,\ \Psi)$-*neutrosophic subalgebra* of X (see [5]) if the following assertions are valid.

$$\begin{aligned}
&x \in T_\Phi(A;\alpha_x),\ y \in T_\Phi(A;\alpha_y) \ \Rightarrow\ x * y \in T_\Psi(A;\alpha_x \wedge \alpha_y),\\
&x \in I_\Phi(A;\beta_x),\ y \in I_\Phi(A;\beta_y) \ \Rightarrow\ x * y \in I_\Psi(A;\beta_x \wedge \beta_y),\\
&x \in F_\Phi(A;\gamma_x),\ y \in F_\Phi(A;\gamma_y) \ \Rightarrow\ x * y \in F_\Psi(A;\gamma_x \vee \gamma_y)
\end{aligned} \tag{2.4}$$

for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y, \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

3 Generalizations of $(\in,\ \in\!\vee q)$-neutrosophic subalgebras

In what follows, let k_T, k_I and k_F denote arbitrary elements of $[0,1)$ unless otherwise specified. If k_T, k_I and k_F are the same number in $[0,1)$, then it is denoted by k, i.e., $k = k_T = k_I = k_F$.

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we consider the following sets:

$$
\begin{aligned}
&T_{q_{k_T}}(A;\alpha) := \{x \in X \mid A_T(x) + \alpha + k_T > 1\},\\
&I_{q_{k_I}}(A;\beta) := \{x \in X \mid A_I(x) + \beta + k_I > 1\},\\
&F_{q_{k_F}}(A;\gamma) := \{x \in X \mid A_F(x) + \gamma + k_F < 1\},\\
&T_{\in\vee\, q_{k_T}}(A;\alpha) := \{x \in X \mid A_T(x) \geq \alpha \text{ or } A_T(x) + \alpha + k_T > 1\},\\
&I_{\in\vee\, q_{k_I}}(A;\beta) := \{x \in X \mid A_I(x) \geq \beta \text{ or } A_I(x) + \beta + k_I > 1\},\\
&F_{\in\vee\, q_{k_F}}(A;\gamma) := \{x \in X \mid A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma + k_F < 1\}.
\end{aligned}
$$

We say $T_{q_{k_T}}(A;\alpha)$, $I_{q_{k_I}}(A;\beta)$ and $F_{q_{k_F}}(A;\gamma)$ are *neutrosophic* q_k*-subsets*; and $T_{\in\vee\, q_{k_T}}(A;\alpha)$, $I_{\in\vee\, q_{k_I}}(A;\beta)$ and $F_{\in\vee\, q_{k_F}}(A;\gamma)$ are *neutrosophic* $(\in \vee\, q_k)$*-subsets*. For $\Phi \in \{\in,\ q,\ q_k,\ q_{k_T},\ q_{k_I},\ q_{k_F},\ \in\vee\, q,\ \in\vee\, q_k,\ \in\vee\, q_{k_T},\ \in\vee\, q_{k_I},\ \in\vee\, q_{k_F}\}$, the element of $T_\Phi(A;\alpha)$ (resp., $I_\Phi(A;\beta)$ and $F_\Phi(A;\gamma)$) is called a *neutrosophic* T_Φ*-point* (resp., *neutrosophic* I_Φ*-point* and *neutrosophic* F_Φ*-point*) with value α (resp., β and γ).

It is clear that

$$T_{\in\vee\, q_{k_T}}(A;\alpha) = T_\in(A;\alpha) \cup T_{q_{k_T}}(A;\alpha), \tag{3.1}$$

$$I_{\in\vee\, q_{k_I}}(A;\beta) = I_\in(A;\beta) \cup I_{q_{k_I}}(A;\beta), \tag{3.2}$$

$$F_{\in\vee\, q_{k_F}}(A;\gamma) = F_\in(A;\gamma) \cup F_{q_{k_F}}(A;\gamma). \tag{3.3}$$

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we consider the following sets:

$$T^*_\in(A;\alpha) := \{x \in X \mid A_T(x) > \alpha\}, \tag{3.4}$$

$$I^*_\in(A;\beta) := \{x \in X \mid A_I(x) > \beta\}, \tag{3.5}$$

$$F^*_\in(A;\gamma) := \{x \in X \mid A_F(x) < \gamma\}. \tag{3.6}$$

Proposition 3.1. *For any neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we have*

$$\alpha \leq \tfrac{1-k}{2} \ \Rightarrow\ T_{q_k}(A;\alpha) \subseteq T^*_\in(A;\alpha), \tag{3.7}$$

$$\beta \leq \tfrac{1-k}{2} \ \Rightarrow\ I_{q_k}(A;\beta) \subseteq I^*_\in(A;\beta), \tag{3.8}$$

$$\gamma \geq \tfrac{1-k}{2} \ \Rightarrow\ F_{q_k}(A;\gamma) \subseteq F^*_\in(A;\gamma), \tag{3.9}$$

$$\alpha > \tfrac{1-k}{2} \ \Rightarrow\ T_\in(A;\alpha) \subseteq T_{q_k}(A;\alpha), \tag{3.10}$$

$$\beta > \tfrac{1-k}{2} \ \Rightarrow\ I_\in(A;\beta) \subseteq I_{q_k}(A;\beta), \tag{3.11}$$

$$\gamma < \tfrac{1-k}{2} \ \Rightarrow\ F_\in(A;\gamma) \subseteq F_{q_k}(A;\gamma). \tag{3.12}$$

