Redefining Imaginary and Complex Numbers, Defining Imaginary and Complex Objects

Huseyin OZEL *
Amateur Mathematician

Abstract

The existing definition of imaginary numbers is solely based on the fact that certain mathematical operation, square operation, would not yield certain type of outcome, negative numbers; hence such operational outcome could only be imagined to exist. Although complex numbers actually form the largest set of numbers, it appears that almost no thought has been given until now into the full extent of all possible types of imaginary numbers. A close look into what further non-existing numbers could be imagined help reveal that we could actually expand the set of imaginary numbers, redefine complex numbers, as well as define imaginary and complex mathematical objects other than merely numbers.
Keywords. Number theory, imaginary numbers, complex numbers, imaginary objects, complex objects.

1 Introduction

Let us formulate the historical initial reference by Gerolamo Cardano and the conventional definition by Rafael Bombelli for imaginary numbers in Definition 1.0.1.

Definition 1.0.1. *Let q and y be arbitrary real numbers, then there is no q satisfying the condition*

$$q^2 = y\,,\ y < 0.$$

Hence some hypothetical numbers, x, are imagined to exist to satisfy this same condition

$$x^2 = y\,,\ x \notin \mathbb{R}\,,\ y \in \mathbb{R}^- \Rightarrow x = \sqrt{y}.$$

Such x are defined as imaginary numbers.

A specific imaginary number, i, as defined in Definition 1.0.2, was introduced by Leonhard Euler [Nahin (1998)] and is used as the unit imaginary number.

Definition 1.0.2. *Let i be an imaginary number such that*

$$i^2 = -1\,,\ i \notin \mathbb{R} \Rightarrow i = \sqrt{-1}.$$

Then, every imaginary number could be represented in terms of i:

$$a \in \mathbb{R}^- \Rightarrow \sqrt{a} = \sqrt{-1} \times \sqrt{-a} = i \times \sqrt{-a}.$$

*Contact: number.theory.paper@gmail.com

Hence i is used as the unit number for imaginary numbers.

The conventional definition for complex numbers, as first formulated by Caspar Wessel [Nahin (1998)], is in the form combining a real component and an imaginary component, as presented in Definition 1.0.3.

Definition 1.0.3. *Let a and b be real numbers and i be the unit imaginary number, then c is defined as a complex number such that*

$$c = a + b\,i.$$

In this paper, we explore further types of imaginary numbers than the only one that is conventionally identified. Accordingly, we offer wider definitions for imaginary and complex numbers overall. Additionally, we demonstrate the need for and define imaginary and complex mathematical objects further than numbers.

This article is organized as follows. Section 2 explores how the mathematical notation and the conventional literal thinking might have hindered identification of various types of imaginary numbers. Section 3 attempts to set boundaries between real and imaginary numbers and offers a generic definition for imaginary numbers. Section 4 underlines the need for identifying imaginary instances of mathematical objects further than only numbers and offers a conclusion on the definition of imaginary objects. In the light of the conclusive definition for imaginary objects, Section 5 proposes new definitions for complex numbers and complex objects. Finally, a brief summary and some remarks in Section 6 conclude the paper.

2 Mathematical Notation, Literal Thinking and Imaginary Numbers

2.1 Mathematical Notation and Imaginary Numbers

We find it notable that the historical definition of imaginary numbers was based on the square operation only:

$$x \notin \mathbb{R}\,,\; y \in \mathbb{R} \mid x^2 = y \wedge y < 0$$

rather than also, for instance, the absolute value operation, such as

$$x \notin \mathbb{R}\,,\; y \in \mathbb{R} \mid |x| = y \wedge y < 0.$$

At a glance, an imaginary number defined as the negative outcome of a square operation seems quite useful in solving certain types of equations, such as the quadratic equations. On the other hand, an imaginary number defined as the negative outcome of an absolute value operation has less promise to be of practical use. However, the reason why human imagination for identifying imaginary numbers that are negative outcomes of absolute value operations is hindered might have more reasons than usefulness only.

The fact that there is a commonly accepted straight-forward notation for the square root operation that is the inverse of the square operation, makes the square operation easy to work with in mathematical statements, i.e. let x and y be arbitrary numbers:

$$x^2 = y \;\Rightarrow x = \sqrt{y}.$$

Whereas the inverse of the absolute value operation requires writing a full conditional statement every time, hence makes it difficult to work with in mathematical statements:

$$|x| = y \ \Rightarrow x = \begin{cases} y & \text{if } y \geq 0, \\ -y & \text{otherwise.} \end{cases}$$

2.2 Identifying Imaginary Numbers Based on Simple Arithmetic Operations

Following Definition 1.0.1 and Definition 1.0.2, let us offer a more structurally formulated definition for imaginary numbers and the unit imaginary number.

Definition 2.2.1. *As there is no real number, q, satisfying the condition*

$$q \times q = y \mid q \in \mathbb{R} \wedge y \in \mathbb{R} \wedge y < 0,$$

some hypothetical numbers, x, are imagined to exist to satisfy this same condition

$$x \times x = y \mid x \notin \mathbb{R} \wedge y \in \mathbb{R} \wedge y < 0.$$

Such x are called imaginary numbers. Observing that

$$x = \sqrt{y} \mid x \notin \mathbb{R} \wedge y \in \mathbb{R} \wedge y < 0,$$

an imaginary number, i, where

$$i = \sqrt{-1},$$

is defined as the unit imaginary number in terms of which every imaginary number could be represented: Let a be an arbitrary negative real number, then the imaginary number $\sqrt{a}$ could be expressed in terms of i, as

$$\sqrt{a} = \sqrt{-a \times -1} = \sqrt{-a} \times i.$$

While the historical definition of imaginary numbers is based on multiplication of a number by itself, the potential for identifying further types of imaginary numbers based on other simple arithmetic operations do not get explored. For instance, division of a number by itself not to yield 1:

$$x \ / \ x \neq 1,$$

does not seem to be explored for identifying a new type for imaginary numbers. It is notable that there is no specifically designated mathematical notation for a number divided by itself; whereas a number multiplied by itself has its own specifically designated mathematical notation as

$$x \times x = x^2$$

and a specifically designated mathematical notation for the inverse operation as

$$x^2 = y \Rightarrow x = \sqrt{y}.$$

The lack of such notation for a number divided by itself might be hindering human imagination to identify the imaginary numbers in this case. Using a dedicated notation for a number divided by itself and for the inverse of that could potentially help us think beyond real number operations hence help us explore further types of imaginary numbers:

$$x \ / \ x = x^{\wedge} \tag{1}$$

$$x^{\wedge} = y \Rightarrow x = \ y_{\wedge} \tag{2}$$

With the specific notation we offer in (1) and (2), we shall examine division of a number by itself for identifying imaginary numbers.

