Intuitionistic continuous, closed and open mappings

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Abstract. First of all, we define an intuitionistic quotient mapping and obtain its some properties. Second, we define some types discontinuities, open and closed mappings. And we investigate relationships among them and give some examples. Finally, we introduce the notions of an intuitionistic subspace and the heredity, and obtain some properties of each concept.

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1. Introduction

In this paper, first of all, we define an intuitionistic quotient mapping and obtain its some properties. Second, we define some types continuities, open and closed mappings. And we investigate relationships among them and give some examples. Finally, we introduce the notions of an intuitionistic subspace and the heredity, and obtain some properties of each concept.

2. Preliminaries

In this section, we list the concepts of an intuitionistic set, an intuitionistic point, an intuitionistic vanishing point and operations of intuitionistic sets and some results obtained by [5, 6, 7, 11].

Definition 2.1 ([5]). Let $X$ be a non-empty set. Then $A$ is called an intuitionistic set (in short, IS) of $X$, if it is an object having the form $A = (A_T, A_F)$, such that $A_T \cap A_F = \emptyset$, where $A_T$ [resp. $A_F$] is called the set of members [resp. nonmembers] of $A$.

In fact, $A_T$ [resp. $A_F$] is a subset of $X$ agreeing or approving [resp. refusing or opposing] for a certain opinion, view, suggestion or policy.

The intuitionistic empty set [resp. the intuitionistic whole set] of $X$, denoted by $\phi_I$ [resp. $X_I$], is defined by $\phi_I = (\phi, X)$ [resp. $X_I = (X, \phi)$].

In general, $A_T \cup A_F \neq X$.

We will denote the set of all ISs of $X$ as $\text{IS}(X)$.

It is obvious that $A = (A, \phi) \in \text{IS}(X)$ for each ordinary subset $A$ of $X$. Then we can consider an IS of $X$ as the generalization of an ordinary subset of $X$. Furthermore, it is clear that $A = (A_T, A_T, A_F)$ is a neutrosophic crisp set in $X$, for each $A \in \text{IS}(X)$. Thus we can consider a neutrosophic crisp set in $X$ as the generalization of an IS of $X$. Moreover, we can consider an intuitionistic set in $X$ as an intuitionistic fuzzy set in $X$ from Remark 2.2 in [11].

Definition 2.2 ([5]). Let $A, B \in \text{IS}(X)$ and let $(A_j)_{j \in J} \subset \text{IS}(X)$.

(i) We say that $A$ is contained in $B$, denoted by $A \subset B$, if $A_T \subset B_T$ and $A_F \supset B_F$.

(ii) We say that $A$ equals to $B$, denoted by $A = B$, if $A \subset B$ and $B \subset A$.

(iii) The complement of $A$ denoted by $A^c$, is an IS of $X$ defined as:

$$A^c = (A_F, A_T).$$

(iv) The union of $A$ and $B$, denoted by $A \cup B$, is an IS of $X$ defined as:

$$A \cup B = (A_T \cup B_T, A_F \cap B_F).$$

(v) The union of $(A_j)_{j \in J}$, denoted by $\bigcup_{j \in J} A_j$ (in short, $\bigcup A_j$), is an IS of $X$ defined as:

$$\bigcup_{j \in J} A_j = (\bigcup_{j \in J} A_{j,T}, \bigcap_{j \in J} A_{j,F}).$$

(vi) The intersection of $A$ and $B$, denoted by $A \cap B$, is an IS of $X$ defined as:

$$A \cap B = (A_T \cap B_T, A_F \cup B_F).$$
(vii) The intersection of \((A_j)_{j \in J}\), denoted by \(\bigcap_{j \in J} A_j\) (in short, \(\bigcap_j A_j\)), is an IS of \(X\) defined as:
\[
\bigcap_{j \in J} A_j = \left(\bigcap_{j \in J} A_{j,T}, \bigcup_{j \in J} A_{j,F}\right).
\]

(viii) \(A - B = A \cap B^c\).

(ix) \([A] = (A_T, A_F^c), \langle > A = (A_F^c, A_F)\).

From Propositions 3.6 and 3.7 in [10], we can easily see that \((IS(X), \cup, \cap, \phi_I, X_I)\) is a Boolean algebra except the following conditions:

\[ A \cup A^c \neq X_I, A \cap A^c \neq \phi_I. \]

However, by Remark 2.12 in [11], \((IS_+(X), \cup, \cap, \phi_I, X_I)\) is a Boolean algebra, where

\[ IS_+(X) = \{ A \in IS(X) : A_T \cup A_F = X\}. \]

**Definition 2.3 ([5]).** Let \(f : X \to Y\) be a mapping, and let \(A \in IS(X)\) and \(B \in IS(Y)\). Then

(i) the image of \(A\) under \(f\), denoted by \(f(A)\), is an IS in \(Y\) defined as:

\[ f(A) = (f(A)_T, f(A)_F), \]

where \(f(A)_T = f(A_T)\) and \(f(A)_F = (f(A_F)^c)^c\).

(ii) the preimage of \(B\), denoted by \(f^{-1}(B)\), is an IS in \(X\) defined as:

\[ f^{-1}(B) = (f^{-1}(B)_T, f^{-1}(B)_F), \]

where \(f^{-1}(B)_T = f^{-1}(B_T)\) and \(f^{-1}(B)_F = f^{-1}(B_F)\).

**Result 2.4.** (See [5], Corollary 2.11) Let \(f : X \to Y\) be a mapping and let \(A, B, C \in IS(X), (A_j)_{j \in J} \subset IS(X)\) and \(D, E, F \in IS(Y), (D_k)_{k \in K} \subset IS(Y)\). Then the followings hold:

1. if \(B \subset C\), then \(f(B) \subset f(C)\) and if \(E \subset F\), then \(f^{-1}(E) \subset f^{-1}(F)\).
2. \(A \subset f^{-1}(f(A))\) and if \(f\) is injective, then \(A = f^{-1}(f(A))\).
3. \(f(f^{-1}(D)) \subset D\) and if \(f\) is surjective, then \(f(f^{-1}(D)) = D\).
4. \(f^{-1}(\bigcup D_k) = \bigcup f^{-1}(D_k), f^{-1}(\bigcap D_k) = \bigcap f^{-1}(D_k)\).
5. \(f(\bigcup A_j) = \bigcup f(A_j), f(\bigcap A_j) \subseteq \bigcap f(A_j)\).
6. \(f(A) = \phi_I\) if and only if \(A = \phi_I\) and hence \(f(\phi_I) = \phi_I\), in particular if \(f\) is surjective, then \(f(X_I) = Y_I\).
7. \(f^{-1}(Y_I) = X_I, f^{-1}(\phi_I) = \phi_I\).
8. if \(f\) is surjective, then \(f(A)^c \subset f(A^c)\) and furthermore, if \(f\) is injective, then \(f(A)^c = f(A^c)\).
9. \(f^{-1}(D^c) = (f^{-1}(D))^c\).

**Definition 2.5** (See [5]). Let \(X\) be a non-empty set, \(a \in X\) and let \(A \in IS(X)\).

(i) The form \((\{a\}, \{a\}^c)\) (resp. \((\phi, \{a\}^c)\)) is called an intuitionistic point (resp. vanishing point) of \(X\) and denoted by \(a_I\) (resp. \(a_{IV}\)).

(ii) We say that \(a_I\) (resp. \(a_{IV}\)) is contained in \(A\), denoted by \(a_I \in A\) (resp. \(a_{IV} \in A\)), if \(a \in A_T\) (resp. \(a \notin A_F\)).

We will denote the set of all intuitionistic points or intuitionistic vanishing points in \(X\) as \(IP(X)\).
Definition 2.6 ([6]). Let $X$ be a non-empty set and let $\tau \subseteq IC(X)$. Then $\tau$ is called an intuitionistic topology (in short IT) on $X$, it satisfies the following axioms:

(i) $\phi, X \in \tau$,
(ii) $A \cap B \in \tau$, for any $A, B \in \tau$,
(iii) $\bigcup_{j \in J} A_j \in \tau$, for each $(A_j)_{j \in J} \subseteq \tau$.

In this case, the pair $(X, \tau)$ is called an intuitionistic topological space (in short, ITS) and each member $O$ of $\tau$ is called an intuitionistic open set (in short, IOS) in $X$. An IS $F$ of $X$ is called an intuitionistic closed set (in short, ICS) in $X$, if $F^c \in \tau$.

It is obvious that $\{\phi, X\}$ is the smallest IT on $X$ and will be called the intuitionistic indiscrete topology and denoted by $\tau_X$. Furthermore, the following two ordinary topologies on $X$ can be defined by (See [3])

$$\tau_1 = \{U_T : U \in \tau\}, \ \tau_2 = \{U_F : U \in \tau\}.$$ 

We will denote the set of all ITs on $X$ as $IT(X)$. For an ITS $X$, we will denote the set of all IOSs [resp. ICSs] on $X$ as $IO(X)$ [resp. $IC(X)$].

Result 2.7 ([6], Proposition 3.5). Let $(X, \tau)$ be an ITS. Then the following two ITs on $X$ can be defined by:

$$\tau_{0,1} = \{\{U : U \in \tau\}\}, \ \tau_{0,2} = \{< U : U \in \tau\}.$$ 

Furthermore, the following two ordinary topologies on $X$ can be defined by (See [3]):

$$\tau_1 = \{U_T : U \in \tau\}, \ \tau_2 = \{U_F : U \in \tau\}.$$ 

We will denote two ITs $\tau_{0,1}$ and $\tau_{0,2}$ defined in Result 2.7 as

$$\tau_{0,1} = \{\} \text{ and } \tau_{0,2} = < \tau.$$ 

Moreover, for an IT $\tau$ on a set $X$, we can see that $(X, \tau_1, \tau_2)$ is a bitopological space by Kelly [9] (Also see Proposition 3.1 in [4]).