Proof. If $\alpha \leq \frac{1-k}{2}$, then $1 - \alpha \geq \frac{1+k}{2}$ and $\alpha \leq 1 - \alpha$. Assume that $x \in T_{q_k}(A;\alpha)$. Then $A_T(x) + k > 1 - \alpha \geq \frac{1+k}{2}$, and so $A_T(x) > \frac{1+k}{2} - k = \frac{1-k}{2} \geq \alpha$. Hence $x \in T^*_\in(A;\alpha)$. Similarly, we have the result (3.8). Suppose that $\gamma \geq \frac{1-k}{2}$ and let $x \in F_{q_k}(A;\gamma)$. Then $A_F(x) + \gamma + k < 1$, and thus $A_F(x) < 1 - \gamma - k \leq 1 - \frac{1-k}{2} - k = \frac{1-k}{2} \leq \gamma$. Hence $x \in F^*_\in(A;\gamma)$. Suppose that $\alpha > \frac{1-k}{2}$. If $x \in T_\in(A;\alpha)$, then

$$A_T(x) + \alpha + k \geq 2\alpha + k > 2 \cdot \tfrac{1-k}{2} + k = 1$$

and so $x \in T_{q_k}(A;\alpha)$. Hence $T_\in(A;\alpha) \subseteq T_{q_k}(A;\alpha)$. Similarly, we can verify that if $\beta > \frac{1-k}{2}$, then $I_\in(A;\beta) \subseteq I_{q_k}(A;\beta)$. Suppose that $\gamma < \frac{1-k}{2}$. If $x \in F_\in(A;\gamma)$, then $A_F(x) \leq \gamma$, and thus

$$A_F(x) + \gamma + k \leq 2\gamma + k < 2 \cdot \tfrac{1-k}{2} + k = 1,$$

that is, $x \in F_{q_k}(A;\gamma)$. Hence $F_\in(A;\gamma) \subseteq F_{q_k}(A;\gamma)$. □

Definition 3.2. A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is called an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-*neutrosophic subalgebra* of X if

$$\begin{aligned} &x \in T_\in(A;\alpha_x),\ y \in T_\in(A;\alpha_y) \ \Rightarrow\ x * y \in T_{\in\vee q_{k_T}}(A;\alpha_x \wedge \alpha_y),\\ &x \in I_\in(A;\beta_x),\ y \in I_\in(A;\beta_y) \ \Rightarrow\ x * y \in I_{\in\vee q_{k_I}}(A;\beta_x \wedge \beta_y),\\ &x \in F_\in(A;\gamma_x),\ y \in F_\in(A;\gamma_y) \ \Rightarrow\ x * y \in F_{\in\vee q_{k_F}}(A;\gamma_x \vee \gamma_y) \end{aligned} \tag{3.13}$$

for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y, \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

An $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra with $k_T = k_I = k_F = k$ is called an $(\in, \in \vee q_k)$-*neutrosophic subalgebra*.

Lemma 3.3 ([5]). *A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is an $(\in, \in)$-neutrosophic subalgebra of X if and only if it satisfies:*

$$(\forall x, y \in X) \begin{pmatrix} A_T(x*y) \geq A_T(x) \wedge A_T(y) \\ A_I(x*y) \geq A_I(x) \wedge A_I(y) \\ A_F(x*y) \leq A_F(x) \vee A_F(y) \end{pmatrix}. \tag{3.14}$$

Theorem 3.4. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of $X \in \mathcal{B}(X)$, then neutrosophic q_k-subsets $T_{q_{k_T}}(A;\alpha)$, $I_{q_{k_I}}(A;\beta)$ and $F_{q_{k_F}}(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$ whenever they are nonempty.*

Proof. Let $x, y \in T_{q_{k_T}}(A;\alpha)$. Then $A_T(x) + \alpha + k_T > 1$ and $A_T(y) + \alpha + k_T > 1$. It follows from Lemma 3.3 that

$$\begin{aligned} A_T(x*y) + \alpha + k_T &\geq (A_T(x) \wedge A_T(y)) + \alpha + k_T \\ &= (A_T(x) + \alpha + k_T) \wedge (A_T(y) + \alpha + k_T) > 1 \end{aligned}$$

and so that $x * y \in T_{q_{k_T}}(A;\alpha)$. Hence $T_{q_{k_T}}(A;\alpha)$ is a subalgebra of X. Similarly, we can prove that $I_{q_{k_I}}(A;\beta)$ is a subalgebra of X. Now let $x, y \in F_{q_{k_F}}(A;\gamma)$. Then $A_F(x)+\gamma+k_F < 1$ and $A_F(y)+\gamma+k_F < 1$, which imply from Lemma 3.3 that

$$\begin{aligned} A_F(x*y) + \gamma + k_F &\leq (A_F(x) \vee A_F(y)) + \gamma + k_F \\ &= (A_F(x) + \gamma + k_F) \vee (A_F(y) + \gamma + k_F) < 1. \end{aligned}$$

Hence $x * y \in F_{q_{k_F}}(A;\gamma)$ and so $F_{q_{k_F}}(A;\gamma)$ is a subalgebra of X. □

Corollary 3.5. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of $X \in \mathcal{B}(X)$, then neutrosophic q_k-subsets $T_{q_k}(A;\alpha)$, $I_{q_k}(A;\beta)$ and $F_{q_k}(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$ whenever they are nonempty.*

If we take $k_T = k_I = k_F = 0$ in Theorem 3.4, then we have the following corollary.

Corollary 3.6 ([5]). *If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of $X \in \mathcal{B}(X)$, then neutrosophic q-subsets $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$ whenever they are nonempty.*

Definition 3.7. A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is called a $(q_{(k_T,k_I,k_F)}, \in \vee q_{(k_T,k_I,k_F)})$-*neutrosophic subalgebra* of X if

$$\begin{aligned} &x \in T_{q_{k_T}}(A;\alpha_x),\ y \in T_{q_{k_T}}(A;\alpha_y) \ \Rightarrow\ x * y \in T_{\in\vee q_{k_T}}(A;\alpha_x \wedge \alpha_y),\\ &x \in I_{q_{k_I}}(A;\beta_x),\ y \in I_{q_{k_I}}(A;\beta_y) \ \Rightarrow\ x * y \in I_{\in\vee q_{k_I}}(A;\beta_x \wedge \beta_y),\\ &x \in F_{q_{k_F}}(A;\gamma_x),\ y \in F_{q_{k_F}}(A;\gamma_y) \ \Rightarrow\ x * y \in F_{\in\vee q_{k_F}}(A;\gamma_x \vee \gamma_y)\end{aligned} \tag{3.15}$$

for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y, \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

A $(q_{(k_T,k_I,k_F)}, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra with $k_T = k_I = k_F = k$ is called a $(q_k, \in \vee q_k)$-*neutrosophic subalgebra*.