Example 2.2.1. *Let us consider a case where*

$$q \ / \ q = y \mid q \in \mathbb{R} \wedge y \in \mathbb{R} \wedge y < 0.$$

We could rewrite this expression as

$$q = \ y_{\wedge} \mid q \in \mathbb{R} \wedge y \in \mathbb{R} \wedge y < 0.$$

There is no real number, q, satisfying this condition. Hence some hypothetical numbers, x, are imagined to exist to satisfy this same condition

$$x = \ y_{\wedge} \mid x \notin \mathbb{R} \wedge y \in \mathbb{R} \wedge y < 0.$$

Let us next consider another case for division of a number by itself.

Example 2.2.2. *Let us consider a case where*

$$q \ / \ q = y \mid q \in \mathbb{R} \wedge y \in \mathbb{R} \wedge y > 0 \wedge y \neq 1.$$

We could rewrite this expression as

$$q = \ y_{\wedge} \mid q \in \mathbb{R} \wedge y \in \mathbb{R} \wedge y > 0 \wedge y \neq 1.$$

There is no real number, q, satisfying this condition. Hence some hypothetical numbers, x, are imagined to exist to satisfy this same condition

$$x = \ y_{\wedge} \mid x \notin \mathbb{R} \wedge y \in \mathbb{R} \wedge y > 0 \wedge y \neq 1.$$

Having identified two different types of imaginary numbers based on division of a number by itself, let us now consider unit imaginary numbers for these two types of imaginary numbers.

For the type of imaginary number identified in Example 2.2.1, a unit imaginary number h, where

$$h = \ -1_{\wedge}$$

could be defined.

For the type of imaginary number identified in Example 2.2.2, on the other hand, a unit imaginary number g, where

$$g = \ 2_{\wedge}$$

could be defined.

Every imaginary number of these two types could be expressed in terms of h and g by using them in combination.

Let w be an imaginary number, and v be a real number such that $w/w = v$ and $v > 0, v \neq 1$, then the imaginary number w could be expressed in terms of g, as

$$w^{\wedge} = v \Rightarrow w = \ v_{\wedge} \Rightarrow w = g^{\log_2 v.}$$

For instance: $w = \ 4_{\wedge} \Rightarrow w = g^2$.

Let t be an imaginary number, and u be a real number such that $t/t = u$ and $u < 0$, then the imaginary number t could be expressed in terms of g and h, as

$$t^{\wedge} = u \Rightarrow t = \ u_{\wedge} \Rightarrow t = h \times g^{\log_2 u.}$$

For instance: $t = \ -4_{\wedge} \Rightarrow t = h \times g^2$.

Let us now turn our attention to subtraction of a number from itself not to yield 0:

$$x - x \neq 0$$

for identifying imaginary numbers. Using a dedicated notation for subtraction of a number from itself could potentially help us in this case as well:

$$x - x = x^{-} \tag{3}$$

$$x^{-} = y \Rightarrow x = \ y_{-} \tag{4}$$

Using the specific notation we offer in (3) and (4), we could examine subtraction of a number from itself for identifying imaginary numbers.

Example 2.2.3. *Let us consider a case where*

$$q - q = y \mid q \in \mathbb{R} \wedge y \in \mathbb{R} \wedge y > 0.$$

We could rewrite this expression as

$$q = \ y_{-} \mid q \in \mathbb{R} \wedge y \in \mathbb{R} \wedge y > 0.$$

There is no real number, q, satisfying this condition. Hence some hypothetical numbers, x, are imagined to exist to satisfy this same condition

$$x = \ y_{-} \mid x \notin \mathbb{R} \wedge y \in \mathbb{R} \wedge y > 0.$$

An imaginary number, f, such that

$$f = 1_{-}$$

could be defined as the unit imaginary number in terms of which every imaginary number of this type could be represented. Let v be an arbitrary positive real number, then the imaginary number w such that $w = v_{-}$ could be expressed in terms of f, as

$$w = f \times v.$$

Let us next consider another case of subtraction of a number from itself.

Example 2.2.4. *Let us consider a case where*

$$q - q = y \mid q \in \mathbb{R} \wedge y \in \mathbb{R} \wedge y < 0.$$

We could rewrite this expression as

$$q = \ y_{-} \mid q \in \mathbb{R} \wedge y \in \mathbb{R} \wedge y < 0.$$

There is no real number, q, satisfying this condition. Hence some hypothetical numbers, x, are imagined to exist to satisfy this same condition

$$x = \ y_{-} \mid x \notin \mathbb{R} \wedge y \in \mathbb{R} \wedge y < 0.$$

An imaginary number, d such that

$$d = -1_{-}$$

could be defined as the unit imaginary number in terms of which every imaginary number of this type could be represented. Let u be an arbitrary negative real number, then the imaginary number t such that $t = u_{-}$ could be expressed in terms of d, as

$$t = d \times u.$$

A remark we could make regarding Examples 2.2.3 and 2.2.4 is that a more generalized condition of

$$x - x = y,$$

where $x \notin \mathbb{R} \wedge y \in \mathbb{R} \wedge y \neq 0$, could be expressed as

$$x(1 - 1) = y \ \Rightarrow x \times 0 = y;$$

however it could not be further derived that

$$x = y \ / \ 0,$$

since such a transition from statement $x \times 0 = y$ into $x = y \ / \ 0$ would rely on dividing both sides of the equation by 0 hence assuming $0/0$ would yield 1 on one side of the equation. We can also verify that $x = y \ / \ 0$ would be incorrect as $(y/0) - (y/0) \neq y$.

Hence while the following statements about f and d are true

$$f \times 0 = 1, \ d \times 0 = -1,$$

it is incorrect to derive the following statements:

$$f = 1 \ / \ 0, \ d = -1 \ / \ 0.$$

A real number divided by 0 is conventionally considered an undefined value; in Appendix A we examine division of a number by zero for identifying imaginary numbers as well.