Definition 2.8 ([7]). Let $X$ be an ITS, $p \in X$ and let $N \in IS(X)$. Then

(i) $N$ is called a neighborhood of $p_I$, if there exists an IOS $G$ in $X$ such that

$$p_I \in G \subset N, \text{ i.e., } p \in G_T \subset N_T \text{ and } G_F \supset N_F,$$

(ii) $N$ is called a neighborhood of $p_{IV}$, if there exists an IOS $G$ in $X$ such that

$$p_{IV} \in G \subset N, \text{ i.e., } G_T \subset N_T \text{ and } p \notin G_F \supset N_F.$$ 

We will denote the set of all neighborhoods of $p_I$ [resp. $p_{IV}$] by $N(p_I)$ [resp. $N(p_{IV})$].

Result 2.9 ([11], Theorem 4.2). Let $(X, \tau)$ be an ITS and let $A \in IS(X)$. Then

(1) $A \in \tau$ if and only if $A \in N(a_I)$, for each $a_I \in A$,
(1) $A \in \tau$ if and only if $A \in N(a_{IV})$, for each $a_{IV} \in A$.

Result 2.10 ([7], Proposition 3.4). Let $(X, \tau)$ be an ITS. We define the families

$$\tau_I = \{G : G \in N(p_I), \text{ for each } p_I \in G\}$$

and

$$\tau_{IV} = \{G : G \in N(p_{IV}), \text{ for each } p_{IV} \in G\}.$$
Then $\tau_I, \tau_{IV} \in IT(X)$.

**Result 2.11** ([7], Proposition 3.5). Let $(X, \tau)$ be an ITS. Then $\tau \subset \tau_I$ and $\tau \subset \tau_{IV}$.

**Result 2.12** ([11], Corollary 4.6). Let $(X, \tau)$ be an ITS and let $IC_{\tau}$ [resp. $IC_{\tau_I}$ and $IC_{\tau_{IV}}$] be the set of all ICSs w.r.t. $\tau$ [resp. $\tau_I$ and $\tau_{IV}$]. Then $IC_{\tau}(X) \subset IC_{\tau_I}(X)$ and $IC_{\tau}(X) \subset IC_{\tau_{IV}}(X)$.

**Definition 2.13** ([6]). Let $(X, \tau)$ be an ITS and let $A \in IS(X)$.

(i) The intuitionistic closure of $A$ w.r.t. $\tau$, denoted by $Icl(A)$, is an IS of $X$ defined as:

$$Icl(A) = \bigcap \{K : K^c \in \tau \text{ and } A \subset K\}.$$

(ii) The intuitionistic interior of $A$ w.r.t. $\tau$, denoted by $Int(A)$, is an IS of $X$ defined as:

$$Int(A) = \bigcup \{G : G \in \tau \text{ and } G \subset A\}.$$

3. Intuitionistic quotient spaces

In this section, we define an intuitionistic quotient mapping and obtain its some properties.

**Definition 3.1** ([6]). Let $X, Y$ be ITSs. Then a mapping $f : X \rightarrow Y$ is said to be continuous, if $f^{-1}(V) \in IO(X)$, for each $V \in IO(Y)$.

The following is the immediate result of the above definition.

**Proposition 3.2.** Let $X, Y$ be ITSs. Then

1. the identity $id : X \rightarrow X$ is continuous,
2. if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous, then $g \circ f : X \rightarrow Z$ is continuous,
3. if $f : X \rightarrow Y$ is a constant mapping, then $f$ is continuous,
4. if $X$ is an intuitionistic discrete space, then $f$ is continuous,
5. if $Y$ is an intuitionistic indiscrete space, then $f$ is continuous.

**Result 3.3** ([6], Proposition 4.4). $f : X \rightarrow Y$ is continuous if and only if $f^{-1}(F) \in IC(X)$, for each $F \in IC(Y)$.

**Result 3.4** ([6], Proposition 4.5). The followings are equivalent:

1. $f : X \rightarrow Y$ is continuous,
2. $f^{-1}(Int(B)) \subset Int(f^{-1}(B))$, for each $B \in IS(Y)$,
3. $Icl(f^{-1}(B)) \subset f^{-1}(Icl(B))$, for each $B \in IS(Y)$.

**Result 3.5** ([15], Theorem 3.1). The followings are equivalent:

1. $f : X \rightarrow Y$ is continuous,
2. $f(Icl(A)) \subset Icl(f(A))$, for each $a \in IS(X)$.

**Definition 3.6.** Let $X, Y$ be ITSs. Then a mapping $f : X \rightarrow Y$ is said to be:

(i) open [6], if $f(A) \in IO(Y)$, for each $A \in IO(X)$,
(ii) closed [15], if $f(F) \in IC(Y)$, for each $F \in IC(X)$.

The following is the immediate result of the above definition.
Proposition 3.7. Let $X,Y$ be an ITSs.

(1) $f : X \to Y$ and $g : Y \to Z$ are open [resp. closed], then $g \circ f : X \to Z$ is open [resp. closed].

(2) If both $X$ and $Y$ are intuitionistic discrete spaces, then $f$ is continuous and open.

Result 3.8 ([15], Theorem 3.2). $f : X \to Y$ be continuous and injective. Then $\text{Int}f(A) \subset f(\text{Int}(A))$, for each $A \in \text{IS}(X)$.

Result 3.9 ([15], Theorem 3.4). Let $X,Y$ be ITSs. Then the followings are equivalent:

(1) $f : X \to Y$ is open,
(2) $f(\text{Int}(A)) \subset \text{Int}(f(A))$, for each $A \in \text{IS}(X)$,
(3) $\text{Int}(f^{-1}(B)) \subset f^{-1}(\text{Int}(B))$, for each $B \in \text{IS}(X)$.

The following is the immediate result of Results 3.8 and 3.9.

Corollary 3.10. $f : X \to Y$ be continuous, open and injective. Then $\text{Int}f(A) = f(\text{Int}(A))$, for each $A \in \text{IS}(X)$.

Result 3.11 ([15], Theorem 3.8). Let $X,Y$ be ITSs and $f : X \to Y$ a mapping. Then $f$ is closed if and only if $\text{Icl}f(A) \subset f(\text{Icl}(A))$, for each $A \in \text{IS}(X)$.

The following is the immediate result of Results 3.5 and 3.11.

Corollary 3.12. Let $X,Y$ be ITSs and $f : X \to Y$ a mapping. Then $f$ is continuous and closed if and only if $\text{Icl}f(A) = f(\text{Icl}(A))$, for each $A \in \text{IS}(X)$.

Proposition 3.13. Let $(X,\tau)$ be an ITS, let $Y$ be a set and let $f : X \to Y$ be a mapping. We define a family $\tau_Y \subset \text{IS}(Y)$ as follows:

$$\tau_Y = \{U \in \text{IS}(Y) : f^{-1}(U) \in \tau\}.$$  

Then

(1) $\tau_Y \in \text{IT}(Y)$,
(2) $f : (X,\tau) \to (Y,\tau_Y)$ is continuous,
(3) if $\sigma$ is an IT on $Y$ such that $f : (X,\tau) \to (Y,\sigma)$ is continuous, then $\tau_Y$ is finer than $\sigma$, i.e., $\sigma \subset \tau_Y$.

Proof. (1) From Result 2.4 and the definition of an IT, we can easily show that $\tau_Y \in \text{IT}(Y)$.

(2) It is obvious that $f : (X,\tau) \to (Y,\tau_Y)$ is continuous, by the definition $\tau_Y$.

(3) Let $U \in \sigma$. Since $f : (X,\tau) \to (Y,\sigma)$ is continuous, $f^{-1}(U) \in \tau$. Then by the definition $\tau_Y$, $U \in \tau_Y$. Thus $\sigma \subset \tau_Y$. \hfill \Box

Definition 3.14. Let $(X,\tau)$ be an ITS, let $Y$ be a set and let $f : X \to Y$ be a surjective mapping. Let $\tau_Y = \{U \in \text{IS}(Y) : f^{-1}(U) \in \tau\}$ be the IT on $Y$ in Proposition 3.13. Then $\tau_Y$ is called the intuitionistic quotient topology on $Y$ induced by $f$. The pair $(Y,\tau_Y)$ is called an intuitionistic quotient space of $X$ and $f$ is called an intuitionistic quotient mapping.

From Proposition 3.13, the intuitionistic quotient mapping $f$ is not only continuous but $\tau_Y$ is the finest topology on $Y$ for which $f$ is continuous. It is easy to prove
that if \((Y, \sigma)\) is an intuitionistic quotient space of \((X, \tau)\) with intuitionistic quotient mapping \(f\), then \(F\) is closed in \(X\) if and only if \(f^{-1}(F)\) is closed in \(X\).

**Proposition 3.15.** Let \((X, \tau)\) and \((Y, \sigma)\) be ITSs, let \(f : X \to Y\) be a continuous surjective mapping and let \(\tau_Y\) be the intuitionistic quotient topology on \(Y\) induced by \(f\). If \(f\) is open or closed, then \(\sigma = \tau_Y\).

**Proof.** Suppose \(f\) is open. Since \(\tau_Y\) is the finest topology on \(Y\) for which \(f\) is continuous, \(\sigma \subseteq \tau_Y\). Let \(U \subseteq \tau_Y\). Then by the definition of \(\tau_Y\), \(f^{-1}(U) \subseteq \tau\). Since \(f\) is open and surjective, \(U = f(f^{-1}(U)) \subseteq \sigma\). Thus \(U \subseteq \sigma\). So \(\tau_Y \subseteq \sigma\). Hence \(\sigma = \tau_Y\).

Suppose \(f\) is open. Then by the similar arguments, we can see that \(\sigma = \tau_Y\). \(\Box\)

From Proposition 3.15, we can easily see that if \(f : (X, \tau) \to (Y, \sigma)\) is open (or closed) continuous surjective, then \(f\) is an intuitionistic quotient mapping.

The following is the immediate result of Definition 3.14.

**Proposition 3.16.** The composition of two intuitionistic quotient mappings is an intuitionistic quotient mapping.

**Theorem 3.17.** Let \((X, \tau)\) be an ITS, let \(Y\) be a set, let \(f : X \to Y\) be a surjection, let \(\tau_Y\) be the intuitionistic quotient topology on \(Y\) induced by \(f\) and let \((Z, \sigma)\) be an ITS. Then a mapping \(g : Y \to Z\) is continuous if and only if \(g \circ f : X \to Z\) is continuous.