Theorem 3.8. *If $A = (A_T, A_I, A_F)$ is a $(q_{(k_T,k_I,k_F)}, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of $X \in \mathcal{B}(X)$, then neutrosophic q_k-subsets $T_{q_{k_T}}(A;\alpha)$, $I_{q_{k_I}}(A;\beta)$ and $F_{q_{k_F}}(A;\gamma)$ are subalgebras of X for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$ whenever they are nonempty.*

Proof. Let $x, y \in T_{q_{k_T}}(A;\alpha)$ for $\alpha \in (\frac{1-k_T}{2}, 1]$. Then $x * y \in T_{\in\vee\, q_{k_T}}(A;\alpha)$, that is, $x * y \in T_{\in}(A;\alpha)$ or $x * y \in T_{q_{k_T}}(A;\alpha)$. If $x * y \in T_{\in}(A;\alpha)$, then $x * y \in T_{q_{k_T}}(A;\alpha)$ by (3.10). Therefore $T_{q_{k_T}}(A;\alpha)$ is a subalgebra of X. Similarly, we prove that $I_{q_{k_I}}(A;\beta)$ is a subalgebra of X. Let $x, y \in F_{q_{k_F}}(A;\gamma)$ for $\gamma \in [0, \frac{1-k_F}{2})$. Then $x*y \in F_{\in\vee\, q_{k_F}}(A;\gamma)$, and so $x*y \in F_{\in}(A;\gamma)$ or $x*y \in F_{q_{k_F}}(A;\gamma)$. If $x*y \in F_{\in}(A;\gamma)$, then $x * y \in F_{q_{k_F}}(A;\gamma)$ by (3.12). Hence $F_{q_{k_F}}(A;\gamma)$ is a subalgebra of X. □

Taking $k_T = k_I = k_F = 0$ in Theorem 3.8 induces the following corollary.

Corollary 3.9 ([5]). *If $A = (A_T, A_I, A_F)$ is a $(q, \in\vee\, q)$-neutrosophic subalgebra of $X \in \mathcal{B}(X)$, then neutrosophic q-subsets $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0, 5)$ whenever they are nonempty.*

We provide characterizations of an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra.

Theorem 3.10. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, the following are equivalent.*

(1) *$A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X.*

(2) *$A = (A_T, A_I, A_F)$ satisfies the following assertion.*

$$(\forall x, y \in X) \left(\begin{array}{l} A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), \frac{1-k_T}{2}\} \\ A_I(x * y) \geq \bigwedge\{A_I(x), A_I(y), \frac{1-k_I}{2}\} \\ A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), \frac{1-k_F}{2}\} \end{array} \right). \tag{3.16}$$

Proof. Let $A = (A_T, A_I, A_F)$ be an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X. Assume that there exist $a, b \in X$ such that

$$A_T(a * b) < \bigwedge\{A_T(a), A_T(b), \tfrac{1-k_T}{2}\}.$$

If $A_T(a) \wedge A_T(b) < \frac{1-k_T}{2}$, then $A_T(a*b) < A_T(a) \wedge A_T(b)$. Hence

$$A_T(a*b) < \alpha_t \leq A_T(a) \wedge A_T(b)$$

for some $\alpha_t \in (0,1]$. It follows that $a \in T_\in(A;\alpha_t)$ and $b \in T_\in(A;\alpha_t)$ but $a*b \notin T_\in(A;\alpha_t)$. Moreover, $A_T(a*b) + \alpha_t < 2\alpha_t < 1-k_T$, and so $a*b \notin T_{q_{k_T}}(A;\alpha_t)$. Thus $a*b \notin T_{\in\vee\, q_{k_T}}(A;\alpha_t)$, a contradiction. If $A_T(a) \wedge A_T(b) \geq \frac{1-k_T}{2}$, then $a \in T_\in(A;\frac{1-k_T}{2})$, $b \in T_\in(A;\frac{1-k_T}{2})$ and $a*b \notin T_\in(A;\frac{1-k_T}{2})$. Also,

$$A_T(a*b) + \tfrac{1-k_T}{2} < \tfrac{1-k_T}{2} + \tfrac{1-k_T}{2} = 1-k_T,$$

i.e., $a*b \notin T_{q_{k_T}}(A;\frac{1-k_T}{2})$. Hence $a*b \notin T_{\in\vee\, q_{k_T}}(A;\frac{1-k_T}{2})$, a contradiction. Consequently,

$$A_T(x*y) \geq \bigwedge\{A_T(x), A_T(y), \tfrac{1-k_T}{2}\}$$

for all $x,y \in X$. Similarly, we know that $A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \frac{1-k_I}{2}\}$ for all $x,y \in X$. Suppose that there exist $a,b \in X$ such that

$$A_F(a*b) > \bigvee\{A_F(a), A_F(b), \tfrac{1-k_F}{2}\}.$$

Then $A_F(a*b) > \gamma_F \geq \bigvee\{A_F(a), A_F(b), \frac{1-k_F}{2}\}$ for some $\gamma_F \in [0,1)$. If $A_F(a) \vee A_F(b) \geq \frac{1-k_F}{2}$, then $A_F(a*b) > \gamma_F \geq A_F(a) \vee A_F(b)$ which implies that $a,b \in F_\in(A;\gamma_F)$ and $a*b \notin F_\in(A;\gamma_F)$. Also, $A_F(a*b) + \gamma_F > 2\gamma_F \geq 1-k_F$, that is, $a*b \notin F_{q_{k_F}}(A;\gamma_F)$. Thus $a*b \notin F_{\in\vee\, q_{k_F}}(A;\gamma_F)$, which is a contradiction. If $A_F(a) \vee A_F(b) < \frac{1-k_F}{2}$, then $a,b \in F_\in(A;\frac{1-k_F}{2})$ and $a*b \notin F_\in(A;\frac{1-k_F}{2})$. Also,

$$A_F(a*b) + \tfrac{1-k_F}{2} > \tfrac{1-k_F}{2} + \tfrac{1-k_F}{2} = 1-k_F$$

and so $a*b \notin F_{q_{k_F}}(A;\frac{1-k_F}{2})$. Hence $a*b \notin F_{\in\vee\, q_{k_F}}(A;\frac{1-k_F}{2})$, a contradiction. Therefore $A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), \frac{1-k_F}{2}\}$ for all $x,y \in X$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X which satisfies the condition (3.16). Let $x,y \in X$ and $\beta_x, \beta_y \in (0,1]$ be such that $x \in I_\in(A;\beta_x)$ and $y \in I_\in(A;\beta_y)$. Then