2.3 Imagining Non-literal Outcomes from Simple Arithmetic Operations

Considering the four simple arithmetic operations for a number on itself, we could notice that the conventional literal thinking, only in terms of real numbers, lead to converting such operations into their real number arithmetic results, e.g. where x is an arbitrary number,

$$x \times x = x^2,$$
$$x - x = 0,$$
$$x \;/\; x = 1,$$
$$x + x = 2 \times x.$$

Once we start considering non-literal results for the simple arithmetic operations, we come to realize that conditions

$$x \times x \neq x^2 \tag{5}$$

and

$$x + x \neq 2 \times x \tag{6}$$

appear to be different cases in their nature than conditions

$$x \;/\; x \neq 1 \tag{7}$$

and

$$x - x \neq 0. \tag{8}$$

The difference is due to the fact that statements (5) and (6) appear to contain the same expressions on each sides of the statements that are merely expressed in different notations; whereas the latter two, (7) and (8), contain operational results on one side of the statements, not just the same mathematical expressions expressed in different notations.

By examining the conditions in statements (7) and (8), we come to identify some types of imaginary numbers as in Examples 2.2.1, 2.2.2, 2.2.3, 2.2.4. Statements (5) and (6), on the other hand, will be discussed in more detail in Sections 3.3 and 3.7 with deeper analysis.

3 Redefining Imaginary Numbers

3.1 A Revised Definition for Imaginary Numbers

Let us reformulate the historically recognized definition of imaginary numbers in Definition 3.1.1, using the relation notation and by denoting the set of hypothetical numbers that are imagined to exist as the absolute complement of real numbers: ℝ'.

Definition 3.1.1. *Let R be a relation on real numbers where*

$$R = \{(q, y) \in (\mathbb{R} \times \mathbb{R}) \mid q = \sqrt{y} \wedge y < 0\}.$$

Since relation R does not map any numbers: $R = \emptyset$, some hypothetical numbers, x, are imagined to exist to satisfy the conditions in updated relation R:

$$R = \{(x, y) \in (\mathbb{R}' \times \mathbb{R}) \mid x = \sqrt{y} \wedge y < 0\}, \text{ hence } R \neq \emptyset.$$

Such x are called imaginary numbers.

Our discussions in Section 2 underline the need for defining imaginary numbers at more generic terms than the historically recognized definition. We argue that any relation with an empty mapping set is a relation on which imaginary numbers could be identified, and we offer this argument in Axiom 3.1.1.

Axiom 3.1.1. *Where a given relation, R, such that*

$$R = \{(q, y) \in (\mathbb{R} \times \mathbb{R})\},$$

does not map any numbers: $R = \emptyset$, some hypothetical numbers could be imagined to exist to satisfy the conditions in updated relation R:

$$R = \{(x, y) \in (\mathbb{R}' \times \mathbb{R})\}.$$

It follows from Definition 3.1.1 and Axiom 3.1.1 that we shall offer a revised definition for imaginary numbers in Proposition 3.1.1.

Proposition 3.1.1. *Let R be a relation on real numbers where*

$$R = \{(q, y) \in (\mathbb{R} \times \mathbb{R})\},$$

such that relation R does not map any numbers: $R = \emptyset$. Then, some hypothetical numbers, x, are imagined to exist to satisfy the conditions in updated relation R:

$$R = \{(x, y) \in (\mathbb{R}' \times \mathbb{R})\}, R \neq \emptyset.$$

Such x are called imaginary numbers.

We should note that it will be possible to identify unit numbers for some imaginary number types, whereas it may not be the case for some others, depending on the nature of the relation conditions.

3.2 Identifying Further Imaginary Numbers Based on the Revised Definition

We shall explore further imaginary number types based on the revised definition for imaginary numbers (Proposition 3.1.1).

Example 3.2.1. *Let R be a relation on real numbers where*

$$R = \{(q, y) \in (\mathbb{R} \times \mathbb{R}) \mid \forall y \mid y > 0 \wedge q > y\},$$

hence relation R does not map any numbers: $R = \emptyset$. We identify one imaginary number to satisfy the conditions in updated relation R as

$$R = \{(x, y) \in (\mathbb{R}' \times \mathbb{R}) \mid \forall y \mid y > 0 \wedge x > y\}.$$

Such imaginary number, x, is denoted by ∞.

Example 3.2.2. *Let R be a relation on real numbers where*

$$R = \{(q, y) \in (\mathbb{R} \times \mathbb{R}) \mid \forall y \mid y < 0 \wedge q < y\},$$

hence relation R does not map any numbers: $R = \emptyset$. *We identify one imaginary number to satisfy the conditions in updated relation R as*

$$R = \{(x, y) \in (\mathbb{R}' \times \mathbb{R}) \mid \forall y \mid y < 0 \wedge x < y\}.$$

Such imaginary number, x*, is denoted by* $-\infty$.

Example 3.2.3. *Let R be a relation on real numbers where*

$$R = \{(q, y) \in (\mathbb{R} \times \mathbb{R}) \mid \forall y \mid q > 0 \wedge y > 0 \wedge q < y\},$$

hence relation R does not map any numbers: $R = \emptyset$. *We identify one imaginary number to satisfy the conditions in updated relation R as*

$$R = \{(x, y) \in (\mathbb{R}' \times \mathbb{R}) \mid \forall y \mid x > 0 \wedge y > 0 \wedge x < y\}.$$

Such imaginary number, x*, is denoted by* $1/\infty$.

Example 3.2.4. *Let R be a relation on real numbers where*

$$R = \{(q, y) \in (\mathbb{R} \times \mathbb{R}) \mid \forall y \mid q < 0 \wedge y < 0 \wedge q > y\},$$

hence relation R does not map any numbers: $R = \emptyset$. *We identify one imaginary number to satisfy the conditions in updated relation R as*

$$R = \{(x, y) \in (\mathbb{R}' \times \mathbb{R}) \mid \forall y \mid x < 0 \wedge y < 0 \wedge x > y\}.$$

Such imaginary number, x*, is denoted by* $-1/\infty$.