**Proof.** Suppose \(g : Y \to Z\) is continuous. Since \(f : (X, \tau) \to (Y, \tau_Y)\) is continuous, by Proposition 3.2 (2), \(g \circ f : (X, \tau) \to (Z, \sigma)\) is continuous.

Suppose \(g \circ f : (X, \tau) \to (Z, \sigma)\) is continuous and let \(V \subseteq \sigma\). Then \((g \circ f)^{-1}(V) \subseteq \tau\) and \((g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))\). Thus by the definition of \(\tau_Y\), \(g^{-1}(V) \subseteq \tau_Y\). So \(g : (Y, \tau_Y) \to (Z, \sigma)\) is continuous. \(\Box\)

**Theorem 3.18.** Let \((X, \tau)\) and \((Y, \sigma)\) be ITSs and let \(p : X \to Y\) be continuous. Then \(p\) is an intuitionistic quotient mapping if and only if for each ITS \((Z, \eta)\) and each mapping \(g : Y \to Z\), the continuity of \(g \circ p\) implies that of \(g\).

**Proof.** The proof is similar to one of an ordinary topological space. \(\Box\)

**Theorem 3.19.** Let \((X, \tau), (Y, \sigma)\) and \((Z, \eta)\) be ITSs, let \(p : (X, \tau) \to (Y, \sigma)\) be an intuitionistic quotient mapping and let \(h : (X, \tau) \to (Z, \eta)\) be continuous. Suppose \(h \circ p^{-1}\) is single-valued, i.e., for each \(y \in Y\), \(h\) is constant on \(p^{-1}(y)\). Then

1. \((h \circ p^{-1}) \circ p = h \circ p^{-1}\) is continuous,
2. \(h \circ p^{-1}\) is open (closed) if and only if \(h(U)\) is open (closed), whenever \(U\) is open (closed) in \(X\) such that \(U = p^{-1}(p(U))\).

**Proof.** (1) Let \(x \in X\). Then \(x_I \in p^{-1}(p(x_I))\). Since \(h\) is constant on \(p^{-1}(p(x_I))\), \(h(x_I) = h(p^{-1}(p(x_I)))\). On the other hand, \(h(p^{-1}(p(x_I))) = ([h \circ p^{-1}] \circ p)(x_I)\). Thus \(h = (h \circ p^{-1}) \circ p\). Since \(h\) is continuous and \(p\) is an intuitionistic quotient mapping, by Theorem 3.18, \(h \circ p^{-1}\) is continuous.

(2) The proof is similar to one of an ordinary topological space. \(\Box\)

**Theorem 3.20.** Let \((X, \tau), (Y, \sigma)\) and \((Z, \eta)\) be ITSs, let \(p : (X, \tau) \to (Y, \sigma)\) be an intuitionistic quotient mapping and let \(g : Y \to Z\) be surjective. Then \(g \circ p\) is an intuitionistic quotient mapping if and only if \(g\) is an intuitionistic quotient mapping.
Proof. The proof is similar to one of an ordinary topological space. □

**Definition 3.21 ([14])**. Let \( X, Y \) be non-empty sets. Then \( R \) is called an intuitionistic relation (in short, IR) from \( X \) to \( Y \), if it is an object having the form

\[
R = (R_T, R_F)
\]

such that \( R_T, R_F \subseteq X \times Y \) and \( R_T \cap R_F = \phi \), where \( R_T \) [resp. \( R_F \)] is called the set of members [resp. nonmembers] of \( R \). In fact, \( R \in IS(X \times Y) \). In general, \( R_T \cup R_F \neq X \).

In particular, \( R \) is called an intuitionistic relation on \( X \), if \( R \in IS(X \times X) \).

The intuitionistic empty relation [resp. the intuitionistic whole relation] on \( X \), denoted by \( \phi_{R,T} \) [resp. \( X_{R,I} \)], is defined by \( \phi_{R,T} = (\phi, X \times X) \) [resp. \( X_{R,I} = (X \times X, \phi) \)].

We will denote the set of all IRs on \( X \) [resp. from \( X \) to \( Y \)] as \( IR(X) \) [resp. \( IR(X \times Y) \)].

It is obvious that if \( R \in IR(X \times Y) \), then \( R_T, R_F \) are ordinary relations from \( X \) to \( Y \) and conversely, if \( R_o \) is an ordinary relation from \( X \) to \( Y \), then \((R_o, R_o^c) \in IR(X \times Y)\).

**Definition 3.22 ([3])**. Let \( X, Y \) be non-empty sets, let \( R \in IR(X \times Y) \) and let \((p, q) \in X \times Y\).

(i) \((p, q)_T \) is said to belong to \( R \), denoted by \((p, q)_T \in R \), if \((p, q) \in R_T \).

(ii) \((p, q)_F \) is said to belong to \( R \), denoted by \((p, q)_F \in R \), if \((p, q) \notin R_F \).

**Definition 3.23 ([14])**. An IR \( R \) is called an intuitionistic equivalence relation (in short, IER) on \( X \), if it satisfies the following conditions:

(i) intuitionistic reflexive, i.e., \( R_T \) is reflexive and \( R_F \) is irreflexive, i.e., \((x, x) \notin R_T \), for each \( x \in X \).

(ii) intuitionistic symmetric, i.e., \( R_T \) and \( R_F \) are symmetric,

(iii) intuitionistic transitive, i.e., \( R_T \circ R_T \subseteq R_T \) and \( R_F \circ R_F \subseteq R_F \), where \( S_T \circ R_T \) and \( S_F \circ R_F \) denotes the ordinary composition and \( S_T \circ R_T = (S_T^c \circ R_T^c)^c \).

We will denote the set of all IERs on \( X \) as \( IE(X) \).

It is obvious that \( R \in IE(X) \) if and only if \( R_T \) is an ordinary equivalence relation on \( X \), \( R_F \subseteq I_X^c \) and \((R_F^c \circ R_F^c)^c \supseteq R_F \).

**Definition 3.24 ([14])**. Let \( R \in IE(X) \) and let \( x \in X \). Then the intuitionistic equivalence class (in short, IEC) of \( x \) modulo \( R \), denoted by \( R_{x_I} \), or \([x_I]\), is an IS in \( X \) defined as:

\[
R_{x_I} = \bigcup \{y_I \in X_I : (x, y)_I \in R \}.
\]

In fact, \( R_{x_I} = \bigcup \{y_I \in X_I : (x, y)_I \in R_T \} \).

We will denote the set of all IECs by \( R \) as \( X/R \) and \( X/R = \{R_{x_I} : x \in X \} \) will be called an intuitionistic quotient set (in short, IQS) of \( X \) by \( R \).

**Result 3.25 ([14], Proposition 4.23)**. Let \( f : X \to Y \) be a mapping. Consider the IR \( R_f \) on \( X \) defined as: for each \((x, y) \in X \times X \), \((x, y)_I \in R_f \) if and only if \( f(x_I) = f(y_I) \). Then \( R_f \in IE(X) \).

In this case, \( R_f \) is called the intuitionistic equivalence relation determined by \( f \).
Proposition 3.26. Let \((X, \tau)\) and \((Y, \sigma)\) be ITSs, let \(f : (X, \tau) \to (Y, \sigma)\) be continuous and let \(R_f\) be the intuitionistic equivalence relation on \(X\) determined by \(f\). Then

1. the intuitionistic natural mapping \(p : (X, \tau) \to (X/R_f, \tau_{X/R_f})\) is an intuitionistic quotient mapping, where \(\tau_{X/R_f}\) denotes the intuitionistic quotient topology on \(X/R_f\).
2. \(f \circ p^{-1}\) is continuous injective,
3. if \(f\) is surjective, then bijective.

Proof. (1) It is obvious.
(2) Suppose \(x_I, y_I \in p^{-1}(z)\), for some \(z = [a_I] \in X/R_f\). Then by the definition of \(R_f\), \(f(x_I) = f(y_I)\). Thus \(f \circ p^{-1}\) is single-valued. So by Theorem 3.19 (1), \(f \circ p^{-1}\) is continuous.

Now suppose \([a_I], [b_I] \in X/R_f\) and \(f \circ p^{-1}([a_I]) = f \circ p^{-1}([b_I])\). Let \(x_I \in p^{-1}([a_I])\) and \(y_I \in p^{-1}([b_I])\). Then \(f(x_I) = f(y_I)\). Thus \((x, y)_I \in R_f\). So \([a_I] = p(x_I) = p(y_I) = [b_I]\). Hence \(f \circ p^{-1}\) is injective.

(3) Suppose \(f\) is surjective and let \(y \in Y\). Then there is \(x \in X\) such that \(f(x) = y\).

Suppose \(Y_I = \bigcup X/R_f\), \([x_I] \in X/R_f\) and \(f \circ p^{-1}([x_I]) = y_I\). Thus \(f \circ p^{-1}\) is surjective. So by (2), \(f \circ p^{-1}\) is bijective.

\(\Box\)

Theorem 3.27. Let \((X, \tau)\) and \((Y, \sigma)\) be ITSs and let \(f : (X, \tau) \to (Y, \sigma)\) be continuous surjective. Then \(f \circ p^{-1} : X/R_f \to Y\) is a homeomorphism if and only if \(f\) is an intuitionistic quotient mapping.

Proof. Suppose \(f \circ p^{-1} : (X/R_f, \tau_{X/R_f}) \to (Y, \sigma)\) is a homeomorphism and let \(\sigma_Y\) be the intuitionistic quotient topology on \(Y\) induced by \(f \circ p^{-1}\). Then by Proposition 3.13, \(\sigma = \sigma_Y\). Thus \(f \circ p^{-1}\) is an intuitionistic quotient mapping. So by Theorem 3.20, \((f \circ p^{-1}) \circ p\) is an intuitionistic quotient mapping. On the other hand, \(f = (f \circ p^{-1}) \circ p\). Hence \(f\) is an intuitionistic quotient mapping.

Suppose \(f : (X, \tau) \to (Y, \sigma)\) is an intuitionistic quotient mapping. Since \(f\) is surjective, by Proposition 3.26 (3), \(f \circ p^{-1}\) is bijective. Let \(U\) be any IOS in \(X/R_f\) such that \(U = p^{-1}(p(U))\). Since \(p^{-1}(p(U)) = f^{-1}(f(U))\), \(f^{-1}(f(U))\) is open in \(X\).