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} \geq \bigwedge\{\beta_x, \beta_y, \tfrac{1-k_I}{2}\}.$$

Suppose that $\beta_x \leq \frac{1-k_I}{2}$ or $\beta_y \leq \frac{1-k_I}{2}$. Then $A_I(x*y) \geq \beta_x \wedge \beta_y$, and so $x*y \in I_\in(A;\beta_x \wedge \beta_y)$. Now, assume that $\beta_x > \frac{1-k_I}{2}$ and $\beta_y > \frac{1-k_I}{2}$. Then $A_I(x*y) \geq \frac{1-k_I}{2}$, and so

$$A_I(x*y) + \beta_x \wedge \beta_y > \tfrac{1-k_I}{2} + \tfrac{1-k_I}{2} = 1-k_I,$$

that is, $x*y \in I_{q_{k_I}}(A;\beta_x \wedge \beta_y)$. Hence $x*y \in I_{\in\vee\, q_{k_I}}(A;\beta_x \wedge \beta_y)$. Similarly, we can verify that if $x \in T_\in(A;\alpha_x)$ and $y \in T_\in(A;\alpha_y)$, then $x*y \in T_{\in\vee\, q_{k_T}}(A;\alpha_x \wedge \alpha_y)$. Finally, let $x,y \in X$ and $\gamma_x, \gamma_y \in [0,1)$ be such that $x \in F_\in(A;\gamma_x)$ and $y \in F_\in(A;\gamma_y)$. Then

$$A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} \leq \bigvee\{\gamma_x, \gamma_y, \tfrac{1-k_F}{2}\}.$$

If $\gamma_x \geq \frac{1-k_F}{2}$ or $\gamma_y \geq \frac{1-k_F}{2}$, then $A_F(x*y) \leq \gamma_x \vee \gamma_y$ and thus $x*y \in F_\in(A;\gamma_x \vee \gamma_y)$. If $\gamma_x < \frac{1-k_F}{2}$ and $\gamma_y < \frac{1-k_F}{2}$, then $A_F(x*y) \leq \frac{1-k_F}{2}$. Hence

$$A_F(x*y) + \gamma_x \vee \gamma_y < \tfrac{1-k_F}{2} + \tfrac{1-k_F}{2} = 1-k_F,$$

that is, $x*y \in F_{q_{k_F}}(A;\gamma_x \vee \gamma_y)$. Thus $x*y \in F_{\in\vee\, q_{k_F}}(A;\gamma_x \vee \gamma_y)$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee\, q_{k_F})$-neutrosophic subalgebra of X. □

Corollary 3.11 ([5]). *A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is an $(\in, \in \vee\, q)$-neutrosophic subalgebra of X if and only if it satisfies:*

$$(\forall x, y \in X) \left(\begin{array}{l} A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), 0.5\} \\ A_I(x * y) \geq \bigwedge\{A_I(x), A_I(y).0.5\} \\ A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \end{array} \right).$$

Proof. It follows from taking $k_T = k_I = k_F = 0$ in Theorem 3.10. □

Theorem 3.12. *Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X \in \mathcal{B}(X)$. Then $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X if and only if neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are subalgebras of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$ whenever they are nonempty.*

Proof. Assume that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X. Let $\beta \in (0, \frac{1-k_I}{2}]$ and $x, y \in I_\in(A;\beta)$. Then $A_I(x) \geq \beta$ and $A_I(y) \geq \beta$. It follows from Theorem 3.10 that

$$A_I(x * y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} \geq \beta \wedge \tfrac{1-k_I}{2} = \beta$$

and so that $x * y \in I_\in(A;\beta)$. Hence $I_\in(A;\beta)$ is a subalgebra of X for all $\beta \in (0, \frac{1-k_I}{2}]$. Similarly, we know that $T_\in(A;\alpha)$ is a subalgebra of X for all $\alpha \in (0, \frac{1-k_T}{2}]$. Let $\gamma \in [\frac{1-k_F}{2}, 1)$ and $x, y \in F_\in(A;\gamma)$. Then $A_F(x) \leq \gamma$ and $A_F(y) \leq \gamma$. Using Theorem 3.10 implies that

$$A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} \leq \gamma \vee \tfrac{1-k_F}{2} = \gamma.$$

Hence $x * y \in F_\in(A;\gamma)$, and therefore $F_\in(A;\gamma)$ is a subalgebra of X for all $\gamma \in [\frac{1-k_F}{2}, 1)$.