3.3 Setting the Boundaries for Imaginary Numbers

The revised definition for imaginary numbers (Proposition 3.1.1) suggests that we could identify imaginary numbers based on any initially given relation where the mapping set is empty. We shall examine three different cases below to analyze whether there should be some limitations or expansions on what kind of relations could qualify for imaginary number identifications.

i. Let R_1 be a relation where

$$R_1 = \{(q, y) \in (\mathbb{R} \times \mathbb{R}) \mid (q \times q = y) \wedge (y \neq q^2)\},$$

hence relation R_1 does not map any numbers: $R_1 = \emptyset$. Could we then identify such x as imaginary numbers to satisfy the conditions in the updated relation R_1 as

$$R_1 = \{(x, y) \in (\mathbb{R}' \times \mathbb{R} \mid (x \times x = y) \wedge (y \neq x^2)\}?$$

Similarly, let R_2 be a relation where

$$R_2 = \{(q, y) \in (\mathbb{R} \times \mathbb{R}) \mid (q + q = y) \wedge (y \neq 2q)\},$$

hence relation R_2 does not map any numbers: $R_2 = \emptyset$. Could we then identify such x as imaginary numbers to satisfy the conditions in the updated relation R_2 as

$$R_2 = \{(x, y) \in (\mathbb{R}' \times \mathbb{R}) \mid (x + x = y) \wedge (y \neq 2x)\}?$$

Such imaginary number identifications would comply with the revised definition offered for imaginary numbers in Proposition 3.1.1. However, they are based on the conditions put forward in (5) and (6) in Section 2.3 and these conditions both appear to be suggesting the inequality of same mathematical expressions expressed in two different notations on each sides of the statements.

Since the integrity of mathematical notations is the foundation for mathematical operations, their consistency is essential for mathematical statements, regardless of whether the subjects are real or imaginary. So, cases against mathematical notation integrity should be avoided in imaginary number identifications. We shall state this in Axiom 3.3.1.

Axiom 3.3.1. *Where a given relation, R, such that*

$$R = \{(q, y) \in (\mathbb{R} \times \mathbb{R})\},$$

does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation, some hypothetical numbers could be imagined to exist to satisfy the conditions in updated relation R:

$$R = \{(x, y) \in (\mathbb{R}\text{'} \times \mathbb{R})\}.$$

A closer look into the conditions put forward in (5) and (6) will reveal more to explore than mathematical notation integrity and we discuss these conditions in more detail in Section 3.7.

ii. Let R_3 be a relation where

$$R_3 = \{(x, y) \in (\mathbb{R} \times \mathbb{R}\text{'}) \mid y = \sqrt{x} \wedge x < 0\}.$$

In its nature, this is the same relation as the one being referred to in the conventional definition of imaginary numbers as specified in Definition 3.1.1, and now represented by R_4 where

$$R_4 = \{(x, y) \in (\mathbb{R}\text{'} \times \mathbb{R}) \mid x = \sqrt{y} \wedge y < 0\}.$$

The same type of imaginary numbers gets identified through either of R_3 and R_4 and this show us that as well as relation mappings from a real number to an imaginary number, $R : \mathbb{R}\text{'} \mapsto \mathbb{R}$, relation mappings from an imaginary number to a real number: $R : \mathbb{R} \mapsto \mathbb{R}\text{'}$ should also be accommodated within the definition of imaginary numbers:

$$R = \{(x, y) \in ((\mathbb{R}\text{'} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}\text{'})) \mid x = \sqrt{y} \wedge y < 0\}.$$

This argument is offered in Lemma 3.3.1.

Lemma 3.3.1. *Let R be a relation on real numbers where*

$$R = \{(q, r) \in (\mathbb{R} \times \mathbb{R})\},$$

such that R does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. Hence, some hypothetical numbers could be imagined to exist to satisfy the conditions in updated relation R:

$$R = \{(x, y) \in ((\mathbb{R}\text{'} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}\text{'}))\}.$$

iii. We shall also explore cases of relation mappings between hypothetical numbers imagined to exist.

Example 3.3.1. *Let R be a relation where*

$$R = \{(q, r) \in (\mathbb{R} \times \mathbb{R}) \mid (q^0 = r) \wedge (r \neq 1)\},$$

hence relation R does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. Then some hypothetical numbers are imagined to exist to satisfy the conditions in updated relation R:

$$R = \{(x, y) \in ((\mathbb{R}' \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}')) \mid (x^0 = y) \wedge (y \neq 1)\}.$$

Among many potential values, we notice that the values $x = \{0, h, g\}$ will all satisfy the conditions in updated relation R, where h is the unit imaginary number defined in Example 2.2.1 and g is the unit imaginary number defined in Example 2.2.2. Hence, we could list some of the relation elements as:

$$R = \{(0, 0^0), (h, -1), (g, 2), ...\}.$$

The first mapping in this listing, $R : 0 \to 0^0$, is from a real number to a hypothetical number imagined to exist: $x \in \mathbb{R}, y \in \mathbb{R}'$. The second and third mappings, $R : h \to -1$ and $R : g \to 2$, are both from a hypothetical number imagined to exist to a real number: $x \in \mathbb{R}', y \in \mathbb{R}$.

Additionally, we note that yet another mapping, $R : \infty \to \infty^0$, which is a mapping from a hypothetical number imagined to exist to another hypothetical number imagined to exist: $x \in \mathbb{R}', y \in \mathbb{R}'$, will also satisfy the conditions in updated relation R if the updated relation R also included mappings between hypothetical numbers imagined to exist:

$$R = \{(x, y) \in ((\mathbb{R}' \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}') \cup (\mathbb{R}' \times \mathbb{R}')) \mid (x^0 = y) \wedge (y \neq 1)\}.$$

This shows us that the imaginary number definition should also accommodate mappings between hypothetical numbers imagined to exist. This is offered in Lemma 3.3.2.

Lemma 3.3.2. *Let R be a relation on real numbers where*

$$R = \{(q, r) \in (\mathbb{R} \times \mathbb{R})\},$$

such that R does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. Hence, some hypothetical numbers could be imagined to exist to satisfy the conditions in updated relation R:

$$R = \{(x, y) \in ((\mathbb{R}' \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}') \cup (\mathbb{R}' \times \mathbb{R}'))\}.$$

It is notable that both 0^0 and ∞^0 are recognized as indeterminate forms and we come to identify them as imaginary numbers as well in Example 3.3.1. Example identifications of all common indeterminate forms as imaginary numbers is offered in Appendix A.

We shall offer an expanded definition for imaginary numbers to reflect our conclusions from Axiom 3.3.1, Lemma 3.3.1 and Lemma 3.3.2.