Since \(f\) is an intuitionistic quotient mapping, \(f(U) \in \tau\). Then by Theorem 3.19 (2), \(f \circ p^{-1}\) is open. Thus \(f \circ p^{-1}\) is a homeomorphism.

\(\Box\)

Now we turn our attention toward another way of defining an intuitionistic quotient space.

Definition 3.28. Let \((X, \tau)\) be an ITS and let \(\Sigma\) be an intuitionistic partition of \(X\). Let \(p : X \to \Sigma\) be the mapping defined by: for each \(x \in X\),

\[p(x_I) = D\] and \(x_I \in D\), for some \(D \in \Sigma\).

If \(\tau_{\Sigma}\) is the intuitionistic quotient topology on \(\Sigma\) induced by \(p\), then \((\Sigma, \tau_{\Sigma})\) is called an intuitionistic quotient space and \(p\) is called the intuitionistic natural mapping of \(X\) onto \(\Sigma\). The set \(\Sigma\) is called an intuitionistic decomposition of \(X\) and the intuitionistic quotient space \((\Sigma, \tau_{\Sigma})\) is called an intuitionistic decomposition space or an intuitionistic identification of \(X\).

Example 3.29. Let \(X = \mathbb{N}\), let \(A = \{n \in \mathbb{N} : n \text{ is odd}\}, \{n \in \mathbb{N} : n \text{ is even}\}\), \(B = \{n \in \mathbb{N} : n \text{ is odd}\}, \{n \in \mathbb{N} : n \text{ is even}\}\) and let \(\Sigma = \{A, B\}\). Consider the mapping \(p : X \to \Sigma\) given by: for each \(n \in X\),
Then clearly, Σ is an intuitionistic partition of X. Let τ be the usual intuitionistic topology on N and consider τN. Then clearly, τN is the intuitionistic discrete topology on N. Thus p−1(A), p−1(B) ∈ τN. So Σ is an intuitionistic decomposition of X.

4. SOME TYPES OF INTUITIONISTIC CONTINUITIES

In this section, we define some types of continuities, open and closed mappings. And we investigate relationships among them and give some examples.

**Definition 4.1.** Let (X, τ), (Y, σ) be an ITSs. Then a mapping f : X → Y is said to:

(i) σ-τ-continuous, if it is continuous in the sense of Definition 3.1,
(ii) στI-continuous, if for each V ∈ σ, f−1(V) ∈ τI,
(iii) στIV-continuous, if for each V ∈ σ, f−1(V) ∈ τIV,
(iv) στ-continuous, if for each V ∈ σI, f−1(V) ∈ τ,
(v) στI-continuous, if for each V ∈ σI, f−1(V) ∈ τI,
(vi) στIV-continuous, if for each V ∈ σIV, f−1(V) ∈ τIV,
(vii) σIVτ-continuous, if for each V ∈ σIV, f−1(V) ∈ τ,
(viii) σIVτI-continuous, if for each V ∈ σIV, f−1(V) ∈ τI,
(ix) σIVτIV-continuous, if for each V ∈ σIV, f−1(V) ∈ τIV.

The followings are the immediate results of Definition 4.1 and Result 2.11.

**Proposition 4.2.** Let (X, τ), (Y, σ) be an ITSs, f : X → Y be a mapping and let p ∈ X.

1. If f is continuous, then it is both στI-continuous and στIV-continuous.
2. If σIτ-continuous, then both στI-continuous and στIV-continuous.
3. σIVτ-continuous, then both σIVτI-continuous and σIVτIV-continuous.

The followings explain relationships among types of intuitionistic continuities.

**Example 4.3.** (See Examples 3.14) (1) Let X = {a, b, c, d} and consider ITs τ on X given by:

τ = {φI, XI, A1, A2, A3, A4},

where

\[ A_1 = \{a, b\}, \quad A_2 = \{c\}, \quad A_3 = \{b, d\}, \quad A_4 = \{a, b, c\}, \quad A_5 = \{d\} \]

Moreover,

\[ τ_I = τ \bigcup \{A_i : i = 5, 6, \cdots , 23\}, \quad τ_{IV} = τ \bigcup \{A_{24}, A_{25}\} \]

where

\[ A_5 = \{c\}, \quad A_6 = \{d\}, \quad A_7 = \{a, b, φ\}, \quad A_8 = \{a, b, c, φ\}, \quad A_9 = \{c, φ\}, \quad A_{10} = \{φ, \{a\}\}, \quad A_{11} = \{φ, \{b\}\}, \quad A_{12} = \{φ, \{c\}\}, \quad A_{13} = \{φ, \{d\}\}, \quad A_{14} = \{φ, \{a, b\}\}, \quad A_{15} = \{φ, \{a, c\}\}, \quad A_{16} = \{φ, \{a, d\}\}, \quad A_{17} = \{φ, \{b, c\}\}, \quad A_{18} = \{φ, \{c, d\}\}, \quad A_{19} = \{φ, \{a, b, c\}\}, \quad A_{20} = \{φ, \{a, b, d\}\}, \quad A_{21} = \{φ, \{a, c, d\}\}, \quad A_{22} = \{φ, \{b, c, d\}\}, \quad A_{23} = \{φ, φ\}, \quad A_{24} = \{a, c\}, \quad A_{25} = \{a, b, d\} \]

Let Y = {1, 2, 3, 4, 5} and let us consider ITS (Y, σ) given by:

\[ σ = \{φI, X_I, B_1, B_2\} \]
where \( B_1 = (\{1, 2, 3\}, \{5\}) \), \( B_2 = (\{3\}, \{4, 5\}) \). Then we can easily find \( \tau_I \) and \( \tau_{IV} \):
\[
\sigma_I = \sigma \cup \{B_3, B_4, B_5, B_6\} \cup \Im,
\]
where \( B_3 = (\{1, 2, 3\}, \phi) \), \( B_4 = (\{3\}, \{4\}) \), \( B_5 = (\{3\}, \{5\}) \), \( B_6 = (\{3\}, \phi) \), \( \Im = \{(\phi, S) : S \subset Y\} \)
and
\[
\sigma_{IV} = \sigma \cup \{B_7, B_8, B_9, B_{10}, B_{11}, B_{12}, B_{13}, B_{14}, B_{15}, B_{16}, B_{17}, B_{18}\},
\]
where \( B_7 = (\{1, 2, 3\}, \{5\}) \), \( B_8 = (\{1, 3\}, \{4, 5\}) \), \( B_9 = (\{2, 3\}, \{4, 5\}) \),
\( B_{10} = (\{1, 2, 3\}, \{4, 5\}) \), \( B_{11} = (\{1, 3\}, \{4\}) \), \( B_{12} = (\{2, 3\}, \{4\}) \),
\( B_{13} = (\{1, 2, 3\}, \{4\}) \), \( B_{14} = (\{1, 3\}, \{5\}) \), \( B_{15} = (\{2, 3\}, \{5\}) \),
\( B_{16} = (\{1, 2, 3\}, \{5\}) \), \( B_{17} = (\{1, 2, 3\}, \phi) \), \( B_{18} = (\{1, 2, 3, 4\}, \phi) \).

Now let \( f : X \rightarrow Y \) be the mapping defined by:
\[
f(a) = f(b) = 1, f(c) = 4, f(d) = 5.
\]

(i) \( f^{-1}(B_1) = A_1 \in \tau, f^{-1}(B_2) = A_{18} \in \tau_I \). Then \( f \) is not continuous but \( \sigma \_\tau_I \)-continuous.

(ii) We can easily see that \( f^{-1}(U) \in \tau_I \), for each \( U \in \sigma_I \). Then \( f \) is \( \sigma_I \_\tau_I \)-continuous.

(iii) \( f^{-1}(B_1), f^{-1}(B_2) = (\{a, b, c\}, \{d\} \notin \tau_{IV} \). Then \( f \) is neither \( \sigma \_\tau_{IV} \)-continuous nor \( \sigma_{IV} \_\tau_{IV} \)-continuous.

(iv) \( f^{-1}(B_8) = (\{a\}, \{c, d\} \notin \tau \). Then \( f \) is not \( \sigma_{IV} \_\tau_I \)-continuous.

(2) Let \( X = (\{a, b, c, d\}, Y = (\{1, 2, 3, 4, 5\}) \) and consider ITs \( \tau \) and \( \sigma \) on \( X \) and \( Y \), respectively given by:
\[
\tau = \{\phi_I, X_1, A_1, A_2, A_3, A_4\}
\]
and
\[
\sigma = \{\phi, Y_1, B_1\},
\]
where \( A_1 = (\{a, b\}, \{d\}) \), \( A_2 = (\{b, d\}, \{a, c\}) \), \( A_3 = (\{b\}, \{a, c, d\}) \), \( A_4 = (\{a, b, d\}, \phi) \)
and \( B_1 = (\{1, 2\}, \{3, 4\}) \).

Then
\[
\tau_I = \tau \cup \{A_i : i = 5, \ldots, 15\} \cup \Im_X \) and \( \tau_{IV} = \tau \cup \{A_{17}\},
\]
where \( A_5 = (\{a, b\}, \phi) \), \( A_6 = (\{b, d\}, \phi) \), \( A_7 = (\{b, d\}, \{a\}) \), \( A_8 = (\{b, d\}, \{c\}) \),
\( A_9 = (\{b\}, \phi) \), \( A_{10} = (\{b\}, \{a\}) \), \( A_{11} = (\{b\}, \{c\}) \), \( A_{12} = (\{b\}, \{d\}) \),
\( A_{13} = (\{b\}, \{a, c\}) \), \( A_{14} = (\{b\}, \{a, d\}) \), \( A_{15} = (\{b\}, \{c, d\}) \),
\( \Im_X = \{(\phi, S) : S \subset X\} \), \( A_{17} = (\{a, b, c\}, \{d\}) \)
and
\[
\sigma_I = \sigma \cup \{B_2, B_3, B_4\} \cup \Im_Y,
\]
\( \sigma_{IV} = \sigma \cup \{B_5\},
\)
where \( B_2 = (\{1, 2\}, \phi) \), \( B_3 = (\{1, 2\}, \{3\}) \), \( B_4 = (\{1, 2\}, \{4\}) \),\n\( \Im_Y = \{(\phi, S) : S \subset Y\} \), \( B_5 = (\{1, 2, 5\}, \{3, 4\}) \).