Conversely, suppose that the nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are subalgebras of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$. If there exist $a, b \in X$ such that

$$A_T(a * b) < \bigwedge\{A_T(a), A_T(b), \tfrac{1-k_T}{2}\},$$

then $a, b \in T_\in(A;\alpha_T)$ by taking $\alpha_T := \bigwedge\{A_T(a), A_T(b), \frac{1-k_T}{2}\}$. Since $T_\in(A;\alpha_T)$ is a subalgebra of X, it follows that $a * b \in T_\in(A;\alpha_T)$, that is, $A_T(a * b) \geq \alpha_T$. This is a contradiction, and hence

$$A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), \tfrac{1-k_T}{2}\}$$

for all $x, y \in X$. Similarly, we can verify that

$$A_I(x * y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\}$$

for all $x, y \in X$. Now, assume that there exist $a, b \in X$ such that

$$A_F(a * b) > \bigvee\{A_F(a), A_F(b), \tfrac{1-k_F}{2}\}.$$

Then $A_F(a * b) > \gamma_F \geq \bigvee\{A_F(a), A_F(b), \frac{1-k_F}{2}\}$ for some $\gamma_F \in [\frac{1-k_F}{2}, 1)$. Hence $a, b \in F_\in(A;\gamma_F)$, and so $a * b \in F_\in(A;\gamma_F)$ since $F_\in(A;\gamma_F)$ is a subalgebra of X. It follows that $A_F(a * b) \leq \gamma_F$ which is a contradiction. Thus

$$A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\}$$

for all $x, y \in X$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X by Theorem 3.10. □

Corollary 3.13. *Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X \in \mathcal{B}(X)$. Then $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q)$-neutrosophic subalgebra of X if and only if neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$ whenever they are nonempty.*

Proof. It follows from taking $k_T = k_I = k_F = 0$ in Theorem 3.12. □

Theorem 3.14. *Every $(\in, \in)$-neutrosophic subalgebra is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra.*

Proof. Straightforward. □

The converse of Theorem 3.14 is not true as seen in the following example.

Example 3.15. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the binary operation $*$ which is given in Table 1 (see [8]).

Table 1: Cayley table for the binary operation "$*$"

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X \in \mathcal{B}_I(X)$ defined by Table 2

Table 2: Tabular representation of "$A = (A_T, A_I, A_F)$"

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.6	0.5	0.2
a	0.7	0.3	0.6
b	0.3	0.6	0.6
c	0.3	0.3	0.4

If $k_T = 0.36$, then

$$T_\in(A;\alpha) = \begin{cases} X & \text{if } \alpha \in (0, 0.3], \\ \{0, a\} & \text{if } \alpha \in (0.3, 0.32]. \end{cases}$$

If $k_I = 0.32$, then

$$I_\in(A;\beta) = \begin{cases} X & \text{if } \beta \in (0, 0.3], \\ \{0, b\} & \text{if } \beta \in (0.3, 0.34]. \end{cases}$$

If $k_F = 0.36$, then

$$F_{\in}(A;\gamma) = \begin{cases} \{0\} & \text{if } \gamma \in [0.32, 0.4), \\ \{0, c\} & \text{if } \gamma \in [0.4, 0.6), \\ X & \text{if } \gamma \in [0.6, 1]. \end{cases}$$

We know that $T_{\in}(A;\alpha)$, $I_{\in}(A;\beta)$ and $F_{\in}(A;\gamma)$ are subalgebras of X for all $\alpha \in (0, 0.32]$, $\beta \in (0, 0.34]$ and $\gamma \in [0.32, 1)$. It follows from Theorem 3.12 that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X for $k_T = 0.36$, $k_I = 0.32$ and $k_F = 0.36$. Since $A_T(0) = 0.6 < 0.7 = A_T(a) \wedge A_T(a)$ and/or $A_I(0) = 0.5 > 0.3 = A_I(c) \vee A_I(c)$, we know that $A = (A_T, A_I, A_F)$ is not an $(\in, \in)$-neutrosophic subalgebra of X by Lemma 3.3.

Definition 3.16. A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is called an $(\in,\ q_{(k_T,k_I,k_F)})$-*neutrosophic subalgebra* of X if

$$\begin{aligned} &x \in T_{\in}(A;\alpha_x),\ y \in T_{\in}(A;\alpha_y) \ \Rightarrow\ x * y \in T_{q_{k_T}}(A;\alpha_x \wedge \alpha_y), \\ &x \in I_{\in}(A;\beta_x),\ y \in I_{\in}(A;\beta_y) \ \Rightarrow\ x * y \in I_{q_{k_I}}(A;\beta_x \wedge \beta_y), \\ &x \in F_{\in}(A;\gamma_x),\ y \in F_{\in}(A;\gamma_y) \ \Rightarrow\ x * y \in F_{q_{k_F}}(A;\gamma_x \vee \gamma_y) \end{aligned} \tag{3.17}$$

for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y, \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

An $(\in,\ q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra with $k_T = k_I = k_F = k$ is called an $(\in,\ q_k)$-*neutrosophic subalgebra*.

Theorem 3.17. *Every* $(\in,\ q_{(k_T,k_I,k_F)})$*-neutrosophic subalgebra is an* $(\in, \in \vee q_{(k_T,k_I,k_F)})$*-neutrosophic subalgebra.*

Proof. Straightforward. □

The converse of Theorem 3.17 is not true as seen in the following example.

Example 3.18. Consider the BCI-algebra $X = \{0, a, b, c\}$ and the neutrosophic set $A = (A_T, A_I, A_F)$ which are given in Example 3.15. Taking $k_T = 0.2$, $k_I = 0.3$ and $k_F = 0.24$ imply that

$$T_{\in}(A;\alpha) = \begin{cases} X & \text{if } \alpha \in (0, 0.3], \\ \{0, a\} & \text{if } \alpha \in (0.3, 0.4], \end{cases}$$

$$I_{\in}(A;\beta) = \begin{cases} X & \text{if } \beta \in (0, 0.3], \\ \{0, b\} & \text{if } \beta \in (0.3, 0.35], \end{cases}$$

and

$$F_{\in}(A;\gamma) = \begin{cases} \{0\} & \text{if } \beta \in [0.38, 0.4), \\ \{0, c\} & \text{if } \beta \in [0.4, 0.6), \\ X & \text{if } \beta \in [0.6, 1). \end{cases}$$

Since X, $\{0\}$, $\{0, a\}$, $\{0, b\}$ and $\{0, c\}$ are subalgebras of X, we know from Theorem 3.12 that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X for $k_T = 0.2$, $k_I = 0.3$ and $k_F = 0.24$. Note that

$$a * b \notin T_{q_{0.2}}(A; 0.25 \wedge 0.4) \text{ for } a \in T_{\in}(A; 0.4) \text{ and } b \in T_{\in}(A; 0.25),$$

$$b * c \notin I_{q_{0.3}}(A; 0.5 \wedge 0.27) \text{ for } b \in I_{\in}(A; 0.5),\ c \in I_{\in}(A; 0.27)$$

and/or

$$a * c \notin F_{q_{0.24}}(A; 0.6 \vee 0.44) \text{ for } a \in F_{\in}(A; 0.6),\ c \in F_{\in}(A; 0.44).$$

Hence $A = (A_T, A_I, A_F)$ is not an $(\in, q_{(0.2,0.3,0.24)})$-neutrosophic subalgebra of X.