3.4 An Expanded Definition for Imaginary Numbers

Proposition 3.4.1. *Let R be a relation on real numbers where*

$$R = \{(q, r) \in (\mathbb{R} \times \mathbb{R})\},$$

such that relation R does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. Then some hypothetical numbers are imagined to exist to satisfy the conditions in updated relation R:

$$R = \{(x, y) \in ((\mathbb{R}' \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}') \cup (\mathbb{R}' \times \mathbb{R}'))\}, R \neq \emptyset.$$

Where relation R comes to map some hypothetical numbers that are imagined to exist, x, to some real numbers, y:

$$x \in \mathbb{R}', y \in \mathbb{R}, R : x \mapsto y,$$

such x are called imaginary numbers.

Where relation R comes to map some real numbers, x, to some hypothetical numbers that are imagined to exist, y:

$$x \in \mathbb{R}, y \in \mathbb{R}', R : x \mapsto y,$$

such y are called imaginary numbers.

Where relation R comes to map some hypothetical numbers that are imagined to exist, x, to some hypothetical numbers that are imagined to exist, y:

$$x \in \mathbb{R}', y \in \mathbb{R}', R : x \mapsto y,$$

such x and y are called imaginary numbers.

3.5 Explicit and Collective Identifications of Imaginary Numbers

Let us consider another example for imaginary number identifications based on a relation mapping from a hypothetical number that is imagined to exist to another hypothetical number that is imagined to exist.

Example 3.5.1. *Let R be a relation on real numbers where*

$$R = \{(q, r) \in (\mathbb{R} \times \mathbb{R}) \mid (q \times r = 0) \wedge (q \neq 0) \wedge (r \neq 0)\},$$

hence relation R does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. For the updated relation R as

$$R = \{(x, y) \in ((\mathbb{R}' \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}') \cup (\mathbb{R}' \times \mathbb{R}')) \mid (x \times y = 0) \wedge (x \neq 0) \wedge (y \neq 0)\},$$

we imagine two hypothetical numbers, x and y, to exist such that

$$x \in \mathbb{R}', y \in \mathbb{R}' \mid (x \times y = 0) \wedge (x \neq 0) \wedge (y \neq 0)$$

hence $R : x \mapsto y$.

Recalling the imaginary number identifications we made in Example 3.3.1, we could notice that although we observe mappings from hypothetical numbers that are imagined to exist to hypothetical numbers that are imagined to exist

$$R : \mathbb{R}' \mapsto \mathbb{R}',$$

in both Example 3.3.1 and Example 3.5.1, the mappings are different in nature.

In the case of Example 3.3.1, ∞^0 is identified as an imaginary number within the mapping $\infty \to \infty^0$. ∞ was already identified as an imaginary number explicitly and independently in Example 3.2.1; it is not being identified again as an imaginary number in Example 3.3.1 collectively with ∞^0. By identifying a mapping from ∞ to ∞^0, we come to identify ∞^0 as a new imaginary number since ∞ was already established as an imaginary number explicitly.

In the case of Example 3.5.1, however, the mapping we identify is from one hypothetical number to another where the existence of both are imagined together at the same time. Neither x nor y, could be identified as imaginary numbers explicitly on their own. We identify x and y collectively as hypothetical numbers that are imagined to exist where their identifications are dependent on each other.

For expressing imaginary numbers that are identified collectively, rather than explicitly, tuples will serve as the ideal containers. So, we could rewrite Example 3.5.1 as in Example 3.5.2.

Example 3.5.2. *Let R be a relation on real numbers where*

$$R = \{(q, r) \in (\mathbb{R} \times \mathbb{R}) \mid (q \times r = 0) \wedge (q \neq 0) \wedge (r \neq 0)\},$$

hence relation R does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. For the updated relation R as

$$R = \{(x, y) \in ((\mathbb{R}' \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}') \cup (\mathbb{R}' \times \mathbb{R}')) \mid (x \times y = 0) \wedge (x \neq 0) \wedge (y \neq 0)\},$$

we identify x and y collectively as a tuple of imaginary numbers, $[x, y]$, to satisfy the conditions

$$x \in \mathbb{R}', y \in \mathbb{R}' \mid (x \times y = 0) \wedge (x \neq 0) \wedge (y \neq 0).$$

Our conclusion from these examples discussed is that the definition for imaginary numbers needs to make the distinction between explicit and collective identifications of imaginary numbers. We offer our conclusion in Lemma 3.5.1.

Lemma 3.5.1. *Let R be a relation on real numbers where*

$$R = \{(q, r) \in (\mathbb{R} \times \mathbb{R})\},$$

such that R does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. Hence, some hypothetical numbers could be imagined to exist to satisfy the conditions in updated relation R:

$$R = \{(x, y) \in ((\mathbb{R}' \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}') \cup (\mathbb{R}' \times \mathbb{R}'))\}.$$

Where

$$x \in \mathbb{R}', y \in \mathbb{R}':$$

If such x were already identified as imaginary numbers, then we could identify such y as imaginary numbers;

else if such y were already identified as imaginary numbers, then we could identify such x as imaginary numbers;

else if neither such x, nor such y were already identified as imaginary numbers, then we could identify a tuple containing such x and such y, $[x, y]$, as a tuple of imaginary numbers.

We shall offer an extensive definition for imaginary numbers to reflect the arguments from Lemma 3.5.1.

3.6 An Extensive Definition for Imaginary Numbers

Proposition 3.6.1. *Let R be a relation on real numbers where*

$$R = \{(q, r) \in (\mathbb{R} \times \mathbb{R})\},$$

such that relation R does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. Then some hypothetical numbers are imagined to exist to satisfy the conditions in updated relation R:

$$R = \{(x, y) \in ((\mathbb{R}' \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}') \cup (\mathbb{R}' \times \mathbb{R}'))\}, R \neq \emptyset.$$

Where relation R comes to map some hypothetical numbers that are imagined to exist, x, to some real numbers, y:

$$x \in \mathbb{R}', y \in \mathbb{R}, R : x \mapsto y,$$

such x are called imaginary numbers.

Where relation R comes to map some real numbers, x, to some hypothetical numbers that are imagined to exist, y:

$$x \in \mathbb{R}, y \in \mathbb{R}', R : x \mapsto y,$$

such y are called imaginary numbers.