Let \( g : X \rightarrow Y \) be the mapping defined by:
\[
g(a) = 3, g(b) = 1, g(c) = 4, g(d) = 2.
\]

(i) \( g^{-1}(B_1) = A_2 \in \tau \). Then \( g \) is continuous.

(ii) \( g^{-1}(B_2) = A_6, g^{-1}(B_3) = A_7, g^{-1}(B_4) = A_8 \in \tau_I \) but \( g^{-1}(B_2) \notin \tau_{IV} \). Then \( g \) is \( \sigma \_\tau_I \)-continuous but not \( \sigma \_\tau_{IV} \)-continuous.

(iii) \( g^{-1}(B_5) = A_2 \in \tau \) but \( g^{-1}(B_2) \notin \tau_I \) and \( g^{-1}(B_5) \notin \tau_{IV} \). Then \( g \) is \( \sigma_{IV} \_\tau \)-continuous but neither \( \sigma_{IV} \_\tau_I \)-continuous nor \( \sigma \_\tau_{IV} \)-continuous.
Theorem 4.4. Let \((X, \tau), (Y, \sigma)\) be the ITSs. Then

1. \(f : (X, \tau) \to (Y, \sigma)\) is continuous if and only if \(f : (X, [\tau]) \to (Y, [\sigma])\) is continuous,
2. \(f : (X, \tau) \to (Y, \sigma)\) is continuous if and only if \(f : (X, < \tau) \to (Y, < \sigma)\) is continuous.

Proof. (1) Suppose \(f : (X, \tau) \to (Y, \sigma)\) is continuous and let \((V_\tau, V_\sigma) \in [\tau, \sigma]_V\). Then by the definition of \([\tau, \sigma]_V\), there is \(V \in \sigma\) such that \([\tau, \sigma] \subseteq \tau \subseteq V \subseteq \sigma\). Thus by the hypothesis, \(f^{-1}(V) \in \tau\). So \([\tau]f^{-1}(V) = f^{-1}([\sigma]) \in [\tau, \sigma]_V\). Hence \(f : (X, [\tau]) \to (Y, [\sigma])\) is continuous.

Conversely, suppose \(f : (X, [\tau]) \to (Y, [\sigma])\) is continuous and let \(V \in \sigma\). Then clearly, \([\tau]V \in [\tau, \sigma]_V\). Thus by the hypothesis, \(f^{-1}([\sigma]) \subseteq f^{-1}(V) \subseteq [\tau]V \subseteq \tau\). So \(f^{-1}(V) \subseteq \tau\). Hence \(f : (X, \tau) \to (Y, \sigma)\).

(2) The proof is similar to (1).

Proposition 4.5. Let \((X, \tau)\) be the ITS such that \(\tau \subset IS_\omega(X)\). Then \(\tau = \tau_{IV}\) and \(\tau = [\tau] = < \tau\).

Proof. By Result 2.11, it is clear that \(\tau \subset \tau_{IV}\). Let \(G \in \tau_{IV}\). By Result 2.10, \(G \in N(p_{IV})\), for each \(p_{IV} \in G\). Then there exists \(U_{p_{IV}} \in \tau\) such that \(p_{IV} \in U_{p_{IV}} \subset G\). Since \(\tau \subset IS_\sigma(X)\), \(p \in (U_{p_{IV}})_{IV}\) and \(p \notin (U_{p_{IV}})_{PIV}\). Thus \((U_{p_{IV}})_{IV} = \bigcup_{p_{IV} \in G, p \in U_{p_{IV}} \sigma, p \notin (U_{p_{IV}})_{PIV}} p\) and \((U_{p_{IV}})_{PIV} = \bigcap_{p_{IV} \in G, p \notin U_{p_{IV}} \sigma} p\). So \(G = \bigcup_{p_{IV} \in G} U_{p_{IV}} \in \tau\), i.e., \(\tau_{IV} \subset \tau\). Hence \(\tau = \tau_{IV}\).

The proof of second part is clear.

The followings are the immediate results of Propositions 4.2 and 4.5.

Corollary 4.6. Let \((X, \tau)\) be the ITS such that \(\tau \subset IS_\sigma(X)\), \((Y, \sigma)\) be an ITS and let \(f : X \to Y\) be a mapping. Then

1. \(f\) is continuous if and only if \(\sigma\tau_{IV}\)-continuous,
2. \(f\) is \(\sigma\tau_{IV}\)-continuous if and only if \(\sigma\tau_{IV}\)-continuous,
3. \(f\) is \(\sigma\tau_{IV}\)-continuous if and only if \(\sigma\tau_{IV}\)-continuous.

The followings are the immediate results of Propositions 4.2, 4.5 and Corollary 4.6.

Corollary 4.7. Let \((X, \tau), (Y, \sigma)\) be the ITSs such that \(\tau \subset IS_\sigma(X)\), \(\sigma \subset IS_\sigma(Y)\) and let \(f : X \to Y\) be a mapping. Then the followings are equivalent:

1. \(f\) is continuous,
2. \(f\) is \(\sigma\tau_{IV}\)-continuous,
3. \(f\) is \(\sigma\tau_{IV}\)-continuous.

Definition 4.8. Let \((X, \tau), (Y, \sigma)\) be an ITSs and let \(p \in Y\). Then a mapping \(f : X \to Y\) is said to be:

1. \(\tau\sigma\)-open, if it is open in the sense of Definition 3.6,
2. \(\tau\sigma\)-closed, if it is closed in the sense of Definition 3.6,
3. \(\tau\sigma\)-open, if \(f(U) \in \sigma_1\), for each \(U \in \tau\),
4. \(\tau\sigma\)-closed, if \(f(F) \in IC_{\sigma_1}(Y)\), for each \(F \in IC_\tau(X)\),
5. \(\tau\sigma_{IV}\)-open, if \(f(U) \in \sigma_{IV}\), for each \(U \in \tau\),
6. \(\tau\sigma_{IV}\)-closed, if \(f(F) \in IC_{\sigma_{IV}}(Y)\), for each \(F \in IC_\tau(X)\),
Proposition 4.9. Let \((X, \tau), (Y, \sigma)\) be an ITs, \(p \in Y\) and let \(f : X \to Y\) be a mapping.

1. If \(f\) is open, then it is both \(\tau\)-\(\sigma_I\)-open and \(\tau\)-\(\sigma_{IV}\)-open.
2. If \(f\) is closed, then it is both \(\tau\)-\(\sigma_I\)-closed and \(\tau\)-\(\sigma_{IV}\)-closed.
3. If \(f\) is \(\tau\)-\(\sigma_I\)-open, then it is both \(\tau\)-\(\sigma_I\)-open and \(\tau\)-\(\sigma_{IV}\)-open.
4. If \(f\) is \(\tau\)-\(\sigma_I\)-closed, then it is both \(\tau\)-\(\sigma_I\)-closed and \(\tau\)-\(\sigma_{IV}\)-closed.
5. If \(f\) is \(\tau\)-\(\sigma_I\)-open, then it is both \(\tau\)-\(\sigma_I\)-open and \(\tau\)-\(\sigma_{IV}\)-open.
6. If \(f\) is \(\tau\)-\(\sigma_I\)-closed, then it is both \(\tau\)-\(\sigma_I\)-closed and \(\tau\)-\(\sigma_{IV}\)-closed.

The followings explain relationships among types of intuitionistic openness and closedness.

Example 4.10. Let \(X = \{1, 2, 3, 4, 5\}\), \(Y = \{a, b, c, d\}\) and consider ITs \((X, \tau)\) and \(\sigma\) on \(X\) and \(Y\), respectively given by:

\[
\tau = \{\phi, X_1, A_1, A_2, A_3, A_4\}, \quad \sigma = \{\phi, Y_1, B_1, B_2, B_3, B_4\},
\]

where

\[
A_1 = \{1, 2, 3\}, A_2 = \{3\}, A_3 = \{3, 4, 5\}, A_4 = \{1, 2, 3\},
B_1 = \{a, b\}, B_2 = \{b\}, B_3 = \{c, d\}, B_4 = \{a, b\}.
\]

Then clearly,

\[
F_I = \{5\}, F_2 = \{4\}, F_3 = \{4, 5\}, F_4 = \{a, b\} \in IC(X)
\]

and

\[
E_1 = \{a, b\}, E_2 = \{c\}, E_3 = \{c, d\}, E_4 = \{a, b\} \in IC(Y).\]