Theorem 3.19. *If $0 \leq k_T < j_T < 1$, $0 \leq k_I < j_I < 1$ and $0 \leq j_F < k_F < 1$, then every $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra is an $(\in, \in \vee q_{(j_T,j_I,j_F)})$-neutrosophic subalgebra.*

Proof. Straightforward. □

The following example shows that if $0 \leq k_T < j_T < 1$, $0 \leq k_I < j_I < 1$ and $0 \leq j_F < k_F < 1$, then an $(\in, \in \vee q_{(j_T,j_I,j_F)})$-neutrosophic subalgebra may not be an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra.

Example 3.20. Let X be the BCI-algebra given in Example 3.15 and let $A = (A_T,\ A_I,\ A_F)$ be a neutrosophic set in X defined by Table 3

Table 3: Tabular representation of "$A = (A_T, A_I, A_F)$"

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.42	0.40	0.44
a	0.40	0.44	0.66
b	0.48	0.36	0.66
c	0.40	0.36	0.33

If $k_T = 0.04$, then

$$T_{\in}(A; \alpha) = \begin{cases} X & \text{if } \alpha \in (0, 0.40], \\ \{0, b\} & \text{if } \alpha \in (0.40, 0.42], \\ \{b\} & \text{if } \alpha \in (0.42, 0.48]. \end{cases}$$

Note that $T_{\in}(A; \alpha)$ is not a subalgebra of X for $\alpha \in (0.42, 0.48]$.

If $k_I = 0.08$, then

$$I_{\in}(A; \beta) = \begin{cases} X & \text{if } \beta \in (0, 0.36], \\ \{0, a\} & \text{if } \beta \in (0.36, 0.40], \\ \{a\} & \text{if } \beta \in (0.40, 0.44], \\ \emptyset & \text{if } \beta \in (0.44, 0.46]. \end{cases}$$

Note that $I_{\in}(A; \beta)$ is not a subalgebra of X for $\beta \in (0.40, 0.44]$.

If $k_F = 0.42$, then

$$F_{\in}(A; \gamma) = \begin{cases} \emptyset & \text{if } \gamma \in [0.29, 0.33), \\ \{c\} & \text{if } \gamma \in [0.33, 0.44), \\ \{0, c\} & \text{if } \gamma \in [0.44, 0.66), \\ X & \text{if } \gamma \in [0.66, 1). \end{cases}$$

Note that $F_{\in}(A;\gamma)$ is not a subalgebra of X for $\gamma \in [0.33, 0.44)$. Therefore $A = (A_T,\ A_I,\ A_F)$ is not an $(\in, \in\vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X for $k_T = 0.04$, $k_I = 0.08$ and $k_F = 0.42$.

If $j_T = 0.16$, then

$$T_{\in}(A;\alpha) = \begin{cases} X & \text{if } \alpha \in (0, 0.40], \\ \{0, b\} & \text{if } \alpha \in (0.40, 0.42]. \end{cases}$$

If $j_I = 0.20$, then

$$I_{\in}(A;\beta) = \begin{cases} X & \text{if } \beta \in (0, 0.36], \\ \{0, a\} & \text{if } \beta \in (0.36, 0.40]. \end{cases}$$

If $j_F = 0.12$, then

$$F_{\in}(A;\gamma) = \begin{cases} \{0, c\} & \text{if } \gamma \in [0.44, 0.66), \\ X & \text{if } \gamma \in [0.66, 1). \end{cases}$$

Therefore $A = (A_T,\ A_I,\ A_F)$ is an $(\in, \in\vee q_{(j_T,j_I,j_F)})$-neutrosophic subalgebra of X for $j_T = 0.16$, $j_I = 0.20$ and $j_F = 0.12$.

Given a subset S of X, consider a neutrosophic set $A_S = (A_{ST}, A_{SI}, A_{SF})$ in X defined by

$$A_S(x) := \begin{cases} (1, 1, 0) & \text{if } x \in S, \\ (0, 0, 1) & \text{otherwise,} \end{cases}$$

that is,

$$A_{ST}(x) := \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

$$A_{SI}(x) := \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$A_{SF}(x) := \begin{cases} 0 & \text{if } x \in S, \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 3.21. *A nonempty subset S of $X \in \mathcal{B}(X)$ is a subalgebra of X if and only if the neutrosophic set $A_S = (A_{ST}, A_{SI}, A_{SF})$ is an $(\in, \in\vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X.*

Proof. Let S be a subalgebra of X. Then neutrosophic $\in$-subsets $T_{\in}(A_{ST};\alpha)$, $I_{\in}(A_{ST};\beta)$ and $F_{\in}(A_{ST};\gamma)$ are obviously subalgebras of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$. Hence $A_S = (A_{ST},\ A_{SI},\ A_{SF})$ is an $(\in, \in\vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X by Theorem 3.12.