Where relation R comes to map some hypothetical numbers that are imagined to exist, x, to some hypothetical numbers that are imagined to exist, y:

$$x \in \mathbb{R}', y \in \mathbb{R}', R : x \mapsto y,$$

if such x were already identified as imaginary numbers then such y are called imaginary numbers,

else if such y were already identified as imaginary numbers then such x are called imaginary numbers,

else if neither such x, nor such y were already identified as imaginary numbers then a tuple containing such x and such y, $[x, y]$, is called a tuple of imaginary numbers.

3.7 Setting the Boundaries for Numbers

Imaginary numbers are numbers, although they are not real ones. Hence, conformance to the properties of numbers is the expected behavior from an imaginary number.

Let us revisit the arguments from Section 2.3 and Section 3.3 **i** where we noted that statements (5) and (6) appear to hold the same expressions on each sides of the statements that are merely expressed in different notations. So far, we assumed that suggesting such inequalities was against the integrity of mathematical notations, hence we avoided these cases within imaginary number definitions.

However, a closer look into these two cases will reveal more detailed findings. For the case in (5), we realise that $x \times x$ and x^2 are indeed exactly the same mathematical expressions just expressed in different notations, hence suggesting such inequality is against the integrity of mathematical notations. Consequently, a relation R_1 where

$$R_1 = \{(q, r) \in (\mathbb{R} \times \mathbb{R}) \mid (q \times q = r) \wedge (r \neq q^2)\},$$

will not be mapping any numbers: $R_1 = \emptyset$, but does include at least one condition that is against the integrity of mathematical notations. Hence, we cannot identify imaginary numbers to satisfy the conditions in updated relation R_1 as

$$R_1 = \{(x, y) \in ((\mathbb{R}' \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}') \cup (\mathbb{R}' \times \mathbb{R}')) \mid (x \times x = y) \wedge (y \neq x^2)\}.$$

On the other hand, for the case in (6), we realise that the expression on the left of the statement, $x + x$, and the expression on the right of the statement, $2 \times x$, are fundamentally different than each other. $x + x$ is the notation for a number added onto itself whereas $2 \times x$ is the notation for number 2 multiplied by another number. Hence, the two expressions are actually produced by two different mathematical operations.

The reason why one normally assumes $x + x$ and $2 \times x$ to yield exactly the same value, is because this is the expected behavior from numbers, based on the multiplicative identity and multiplicative distributive properties of numbers:

$$x + x = (1 \times x) + (1 \times x) = (1 + 1) \times x = 2 \times x.$$

Hence suggesting the inequality $x + x \neq 2 \times x$ is a condition that is against either one or the both of the multiplicative identity property and the multiplicative distributive property of numbers. This is a different case than integrity of mathematical notations, this is about conformity of a given condition to the properties of numbers.

Given that an imaginary number is still a number, should it be allowed for an imaginary number to violate the properties of numbers? This actually is a philosophical question we run into.

It could be argued that all of types of imaginary numbers we exemplified so far actually violate some properties of numbers in one way or another. To exemplify some:

> The imaginary numbers conventionally recognized, with the unit number i, are defined based on the violation of the multiplicative product of two opposites property, and the multiplicative product by negative signs property.
>
> The imaginary numbers defined in Section 2.2, with the unit numbers h and g are defined based on the violation of the multiplicative identity property, the multiplicative inverse property, the multiplication property of equality, the division property of equality, and the cancellation property of multiplication.
>
> Other imaginary numbers defined in Section 2.2, with the unit numbers f and d are defined based on the violation of the additive identity property, the additive inverse property, the addition property of equality, the subtraction property of equality, the zero property of multiplication, and the cancellation property of addition.

If we are identifying imaginary numbers despite violations of some properties of numbers in their definitions, should we as well identify imaginary numbers based on the condition $x + x \neq 2 \times x$ that is in violation of multiplicative identity and multiplicative distributive properties of numbers?

Our answer to this question is that in order to be able to judge whether a given condition could be a base for identifying imaginary numbers, we need to understand what kind of properties that given condition violates and then check whether those properties are some fundamental ones for numbers. Where the violated properties are some fundamental properties of numbers, obviously we could not identify imaginary numbers based on that given condition.

However, our main argument throughout this paper has been that human imagination should not be hindered or limited. Based on this philosophy, we observe that imagination can, of course, happen on a given condition that violates fundamental properties of numbers, but such imagined objects to satisfy the given condition would then fall into the field of mathematical object types that are not numbers.

Let R_2 be a relation where

$$R_2 = \{(q, r) \in (\mathbb{R} \times \mathbb{R}) \mid (q + q = r) \land (r \neq 2q)\},$$

that does not map any numbers: $R_2 = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. As relation R_2 does include at least one condition that is against the fundamental properties of numbers, some hypothetical numbers could not be imagined to exist to satisfy the conditions in updated relation R_2 as

$$R_2 = \{(x, y) \in ((\mathbb{R}' \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}') \cup (\mathbb{R}' \times \mathbb{R}')) \mid (x + x = y) \land (y \neq 2x)\}.$$

However, some number-like mathematical objects could be identified to satisfy these conditions.

We shall formulate this argument by offering Lemma 3.7.1.

Lemma 3.7.1. *Let R be a relation on real numbers where*

$$R = \{(q, r) \in (\mathbb{R} \times \mathbb{R})\},$$

such that R does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. Hence, some hypothetical object instances could be imagined to exist to satisfy the conditions in updated relation R:

$$R = \{(x, y) \in ((\mathbb{R}' \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}') \cup (\mathbb{R}' \times \mathbb{R}'))\}.$$

If relation R does not include at least one condition that is against the fundamental properties of numbers, such hypothetical object instances that are imagined to exist are called imaginary numbers. Otherwise such hypothetical object instances that are imagined to exist are called instances of a non-predefined mathematical object type.

Obviously, it needs to be established what makes numbers numbers. Hence any behavior outside the fundamental properties of numbers would fall into the field of non-predefined mathematical object types. While such definition for numbers is outside the scope of this paper, we could offer an exemplary one in Axiom 3.7.1 as a guideline for our purposes in this paper.