Furthermore, \(\tau_I = \tau \cup \{A_5, A_6\} \cup \exists X\), \(\tau_{IV} = \tau \cup \{A_7, \cdots, A_{18}\}\) and \(\sigma_I = \sigma \cup \{B_5, B_6\} \cup \exists Y\), \(\sigma_{IV} = \sigma \cup \{B_7, \cdots, B_{13}\}\).
where $A_5 = \{(3), \phi\}$, $A_6 = \{(3), \{5\}\}$, $\mathcal{S}_X = \{ (\phi, S) : S \subset X \}$, 
$A_7 = \{(1,2,3,4), \{5\}\}$, $A_8 = \{(1,3), \{4\}\}$, $A_9 = \{(2,3), \{4\}\}$, 
$A_{10} = \{(3,5), \{4\}\}$, $A_{11} = \{(1,2,3), \{4\}\}$, $A_{12} = \{(2,3,5), \{4\}\}$, 
$A_{13} = \{(1,2,3,5), \{4\}\}$, $A_{14} = \{(1,3), \{4,5\}\}$, $A_{15} = \{(2,3), \{4,5\}\}$, 
$A_{16} = \{(1,2,3), \{4,5\}\}$, $A_{17} = \{(1,2,3,4), \phi\}$, $A_{18} = \{(1,2,3,5), \phi\}$
and
$B_5 = \{ (b), \phi \}$, $B_6 = \{ (b), \{d\} \}$, $\mathcal{S}_Y = \{ (\phi, S) : S \subset Y \}$, 
$B_7 = \{ (a,b,c), \{d\} \}$, $B_8 = \{ (a,b), \{c\} \}$, $B_9 = \{ (b,d), \{c\} \}$, 
$B_{10} = \{ (a,b,d), \{c\} \}$, $B_{11} = \{ (a,b), \{c,d\} \}$, $B_{12} = \{ (a,b,c), \phi \}$, 
$B_{13} = \{ (a,b,d), \phi \}$.
Thus $\mathcal{I}C_{\tau_1}(X) = \mathcal{I}C(X) \cup \{ F_5, F_6 \} \cup \mathcal{S}_X$, 
$\mathcal{I}C_{\tau_4}(X) = \mathcal{I}C(X) \cup \{ F_7, \ldots, F_{18} \}$
and
$\mathcal{I}C_{\sigma_1}(Y) = \mathcal{I}C_Y \cup \{ E_5, E_6 \} \cup \mathcal{S}_Y$, 
$\mathcal{I}C_{\sigma_1}(Y) = \mathcal{I}C_Y \cup \{ E_7, \ldots, E_{13} \}$,
where $F_5 = (\phi, \{3\})$, $F_6 = \{(5), \{3\}\}$, $\mathcal{S}_Y = \{ (S, \phi) : S \subset Y \}$, 
$F_7 = \{(5), \{1,2,3,4\}\}$, $F_8 = \{(4), \{1,3\}\}$, $F_9 = \{(4), \{2,3\}\}$, 
$F_{10} = \{(4), \{3,5\}\}$, $F_{11} = \{(4), \{1,2,3\}\}$, $F_{12} = \{(4), \{2,3,5\}\}$, 
$F_{13} = \{(4), \{1,2,3,5\}\}$, $F_{14} = \{(4,5), \{1,3\}\}$, $F_{15} = \{(4,5), \{2,3\}\}$, 
$F_{16} = \{(4,5), \{1,2,3\}\}$, $F_{17} = (\phi, \{1,2,3,4\})$, $F_{18} = (\phi, \{1,2,3,5\})$
and
$E_5 = (\phi, \{b\})$, $E_6 = \{(d), \{b\}\}$, $\mathcal{S}_Y = \{ (S, \phi) : S \subset Y \}$, 
$E_7 = \{(d), \{a,b,c\}\}$, $E_8 = \{(c), \{a,b\}\}$, $E_9 = \{(c), \{b,d\}\}$, 
$E_{10} = \{(c), \{a,b,d\}\}$, $E_{11} = \{(c,d), \{a,b\}\}$, $E_{12} = (\phi, \{a,b,c\})$, 
$E_{13} = (\phi, \{a,b,d\})$.

Let $f$, $g$, $h : X \to Y$ be the mappings defined by:

$f(1) = a$, $f(2) = f(3) = b$, $f(4) = c$, $f(5) = d$,

$g(1) = a$, $g(2) = g(5) = d$, $g(3) = b$, $g(4) = c$,

$h(1) = h(2) = a$, $h(3) = b$, $h(4) = c$, $h(5) = d$.

Then we can easily check the followings:

(i) $f$ is both open and $\tau_1$-closed but not closed; $f$ is both $\tau_1$-$\sigma_1$-open and $\tau_1$-$\sigma_1$-closed; $f$ is $\tau_{1V}$-$\sigma_1$-open but not $\tau_{1V}$-$\sigma_{1V}$-closed.

(ii) $g$ is $\tau$-$\sigma_1$-$\sigma_1$-open but neither open nor $\tau$-$\sigma_1$-open; $g$ is $\tau_1$-$\sigma_{1V}$-open but neither $\tau_1$-$\sigma_1$-open nor $\tau_1$-$\sigma_{1V}$-open; $g$ is both closed and $\tau_1$-$\sigma_1$-closed but neither $\tau_1$-$\sigma_1$-closed nor $\tau_1$-$\sigma_{1V}$-closed; $g$ is $\tau_{1V}$-$\sigma_{1V}$-closed but neither $\tau_1$-$\sigma_1$-closed nor $\tau_{1V}$-$\sigma_1$-closed.

(iii) $h$ is both open and closed; $h$ is both $\tau_1$-$\sigma_1$-open and $\tau_1$-$\sigma_1$-closed; $h$ is both $\tau_{1V}$-$\sigma_{1V}$-open and $\tau_{1V}$-$\sigma_{1V}$-closed.

**Example 4.11.** Let $X = \{1,2,3,4\}$, $Y = \{a,b,c\}$ and consider ITs $(X, \tau)$ and $\sigma$ on $X$ and $Y$, respectively given by:

$\tau = \{ \phi_1, X_1, A_1, A_2, A_3, A_4 \}$, $\sigma = \{ \phi_1, Y_1, B_1, B_2, B_3, B_4 \}$,

where

$A_1 = \{(1,2), \{3\}\}$, $A_2 = \{(1,4), \{3\}\}$, $A_3 = \{(1), \{2,3\}\}$, $A_4 = \{(1,2,4), \{3\}\}$,
$B_1 = \{(a,b), \{c\}\}$, $B_2 = \{(b), \{a\}\}$, $B_3 = \{(b), \{a,c\}\}$, $B_4 = \{(a,b), \phi\}$.
Then clearly, $F_1 = (\{3\}, \{1, 2\})$, $F_2 = (\{3\}, \{1, 4\})$, $F_3 = (\{2, 3\}, \{2\})$, $F_4 = (\{3\}, \{1, 2, 4\}) \in IC(X)$ and $E_1 = (\{c\}, \{a, b\})$, $E_2 = (\{a\}, \{b\})$, $E_3 = (\{a, c\}, \{b\})$, $E_4 = (\phi, \{a, b\}) \in IC(Y)$.

Furthermore, $\tau_I = \tau \cup \{A_5, \cdots, A_{12}\} \cup \exists_X$, $\tau_{IV} = \tau \cup \{A_{13}\}$ and $\sigma_I = \sigma \cup \{B_5, B_6\} \cup \exists_Y$, $\sigma_{IV} = \sigma \cup \{B_7\}$, where $A_5 = (\{1, 2\}, \phi)$, $A_6 = (\{1, 4\}, \{2\})$, $A_7 = (\{1, 4\}, \{3\})$, $A_8 = (\{1, 4\}, \phi)$, $A_9 = (\{1\}, \{2\})$, $A_{10} = (\{1\}, \{3\})$, $A_{11} = (\{1\}, \phi)$, $A_{12} = (\{1, 2, 4\}, \phi)$, $A_{13} = (\{1, 2, 4\}, \{3\})$ and $B_5 = (\{b\}, \phi)$, $B_6 = (\{b\}, \{c\})$, $B_7 = (\{a, b, c\}, \{d\})$.

Thus $IC_{\tau_I}(X) = IC(X) \cup \{F_5, \cdots, F_{12}\} \cup \exists_X^\tau$, $IC_{\tau_{IV}}(X) = IC(X) \cup \{F_{13}\}$ and $IC_{\sigma_I}(Y) = IC_Y \cup \{E_5, E_6\} \cup \exists_Y$, $IC_{\sigma_{IV}}(Y) = IC_Y \cup \{E_7\}$, where $F_5 = (\phi, \{1, 2\})$, $F_6 = (\{2\}, \{1, 4\})$, $F_7 = (\{3\}, \{1, 4\})$, $F_8 = (\phi, \{1, 4\})$, $F_9 = (\{2\}, \{1\})$, $F_{10} = (\{3\}, \{1\})$, $F_{11} = (\phi, \{1\})$, $F_{12} = (\phi, \{1, 2, 4\})$, $E_5 = (\phi, \{b\})$, $E_6 = (\{c\}, \{b\})$, $E_7 = (\{d\}, \{a, b, c\})$.

Let $f : X \to Y$ be the mappings defined by: $f(1) = f(2) = b$, $f(3) = f(4) = a$.

Then we can easily check that: $f$ is $\tau$-$\sigma_I$-open but neither $\tau$-$\sigma_I$-closed nor open. In fact, $f$ is neither the remainder’s type open nor the remainder’s type closed.

5. **Intuitionistic Subspaces**

In this section, we introduce the notions of an intuitionistic subspace and the heredity, and obtain some properties of each concept.

**Definition 5.1** ([6]). Let $(X, \tau)$ be an ITS.

i) A subfamily $\beta$ of $\tau$ is called an intuitionistic base (in short, IB) for $\tau$, if for each $A \in \tau$, $A = \phi_I$ or there exists $\beta' \subset \beta$ such that $A = \bigcup \beta'$.

ii) A subfamily $\sigma$ of $\tau$ is called an intuitionistic subbase (in short, ISB) for $\tau$, if the family $\beta = \bigcap \sigma' : \sigma' \text{ is a finite subset of } \sigma$ is a base for $\tau$.

In this case, the IT $\tau$ is said to be generated by $\sigma$. In fact, $\tau = \{\phi_I\} \cup \{\bigcup \beta' : \beta' \subset \beta\}$.

**Example 5.2.** (1) ([6], Example 3.10) Let $\sigma = \{((a, b), (-\infty, a]) : a, b \in \mathbb{R}\}$ be the family of ISSs in $\mathbb{R}$. Then $\sigma$ generates an IT $\tau$ on $\mathbb{R}$, which is called the “usual left intuitionistic topology” on $\mathbb{R}$. In fact, the IB $\beta$ for $\tau$ can be written in the form $\beta = \{\mathbb{R}_I\} \cup \sigma$ and $\tau$ consists of the following ISSs in $\mathbb{R}$:

\[ \phi_I, \mathbb{R}_I; \]
(\cup(a_j, b_j), (-\infty, c])
where \(a_j, b_j, c \in \mathbb{R}, \{a_j : j \in J\}\) is bounded from below, \(c < \inf\{a_j : j \in J\}\);
(\cup(a_j, b_j), \phi),
where \(a_j, b_j \in \mathbb{R}, \{a_j : j \in J\}\) is not bounded from below.
Similarly, one can define the “usual right intuitionistic topology” on \(\mathbb{R}\) using an analogue construction.