Conversely, assume that $A_S = (A_{ST},\ A_{SI},\ A_{SF})$ is an $(\in, \in\vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X. Let $x, y \in S$. Then

$$A_{ST}(x * y) \geq \bigwedge\{A_{ST}(x), A_{ST}(y), \tfrac{1-k_T}{2}\} = 1 \wedge \tfrac{1-k_T}{2} = \tfrac{1-k_T}{2},$$

$$A_{SI}(x*y) \geq \bigwedge\{A_{SI}(x), A_{SI}(y), \tfrac{1-k_I}{2}\} = 1 \wedge \tfrac{1-k_I}{2} = \tfrac{1-k_I}{2}$$

and

$$A_{SF}(x*y) \leq \bigvee\{A_{SF}(x), A_{SF}(y), \tfrac{1-k_F}{2}\} = 0 \vee \tfrac{1-k_F}{2} = \tfrac{1-k_F}{2},$$

which imply that $A_{ST}(x*y) = 1$, $A_{SI}(x*y) = 1$ and $A_{SF}(x*y) = 0$. Hence $x*y \in S$, and so S is a subalgebra of X. □

Theorem 3.22. *Let S be a subalgebra of $X \in \mathcal{B}(X)$. For every $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$, there exists an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra $A = (A_T, A_I, A_F)$ of X such that $T_\in(A;\alpha) = S$, $I_\in(A;\beta) = S$ and $F_\in(A;\gamma) = S$.*

Proof. Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by

$$A(x) := \begin{cases} (\alpha, \beta, \gamma) & \text{if } x \in S, \\ (0, 0, 1) & \text{otherwise,} \end{cases}$$

that is,

$$A_T(x) := \begin{cases} \alpha & \text{if } x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

$$A_I(x) := \begin{cases} \beta & \text{if } x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$A_F(x) := \begin{cases} \gamma & \text{if } x \in S, \\ 1 & \text{otherwise.} \end{cases}$$

Obviously, $T_\in(A;\alpha) = S$, $I_\in(A;\beta) = S$ and $F_\in(A;\gamma) = S$. Suppose that

$$A_T(a*b) < \bigwedge\{A_T(a), A_T(b), \tfrac{1-k_T}{2}\}$$

for some $a, b \in X$. Since $\#\mathrm{Im}(A_T) = 2$, it follows that $\bigwedge\{A_T(a), A_T(b), \frac{1-k_T}{2}\} = \alpha$ and $A_T(a*b) = 0$. Hence $A_T(a) = \alpha = A_T(b)$, and so $a, b \in S$. Since S is a subalgebra of X, we have $a*b \in S$. Thus $A_T(a*b) = \alpha$, a contradiction. Therefore

$$A_T(x*y) \geq \bigwedge\{A_T(x), A_T(y), \tfrac{1-k_T}{2}\}$$

for all $x, y \in X$. Similarly, we can verify that

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\}$$

for all $x, y \in X$. Assume that there exist $a, b \in X$ such that

$$A_F(a*b) > \bigvee\{A_F(a), A_F(b), \tfrac{1-k_F}{2}\}.$$

Then $A_F(a*b) = 1$ and $\bigvee\{A_F(a), A_F(b), \frac{1-k_F}{2}\} = \gamma$ since $\#\text{Im}(A_F) = 2$. It follows that $A_F(a) = \gamma = A_F(b)$ and so that $a, b \in S$. Hence $a*b \in S$, and so $A_F(a*b) = \gamma$, which is a contradiction. Thus

$$A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\}$$

for all $x, y \in X$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X by Theorem 3.10. □

Corollary 3.23. *Let S be a subalgebra of $X \in \mathcal{B}(X)$. For every $\alpha \in (0, 0.5]$, $\beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$, there exists an $(\in, \in\vee q)$-neutrosophic subalgebra $A = (A_T, A_I, A_F)$ of X such that $T_\in(A;\alpha) = S$, $I_\in(A;\beta) = S$ and $F_\in(A;\gamma) = S$.*

Proof. It follows from taking $k_T = k_I = k_F = 0$ in Theorem 3.22. □

Theorem 3.24. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, the following are equivalent.*

(1) *$A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X.*

(2) *The neutrosophic $(\in\vee q_k)$-subsets $T_{\in\vee q_{k_T}}(A;\alpha)$, $I_{\in\vee q_{k_I}}(A;\beta)$ and $F_{\in\vee q_{k_F}}(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$.*

Proof. Assume that $A = (A_T,\ A_I,\ A_F)$ is an $(\in, \in\vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X. Let $x, y \in I_{\in\vee q_{k_I}}(A;\beta)$ for $\beta \in (0, 1]$. Then $A_I(x) \geq \beta$ or $A_I(x) + \beta + k_I > 1$, and $A_I(y) \geq \beta$ or $A_I(y) + \beta + k_I > 1$. Using Theorem 3.10, we have

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\}.$$

Case 1. $A_I(x) \geq \beta$ and $A_I(y) \geq \beta$. If $\beta > \frac{1-k_I}{2}$, then

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} = \tfrac{1-k_I}{2},$$

and so $A_I(x*y) + \beta > \frac{1-k_I}{2} + \frac{1-k_I}{2} = 1 - k_I$. Hence $x*y \in I_{q_{k_I}}(A;\beta)$. If $\beta \leq \frac{1-k_I}{2}$, then

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} \geq \beta,$$

and thus $x*y \in I_\in(A;\beta)$. Hence $x*y \in I_\in(A;\beta) \cup I_{q_{k_I}}(A;\beta) = I_{\in\vee q_{k_I}}(A;\beta)$.

Case 2. $A_I(x) \geq \beta$ and $A_I(y) + \beta + k_I > 1$. If $\beta > \frac{1-k_I}{2}$, then

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} = A_I(y) \wedge \tfrac{1-k_I}{2} > (1-\beta-k_I) \wedge \tfrac{1-k_I}{2} = 1-\beta-k_I,$$

and so $x*y \in I_{q_{k_I}}(A;\beta)$. If $\beta \leq \frac{1-k_I}{2}$, then

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} \geq \bigwedge\{\beta, 1-\beta-k_I, \tfrac{1-k_I}{2}\} = \beta,$$

and thus $x*y \in I_\in(A;\beta)$. Therefore $x*y \in I_{\in\vee q_{k_I}}(A;\beta)$.

Case 3. $A_I(x) + \beta + k_I > 1$ and $A_I(y) \geq \beta$. We have $x*y \in I_{\in\vee q_{k_I}}(A;\beta)$ by the similar way to the Case 2.