Axiom 3.7.1. *Numbers are mathematical objects that are mainly useful for counting, measuring and labeling things and that show the following properties: the distributive property of multiplication, the commutative property of addition, the commutative property of multiplication, the associative property of addition, the associative property of multiplication, the substitution property, the reflexive property of* *equality, the symmetric property of equality, the transitive property of equality, the trichotomy property, the placement of negative signs property, and the undefined division by zero property.*

By the act of identifying imaginary numbers, we are actually drawing the lines around the properties of real numbers as well. Numbers is the super set of objects for real and imaginary numbers that are two disjoint subsets. Mathematical objects imagined on a given condition that violates the fundamental properties of real numbers but the fundamental properties of numbers are imaginary numbers, while mathematical objects imagined on a given condition that violates the fundamental properties of numbers are some number-like objects of a non-predefined type.

It will be useful to offer an exemplary definition for real numbers as well, again as a guideline for our purposes in this paper while a conclusive definition is outside the scope of this paper. Axiom 3.7.2 offers such definition.

Axiom 3.7.2. *Real numbers are numbers that show the following properties, in addition to the fundamental properties of numbers: the additive identity property, the multiplicative identity property, the additive inverse property, the multiplicative inverse property, the zero property of multiplication, the closure property of addition, the closure property of multiplication, the addition property of equality, the subtraction property of equality, the multiplication property of equality, the division property of equality, the cancellation property of multiplication, the cancellation property of addition, the multiplicative product by negative signs property, and the multiplicative product of two opposites property.*

These definitions for numbers and for real numbers are by no means conclusive ones; they are solely exemplary for the purposes of this paper. Detailed definitions, that are outside the scope of this paper, would be the subject to a more detailed analysis of properties of numbers. What we aim to emphasize here with the exemplary definitions is that such detailed definitions are necessary in order to be able to decide what mathematical object instances belong to which object type.

Additionally, we do recognize that imaginary number identifications bring out more consequences than just the necessity to define the boundaries of numbers and real numbers. There also are some consequences around needing more rules on mathematical operations, for instance. This is being referred to in Appendix B.

So, if we revisit out latest example: Let R_2 be a relation where

$$R_2 = \{(q, r) \in (\mathbb{R} \times \mathbb{R}) \mid (q + q = r) \wedge (r \neq 2q)\},$$

that does not map any numbers: $R_2 = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation but does include at least one condition that is against the fundamental properties of numbers. While we could not identify such x as imaginary numbers, we could identify them as instances of a non-predefined mathematical object type to satisfy the conditions in updated relation R_2 as

$$R_2 = \{(x, y) \in ((\mathbb{R}' \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}') \cup (\mathbb{R}' \times \mathbb{R}')) \mid (x + x = y) \wedge (y \neq 2x)\}.$$

Next, we shall also exemplify collective identifications of number-like mathematical objects in Example 3.7.1.

Example 3.7.1. *Let R be a relation on real numbers where*

$$R = \{(q, r) \in (\mathbb{R} \times \mathbb{R}) \mid (q \times r \neq r \times q)\},$$

hence relation R does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. However, relation R does include at least one condition that is against the fundamental properties of numbers. For the updated relation R as

$$R = \{(x, y) \in ((\mathbb{R}' \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}') \cup (\mathbb{R}' \times \mathbb{R}')) \mid (x \times y \neq y \times x)\},$$

we identify x and y collectively as a tuple of instances of a non-predefined mathematical object type, $[x, y]$, to satisfy the conditions

$$x \in \mathbb{R}', y \in \mathbb{R}' \mid (x \times y \neq y \times x).$$

If we contrast Example 3.7.1 to the previous Example 3.5.2 from Section 3.5, we note that we come to identify a tuple of instances of a non-predefined mathematical object type in Example 3.7.1 whereas we identified a tuple of imaginary numbers in Example 3.5.2.

We shall offer a comprehensive definition for imaginary numbers to reflect our conclusions as stated in Lemma 3.7.1.

3.8 A Comprehensive Definition for Imaginary Numbers

Proposition 3.8.1. *Let R be a relation on real numbers where*

$$R = \{(q, r) \in (\mathbb{R} \times \mathbb{R})\},$$

such that relation R does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. Then some hypothetical object instances are imagined to exist to satisfy the conditions in updated relation R:

$$R = \{(x, y) \in ((\mathbb{R}' \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}') \cup (\mathbb{R}' \times \mathbb{R}'))\}, R \neq \emptyset.$$

Where relation R comes to map some hypothetical object instances that are imagined to exist, x, to some real numbers, y:

$$x \in \mathbb{R}', y \in \mathbb{R}, R : x \mapsto y,$$

if relation R does not include any condition that is against the fundamental properties of numbers, such x are called imaginary numbers;

else if relation R does include at least one condition that is against the fundamental properties of numbers, such x are called instances of a non-predefined mathematical object type.

Where relation R comes to map some real numbers, x, to some hypothetical object instances that are imagined to exist, y:

$$x \in \mathbb{R}, y \in \mathbb{R}', R : x \mapsto y,$$

if relation R does not include any condition that is against the fundamental properties of numbers, such y are called imaginary numbers;

else if relation R does include at least one condition that is against the fundamental properties of numbers, such y, are called instances of a non-predefined mathematical object type.

Where relation R comes to map some hypothetical object instances that are imagined to exist, x, to some hypothetical object instances that are imagined to exist, y:

$$x \in \mathbb{R}', y \in \mathbb{R}', R : x \mapsto y,$$

if relation R does not include any condition that is against the fundamental properties of numbers,

> *and if such x were already identified as imaginary numbers then such y are called imaginary numbers,*
>
> *otherwise if such y were already identified as imaginary numbers then such x are called imaginary numbers,*
>
> *otherwise if neither such x, nor such y were already identified as imaginary numbers then a tuple containing such x and such y, $[x, y]$ is called a tuple of imaginary numbers;*

else if relation R does include at least one condition that is against the fundamental properties of numbers,

> *and if such x were already identified as imaginary numbers or as instances of a non-predefined mathematical object type then such y are called instances of a non-predefined mathematical object type,*
>
> *otherwise if such y were already identified as imaginary numbers or as instances of a non-predefined mathematical object type then such x are called instances of non-predefined mathematical object type,*
>
> *otherwise if neither such x, nor such y were already identified as imaginary numbers or as instances of a non-predefined mathematical object type then a tuple containing such x and such y, $[x, y]$, is called a tuple of instances of a non-predefined mathematical object type.*

3.9 Imagination Context

Starting off by considering a relation R such as

$$R = \{(q, r) \in (\mathbb{R} \times \mathbb{R})\}, \tag{9}$$

that does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation, we come to identify imaginary numbers or instances of a non-predefined mathematical object type to satisfy the conditions in updated relation R as

$$R = \{(x, y) \in ((\mathbb{R}' \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}') \cup (\mathbb{R}' \times \mathbb{R}'))\}. \tag{10}$$

So, in a typical imaginary number identification, we first set our conditions for all real numbers as in (9). Consequently, our imagination context becomes the sets of combinations of real and imaginary numbers as relation domains and ranges, as specified in (10).