(2) ([6], Example 3.11) Consider the family \(\sigma\) of ISs in \(\mathbb{R}\)
\[
\sigma = \{(a, b), (-\infty, a_1] \cup [b_1, \infty) : a, b, a_1, b_1 \in \mathbb{R}, a_1 \leq a, b_1 \leq b\}.
\]
Then \(\sigma\) generates an IT \(\tau\) on \(\mathbb{R}\), which is called the “usual intuitionistic topology” on \(\mathbb{R}\). In fact, the IB \(\beta\) for \(\tau\) can be written in the form \(\beta = \{\mathbb{R}_I\} \cup \sigma\) and the elements of \(\tau\) can be easily written down as in the above example.

(3) ([11], Example 3.10 (3)) Consider the family \(\sigma_{[0,1]}\) of ISs in \(\mathbb{R}\)
\[
\sigma_{[0,1]} = \{([a, b], (-\infty, a) \cup (b, \infty)) : a, b \in \mathbb{R} \text{ and } 0 \leq a \leq b \leq 1\}.
\]
Then \(\sigma_{[0,1]}\) generates an IT \(\tau_{[0,1]}\) on \(\mathbb{R}\), which is called the “usual unit closed interval intuitionistic topology” on \(\mathbb{R}\). In fact, the IB \(\beta_{[0,1]}\) for \(\tau_{[0,1]}\) can be written in the form \(\beta_{[0,1]} = \{\mathbb{R}_I\} \cup \sigma_{[0,1]}\) and the elements of \(\tau\) can be easily written down as in the above example.

In this case, \(([0, 1], \tau_{[0,1]})\) is called the “intuitionistic usual unit closed interval” and will be denoted by \([0, 1]_I\), where \([0, 1]_I = ([0, 1], (-\infty, 0) \cup (1, \infty))\).

**Definition 5.3** ([11]). Let \(a, b \in \mathbb{R}\) such that \(a \leq b\). Then
(i) (the closed interval) \([a, b]_I = ([a, b], (-\infty, a) \cup (b, \infty)), \)
(ii) (the open interval) \((a, b)_I = ((a, b), (-\infty, a] \cup [b, \infty)), \)
(iii) (the half open interval or the half closed interval)
\((a, b)_I = ((a, b), (-\infty, a] \cup [b, \infty)), [a, b)_I = ([a, b), (-\infty, a) \cup [b, \infty)), \)
(iv) (the half intuitionistic real line)
\((-\infty, a]_I = ((-\infty, a], (a, \infty)), (-\infty, a)_I = ((-\infty, a], [a, \infty)), \)
\([a, \infty)_I = ([a, \infty), (-\infty, a)), (a, \infty)_I = ((a, \infty), (-\infty, a))\),
(v) (the intuitionistic real line) \((-\infty, \infty)_I = ((-\infty, \infty), \phi) = \mathbb{R}_I\).

**Definition 5.4.** Let \((X, \tau)\) be a ITS and let \(A \in IS(X)\). Then the collection
\[
\tau_A = \{U \cap A : U \in \tau\}
\]
is called the subspace topology or relative topology on \(A\).

**Example 5.5.** (1) Let \(\tau = \{U \subset \mathbb{R} : 0_I \in U \text{ or } U = \phi_I\}\) and let
\(A = ([1, 2], ((-\infty, 1), (2, \infty)) \in IS(\mathbb{R})).\)
Then we can easily show that \(\tau\) is an IT on \(\mathbb{R}\) and \(\tau_A\) is the subspace topology on \(A\).

(2) Let \(X = \{a, b, c, d\}\) be a set and consider the IT \(\tau\) given by:
\[
\tau = \{\phi_I, X_I, A_1, A_2, A_3, A_4\},
\]
where \(A_1 = \{(a, b), \{c\}\}, A_2 = \{(a, c), \{b, d\}\}, A_3 = \{(a), \{b, c, d\}\}, A_4 = \{(a, b, c), \phi\}.
Let \(A = \{(a, d), \{b, c\}\}.\) Then
\[
\tau_A = \{\phi_I \cap A, X_I \cap A, A_1 \cap A, A_2 \cap A, A_3 \cap A, A_4 \cap A\}
\]
\[= \{\phi_I, A, \{(a), \{b, c\}\}, \{(a), \{b, c, d\}\}, \{(a), \{d\}\}\}.
\]
(3) Let $(\mathbb{R}, \tau)$ be the usual intuitionistic topological space. Consider 

$$A = ([0, 1], (-\infty, 0) \cup (1, \infty)) \in IS(\mathbb{R}).$$

Then $\tau_A = \tau_{[0,1]}$.

(4) Let $\tau$ be the usual intuitionistic topology on $\mathbb{R}$ and let $U \subset [0, 1]$ such that $0_I, 1_I \notin U$. Then $U \in \tau_{[0,1]}$ if and only if $U \in \tau$. Suppose $0 < b < 1$, for $b \in \mathbb{R}$. Consider $(-1, b)_I = ((-1, b), (-\infty, b) \cup [b, \infty))$ and $(b, 2)_I = ((b, 2), (-\infty, b) \cup [2, \infty))$. Then $(-1, b)_I \cap [0, 1]_I = [0, b)_I \cap \tau_{[0,1]}$ and $(b, 2)_I \cap [0, 1]_I = (b, 1)_I \cap \tau_{[0,1]}$. Thus 

$$\beta = \left\{(a, b)_I : 0 < a < b < 1\right\} \cup \{0, b)_I : 0 < b < 1\right\} \cup \{(b, 1)_I : 0 < b < 1\}$$

is a base for $\tau_{[0,1]}$.

(5) Let $\tau = \{U \subset IS(\mathbb{R}) : 0_I \notin U \text{ or } U = \phi_I\}$. Then we can easily prove that $\tau$ is an IT on $\mathbb{R}$. Let $A = [1, 2)_I \in IS(\mathbb{R})$ and let $x_I, x_{IV} \in A$. Then clearly, 

$$\{0_I, x_I, x_{IV}\} \in \tau$$

and each member of $\tau_A$ is called relatively open set (in short, an open set in $A$).

**Example 5.8.** (1) Let $(\mathbb{R}, \tau)$ be the usual intuitionistic topological space. Then $\tau_{\mathbb{R}}$ is the intuitionistic discrete topology on $\mathbb{Z}$.

(2) If $\tau$ is the intuitionistic discrete topology on a set $X$ and $A \in IS(X)$, then $\tau_A$ is the intuitionistic discrete topology on $A$.

(3) If $\tau$ is the intuitionistic indiscrete topology on a set $X$ and $A \in IS(X)$, then $\tau_A$ is the intuitionistic indiscrete topology on $A$.

The followings are the immediate results of Definition 5.4.

**Proposition 5.6.** Let $(X, \tau)$ be an ITS and let $A \in IS(X)$. Then $\tau_A$ is an IT on $A$.

**Definition 5.7.** Let $(X, \tau)$ be a ITS, let $A \in IS(X)$ and let $\tau_A$ be the subspace topology on $A$. Then the pair $(A, \tau_A)$ is a subspace of $(X, \tau)$ and each member of $\tau_A$ is called relatively open set (in short, an open set in $A$).

**Example 5.8.** (1) Let $(\mathbb{R}, \tau)$ be the usual intuitionistic topological space. Then $\tau_{\mathbb{R}}$ is the intuitionistic discrete topology on $\mathbb{Z}$.

(2) If $\tau$ is the intuitionistic discrete topology on a set $X$ and $A \in IS(X)$, then $\tau_A$ is the intuitionistic discrete topology on $A$.

(3) If $\tau$ is the intuitionistic indiscrete topology on a set $X$ and $A \in IS(X)$, then $\tau_A$ is the intuitionistic indiscrete topology on $A$.

The followings are the immediate results of Definition 5.4.

**Proposition 5.9.** Let $(X, \tau)$ be an ITS and let $A, B \in IS(X)$ such that $A \subset B$. Then $\tau_A = (\tau_B)_A$ where $(\tau_B)_A$ denotes the subspace topology on $A$ by $\tau_B$.

**Proposition 5.10.** Let $(X, \tau)$ be an ITS, let $A \in IS(X)$ and let $\beta$ be a base for $\tau$. Then $\beta_A = \{B \cap A : B \in \beta\}$ is a base for $\tau_A$.

**Proposition 5.11.** Let $(X, \tau)$ be an ITS and let $A \in \tau$. If $U \in \tau_A$, then $U \in \tau$.

**Theorem 5.12.** Let $(X, \tau)$ be an ITS, let $A, B \in IS_+(X)$ such that $B \subset A$. Then $B$ is closed in $(A, \tau_A)$ if and only if there exists $F \in IC(X)$ such that $B = A \cap F$.

**Proof.** Suppose $B$ is closed in $(A, \tau_A)$. Then $A - B \in \tau_A$. Thus there exists $U \in \tau$ such that $A - B = A \cap B^c = A \cap U$, i.e., $A_T \cap B_F = A_T \cap U_T$ and $A_F \cup B_T = A_F \cup U_F$. Since $B \subset A$ and $A, B \in IS_+(X)$, we have $B_T = A_T \cap U_T$ and $B_F = A_F \cup U_T$, i.e., $B = A \cap U^c$. Since $U \in \tau$, $U^c \in IC(X)$. So $B$ is closed in $A$.

Conversely, suppose there exists $F \in IC(X)$ such that $B = A \cap F$. Then $F^c \in \tau$. Since $A, B \in IS_+(X)$, it is clear that $A - B = A \cap F^c$. Thus $A - B \in \tau_A$. So $B$ is closed in $A$. \[\square\]
The following is the immediate result of Theorem 5.12.

**Corollary 5.13.** Let \((X, \tau)\) be an ITS such that \(\tau \subset IS_\ast(X)\), let \(A \in IC(X)\) and let \(B \in IS_\ast(X)\). If \(B\) is closed in \(A\), then \(B \in IC(X)\).

**Proposition 5.14.** Let \((X, \tau)\) be an ITS such that \(\tau \subset IS_\ast(X)\), let \(A, B \in IS_\ast(X)\) such that \(B \subset A\). Then \(cl_{\tau_\ast}(B) = A \cap Icl(B)\), where \(cl_{\tau_\ast}(B)\) denotes the closure of \(B\) in \((A, \tau_A)\).