Case 4. $A_I(x) + \beta + k_I > 1$ and $A_I(y) + \beta + k_I > 1$. If $\beta > \frac{1-k_I}{2}$, then $1 - \beta - k_I < \frac{1-k_I}{2}$, and so

$$A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} > 1-\beta-k_I,$$

i.e., $x * y \in I_{q_{k_I}}(A;\beta)$. If $\beta \leq \frac{1-k_I}{2}$, then

$$A_I(x * y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\} \geq (1-\beta-k_I) \wedge \tfrac{1-k_I}{2} = \tfrac{1-k_I}{2} \geq \beta,$$

i.e., $x * y \in I_{\in}(A;\beta)$. Hence $x * y \in I_{\in\vee\, q_{k_I}}(A;\beta)$. Consequently, $I_{\in\vee\, q_{k_I}}(A;\beta)$ is a subalgebra of X. Similarly, we can prove that if $x, y \in T_{\in\vee\, q_{k_T}}(A;\alpha)$ for $\alpha \in (0,1]$, then $x * y \in T_{\in\vee\, q_{k_T}}(A;\alpha)$, that is, $T_{\in\vee\, q_{k_T}}(A;\alpha)$ is a subalgebra of X. Let $x, y \in F_{\in\vee\, q_{k_F}}(A;\gamma)$ for $\gamma \in [0,1)$. Then $A_F(x) \leq \gamma$ or $A_F(x) + \gamma + k_F < 1$, and $A_F(y) \leq \gamma$ or $A_F(y) + \gamma + k_F < 1$. Using Theorem 3.10, we have

$$A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\}.$$

Case 1. $A_F(x) \leq \gamma$ and $A_F(y) \leq \gamma$. If $\gamma < \frac{1-k_F}{2}$, then

$$A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} = \tfrac{1-k_F}{2},$$

and so $A_F(x * y) + \gamma < \frac{1-k_F}{2} + \frac{1-k_F}{2} = 1 - k_F$. Hence $x * y \in F_{q_{k_F}}(A;\gamma)$. If $\gamma \geq \frac{1-k_F}{2}$, then

$$A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} \leq \gamma,$$

and thus $x * y \in F_{\in}(A;\gamma)$. Hence $x * y \in F_{\in}(A;\gamma) \cup F_{q_{k_F}}(A;\gamma) = F_{\in\vee\, q_{k_F}}(A;\gamma)$.

Case 2. $A_F(x) \leq \gamma$ and $A_F(y) + \gamma + k_F < 1$. If $\gamma < \frac{1-k_F}{2}$, then

$$A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} = A_F(y) \vee \tfrac{1-k_F}{2} < (1-\gamma-k_F) \vee \tfrac{1-k_F}{2} = 1-\gamma-k_F,$$

and so $x * y \in F_{q_{k_F}}(A;\gamma)$. If $\gamma \geq \frac{1-k_F}{2}$, then

$$A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} \leq \bigvee\{\gamma, 1-\gamma-k_F, \tfrac{1-k_F}{2}\} = \gamma,$$

and thus $x * y \in F_{\in}(A;\gamma)$. Therefore $x * y \in F_{\in\vee\, q_{k_F}}(A;\gamma)$.

Similarly, if $A_I(x) + \beta + k_I < 1$ and $A_I(y) \leq \beta$, then $x * y \in F_{\in\vee\, q_{k_F}}(A;\gamma)$.

Finally, assume that $A_F(x) + \gamma + k_F < 1$ and $A_F(y) + \gamma + k_F < 1$. If $\gamma < \frac{1-k_F}{2}$, then $1-\gamma-k_F > \frac{1-k_F}{2}$, and so

$$A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} < 1-\gamma-k_F,$$

i.e., $x * y \in F_{q_{k_F}}(A;\gamma)$. If $\gamma \geq \frac{1-k_F}{2}$, then

$$A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} \leq (1-\gamma-k_F) \vee \tfrac{1-k_F}{2} = \tfrac{1-k_F}{2} \leq \gamma,$$

i.e., $x * y \in F_{\in}(A;\gamma)$. Hence $x * y \in F_{\in\vee\, q_{k_F}}(A;\gamma)$. Therefore $F_{\in\vee\, q_{k_F}}(A;\gamma)$ is a subalgebra of X.

Conversely, suppose that (2) is valid. If it is possible, let

$$A_T(x * y) < \alpha \leq \bigwedge\{A_T(x), A_T(y), \tfrac{1-k_T}{2}\}$$

for some $\alpha \in (0, \frac{1-k_T}{2})$. Then $x, y \in T_{\in}(A;\alpha) \subseteq T_{\in\vee\, q_{k_T}}(A;\alpha)$, which implies that $x * y \in T_{\in\vee\, q_{k_T}}(A;\alpha)$. Thus $A_T(x * y) \geq \alpha$ or $A_T(x * y) + \alpha + k_T > 1$, a contradiction. Hence

$$A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), \tfrac{1-k_T}{2}\}$$

for all $x, y \in X$. Similarly, we can verify that

$$A_I(x * y) \geq \bigwedge\{A_I(x), A_I(y), \tfrac{1-k_I}{2}\}$$

for all $x, y \in X$. Now assume that there exist $a, b \in X$ and $\gamma \in (\frac{1-k_F}{2}, 1)$ such that

$$A_F(a * b) > \gamma \geq \bigvee\{A_F(a), A_F(b), \tfrac{1-k_F}{2}\}.$$

Then $a, b \in F_{\in}(A; \gamma) \subseteq F_{\in\vee\, q_{k_F}}(A; \gamma)$, which implies that $a * b \in F_{\in\vee\, q_{k_F}}(A; \gamma)$. Thus $A_F(a * b) \leq \gamma$ or $A_F(a * b) + \gamma + k_F < 1$, which is a contradiction. Hence

$$A_F(x * y) \geq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\}$$

for all $x, y \in X$. Using Theorem 3.10, we conclude that $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q_{(k_T, k_I, k_F)})$-neutrosophic subalgebra of X. □

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