**Proposition 5.16.** Let \((X, \tau), (Y, \sigma)\) be ITSs and let \(A \in IS(X), B \in IS(Y)\).

1. The inclusion mapping \(i : A \to X\) is continuous.
2. If \(f : X \to Y\) is continuous, then \(f|_A : A \to Y\) is continuous.
3. If \(f : X \to B\) is continuous, then the mapping \(g : X \to Y\) defined by \(g(x) = f(x)\), for each \(x \in X\) is continuous.
4. If \(f : X \to Y\) is continuous and \(f(X) \subset B\), then the mapping \(g : X \to B\) defined by \(g(x) = f(x)\), for each \(x \in X\) is continuous.

**Proof.** (1) Let \(U \in \tau\). Then clearly, \(A \cap U \in \tau_A\) and \(i^{-1}(U) = A \cap U\). Thus \(i\) is continuous.

(2) Let \(U \in \sigma\). Then clearly, \(f^{-1}(U) \in \tau\). Thus \(A \cap f^{-1}(U) \in \tau_A\) and \((f|_A)^{-1}(U) = A \cap f^{-1}(U)\). Thus \((f|_A)^{-1}(U) \in \tau_A\). So \(f|_A\) is continuous.

(3) Let \(U \in \sigma\). Then clearly, \(B \cap U \in \sigma_B\). Since \(f : X \to B\) is continuous, \(f^{-1}(B \cap U) = f^{-1}(U) \in \tau\). Since \(g(x) = f(x)\), for each \(x \in X\), \(g^{-1}(U) = f^{-1}(U)\). Thus \(g^{-1}(U) \in \tau\). So \(g\) is continuous.

(4) Let \(U \in \sigma_B\). Then there is \(V \in \sigma\) such that \(U = B \cap V\). Since \(f : X \to Y\) is continuous, \(f^{-1}(V) \in \tau\). On the other hand,
\[
g^{-1}(U) = g^{-1}(B) \cap g^{-1}(V) = X \cap f^{-1}(V) = f^{-1}(V).
\]
Thus $g^{-1}(U) \in \tau$. So $g$ is continuous. \hfill \square

**Proposition 5.17.** Let $X, Y$ be ITSs, let $f : X \to Y$ be a mapping, let $\{U_j : j \in J\} \subset IO(X)$ such that $X_I = \bigcup_{j \in J} U_j$ and let $f|_{U_j} : U_j \to Y$ is continuous, for each $j \in J$. Then so is $f$.

**Proof.** Let $V \in IO(Y)$ and let $j \in J$. Then by the hypothesis, $(f|_{U_j})^{-1}(V) \in IO(U_j)$. Since $U_j \in IO(X)$, by Proposition 5.16 (2), $(f|_{U_j})^{-1}(V) \in IO(X)$. Thus $f^{-1}(V) = \bigcup_{j \in J}(f|_{U_j})^{-1}(V) \in IO(X)$. So $f$ is continuous. \hfill \square

**Proposition 5.18.** Let $(X, \tau)$ be an ITS such that $\tau \subset IS_0(X)$, let $(Y, \sigma)$ be an ITS, let $A, B \in IC(X)$ such that $X_I = A \cup B$ and let $f : A \to Y$, $g : B \to Y$ be continuous such that $f(x) = g(x)$, for each $x \in A_T \cap B_T$. Define $h : X \to Y$ as follows:

$$h(x) = g(x), \forall x \in A_T \text{ and } h(x) = g(x), \forall x \in B_T.$$  

Then $h$ is continuous.

**Proof.** Let $F \in IC(Y)$. Since $f : A \to Y$ and $g : B \to Y$ are continuous, by Result 3.3, $f^{-1}(F)$ is closed in $A$ and $g^{-1}(F)$ is closed in $B$. Since $A, B \in IC(X)$, by Corollary 5.13, $f^{-1}(F), g^{-1}(F) \in IC(X)$. On the other hand, $h^{-1}(F) = f^{-1}(F) \cup g^{-1}(F)$. Thus $h^{-1}(F) \in IC(X)$. Thus by Result 3.3, $h$ is continuous. \hfill \square

**Definition 5.19.** An intuitionistic topological property $P$ is said to be hereditary if every subspace of an ITS with $P$ also has $P$.

For separation axioms in intuitionistic topological spaces, see [3, 12].

**Proposition 5.20.** (1) $T_0(i)$ is hereditary, i.e., every subspace of a $T_0(i)$-space is $T_0(i)$.

(2) $T_1(i)$ is hereditary, i.e., every subspace of a $T_1(i)$-space is $T_1(i)$.

(3) $T_2(i)$ is hereditary, i.e., every subspace of a $T_2(i)$-space is $T_2(i)$.

**Proof.** Let $(X, \tau)$ be an ITS and let $A \in IS(X)$.

(1) Suppose $(X, \tau)$ is $T_0(i)$ and let $x_I \neq y_I \in A$. Then clearly, $x \neq y \in X$. Thus by the hypothesis, there exists $U \in \tau$ such that $x_I \in U, y_I \notin U$ or $x_I \notin U, y_I \in U$. Let $V = A \cap U$. Then clearly, $V \in \tau_A$. Moreover, $x_I \in V, y_I \notin V$ or $x_I \notin V, y_I \in V$. Thus $(A, \tau_A)$ is $T_0(i)$.

(2) Suppose $(X, \tau)$ is $T_1(i)$ and let $x_I \neq y_I \in A$. Then clearly, $x \neq y \in X$. Thus by the hypothesis, there exists $G, H \in \tau$ such that $x_I \in G, y_I \notin G$ and $x_I \notin H, y_I \in H$. Let $U = A \cap G$ and let $V = A \cap H$. Then clearly, $U, V \in \tau_A$. Moreover, $x_I \in U, y_I \notin U$ or $x_I \notin V, y_I \in V$. Thus $(A, \tau_A)$ is $T_1(i)$.

(3) Suppose $(X, \tau)$ is $T_2(i)$ and let $x_I \neq y_I \in A$. Then clearly, $x \neq y \in X$. Thus by the hypothesis, there exists $G, H \in \tau$ such that $x_I \in G, y_I \in H$ and $G \cap H = \phi_I$. Let $U = A \cap G$ and let $V = A \cap H$. Then clearly, $U, V \in \tau_A$. Since $G \cap H = \phi_I$, $U \cap V = \phi_I$. Moreover, $x_I \in U$ and $y_I \in V$. So $(A, \tau_A)$ is $T_2(i)$. \hfill \square

**Proposition 5.21.** Let $(X, \tau)$ be an ITS such that $\tau \subset IS_0(X)$.

(1) $T_3(i)$ is hereditary, i.e., every subspace of a $T_3(i)$-space is $T_3(i)$.

(2) An intuitionistic complete regularity is hereditary, i.e., every subspace of intuitionistic complete regular space is intuitionistic complete regular.
Proof. (1) Suppose $(X, \tau)$ be $T_3(i)$ and let $A \in IS_\tau(X)$. Since $(X, \tau)$ is $T_1(i)$, by Proposition 5.20 (2), $(A, \tau_A)$ is $T_1(i)$. Let $B$ be closed in $(A, \tau_A)$ such that $x_I \in B^c$. Then by Theorem 5.12, there exists $F \in IC(X)$ such that $B = A \cap F$. Since $x_I \in B^c$, $x_I \in F^c$. Thus by hypothesis, there exist $U, V \in \tau$ such that $F \subseteq U$, $x_I \in V$ and $U \cap V = \phi_I$. So $A \cap U, A \cap V \in \tau_A$ and $(A \cap U) \cap (A \cap V) = \phi_I$. Moreover, $F \subseteq A \cap U$ and $x_I \in A \cap V$. Hence $(A, \tau_A)$ is $T_3(i)$.

(2) Suppose $(X, \tau)$ be an intuitionistic complete regular space and let $A \in IS_\tau(X)$. Since $(X, \tau)$ is $T_1(i)$, by Proposition 5.20 (2), $(A, \tau_A)$ is $T_1(i)$. Let $B$ be closed in $A$ such that $x_I \in B^c$. Then by Theorem 5.12, there exists $F \in IC(X)$ such that $B = A \cap F$. Since $x_I \in B^c$, $x_I \in F^c$. Thus by hypothesis, there exists a continuous mapping $f : X \to [0, 1]$ such that $f(x_I) = 1_I$ and $f(y_I) = 0_I$, for each $y_I \in F$. Since $f : X \to [0, 1]$ is continuous, by Proposition 5.16 (2), $f|_A : A \to [0, 1]$ is continuous. Let $y_I \in B$. Since $B = A \cap F$, $y_I \in F$. So $f|_A(y_I) = f(y_I) = 0_I$. Moreover, $f|_A(x_I) = f(x_I) = 1_I$. Hence $(A, \tau_A)$ is intuitionistic complete regular. \hfill $\square$

Proposition 5.22. Let $(X, \tau)$ be an ITS such that $\tau \subseteq IS_\tau(X)$ and let $A \in IC(X)$. If $(X, \tau)$ is $T_4(i)$, then $(A, \tau_A)$ is $T_4(i)$.

Proof. Suppose $(X, \tau)$ is $T_4(i)$ and let $A \in IC(X)$. Since $(X, \tau)$ is $T_1(i)$, by Proposition 5.20 (2), $(A, \tau_A)$ is $T_1(i)$. Let $B$ and $C$ be closed in $A$ such that $B \cap C = \phi_I$. Then by Theorem 5.12, there exists $F_1, F_2 \in IC(X)$ such that $B = A \cap F_1$ and $C = A \cap F_2$. Since $A \in IC(X)$, $B, C \in IC(X)$. Thus by the hypothesis, $U, V \in \tau$ such that $B \subseteq U$, $C \subseteq V$ and $U \cap V = \phi_I$. So $A \cap U, A \cap V \in \tau_A$ and $(A \cap U) \cap (A \cap V) = \phi_I$. Moreover, $B \subseteq A \cap U$ and $C \subseteq A \cap V$. Hence $(A, \tau_A)$ is $T_4(i)$. \hfill $\square$

6. Conclusions

In this paper, we mainly dealt with some properties of quotient mappings, various types of continuities, open and closed mappings in intuitionistic topological spaces. In particular, we defined continuities, open and closed mappings under the global sense but did not define them under the local sense.

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