Let us consider an imagination example based on Fermat's Last Theorem.

Example 3.9.1. *Let R be a relation on real numbers such that*

$$R = \{(q, r, s) \in (\mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+) \mid (n \in \mathbb{Z}) \wedge (n > 2) \wedge (q^n + r^n = s^n)\}.$$

Hence relation R does not map any numbers: $R = \emptyset$ and does not include at least one condition that is *against the integrity of any mathematical notation. For the updated relation R as*

$$
\begin{aligned}
R = \{(x, y, x) \in \\
&((\mathbb{Z}^{+}{}' \times \mathbb{Z}^{+} \times \mathbb{Z}^{+}) \cup \\
&(\mathbb{Z}^{+} \times \mathbb{Z}^{+}{}' \times \mathbb{Z}^{+}) \cup \\
&(\mathbb{Z}^{+} \times \mathbb{Z}^{+} \times \mathbb{Z}^{+}{}') \cup \\
&(\mathbb{Z}^{+}{}' \times \mathbb{Z}^{+}{}' \times \mathbb{Z}^{+}) \cup \\
&(\mathbb{Z}^{+}{}' \times \mathbb{Z}^{+} \times \mathbb{Z}^{+}{}') \cup \\
&(\mathbb{Z}^{+} \times \mathbb{Z}^{+}{}' \times \mathbb{Z}^{+}{}') \cup \\
&(\mathbb{Z}^{+}{}' \times \mathbb{Z}^{+}{}' \times \mathbb{Z}^{+}{}'))\}, \\
&\mid (n \in \mathbb{Z}) \wedge (n > 2) \wedge (x^n + y^n = z^n)\},
\end{aligned}
$$

we identify x, y and z collectively as a tuple of imaginary positive natural numbers, $[x, y, z]$, to satisfy the conditions

$$x \in \mathbb{Z}^{+}{}', y \in \mathbb{Z}^{+}{}', z \in \mathbb{Z}^{+}{}' \mid (n \in \mathbb{Z}) \wedge (n > 2) \wedge (x^n + y^n = z^n).$$

Example 3.9.1 seems to present a perfectly valid case for imagination and we could make the following observations based on this example:

i. Imagination does not necessarily have to happen over a binary relation, it can as well be on a relation with multiple domains and ranges, as featured in Example 3.9.1:

$$R = \{(x, y, z) \in (\mathbb{Z}^{+\prime} \times \mathbb{Z}^{+\prime} \times \mathbb{Z}^{+\prime})\},$$

which is a ternary relation.

ii. The entirety of an object type does not have to be included within the imagination context, the imagination context can be on a subset of an overall object type. For the case in Example 3.9.1, the imagination did not include all numbers, but only positive natural numbers, since the given relation R was defined over $\mathbb{Z}^+$ instead of $\mathbb{R}$. The numbers we come to imagine in this example are not just imaginary numbers, but they are imaginary positive natural numbers.

A point to note about Example 3.9.1 is that this example helps us underline the fact that the notation we have been using to denote the set of hypothetical numbers that are imagined to exist as the absolute complement of a given set of numbers is not really adequate for our purposes. In the case of Example 3.9.1, we use $\mathbb{Z}^{+\prime}$ to denote the set of hypothetical positive natural numbers that are imagined to exist, however this notation, in fact, denotes any number outside the set of positive natural numbers, not our intended set of hypothetical positive natural numbers that are imagined to exist. So, we need a dedicated notation to denote specific sets of hypothetical numbers that are imagined to exist. Such notation is offered in Proposition 3.9.1.

Proposition 3.9.1. *Where K denotes the set of all possible numbers within a subset of real numbers, K^i denotes the set of all hypothetical object instances that are imagined to exist for this specific subset of numbers.*

In the light of observations **i** and **ii**, and the specific notation we offer in Proposition 3.9.1, we could write an expanded specification for the imagination context as in Lemma 3.9.1.

Lemma 3.9.1. *Let R be a relation on some arbitrary subsets of real numbers, $\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n$ represent the sets containing all possible numbers within those subsets, and n be a positive integer greater than 1 where*

$$R = \{(a_1, a_2, ..., a_n) \in (R_1 \times R_2 \times ... \times R_n)\}$$

such that relation R does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. Then some hypothetical object instances are imagined to exist to satisfy the conditions in updated relation R:

$$\begin{aligned} R = \{&(a_1, a_2, ..., a_n) \in \\ &((\mathbb{R}_1^i \times \mathbb{R}_2 \times ... \times \mathbb{R}_n) \cup (\mathbb{R}_1 \times \mathbb{R}_2^i \times ... \times \mathbb{R}_n) \cup ... \cup (\mathbb{R}_1 \times \mathbb{R}_2 \times ... \times \mathbb{R}_n^i) \cup \\ &(\mathbb{R}_1^i \times \mathbb{R}_2^i \times ... \times \mathbb{R}_n) \cup (\mathbb{R}_1^i \times \mathbb{R}_2 \times \mathbb{R}_3^i \times ... \times \mathbb{R}_n) \cup ... \cup (\mathbb{R}_1^i \times \mathbb{R}_2 \times ... \times \mathbb{R}_n^i) \cup \\ &(\mathbb{R}_1 \times \mathbb{R}_2^i \times \mathbb{R}_3^i \times ... \times \mathbb{R}_n) \cup (\mathbb{R}_1 \times \mathbb{R}_2^i \times \mathbb{R}_3 \times \mathbb{R}_4^i \times ... \times \mathbb{R}_n) \cup ... \cup (\mathbb{R}_1 \times \mathbb{R}_2^i \times ... \times \mathbb{R}_n^i) \cup \\ &... \cup \\ &(\mathbb{R}_1^i \times \mathbb{R}_2^i \times \mathbb{R}_3^i \times ... \times \mathbb{R}_n) \cup (\mathbb{R}_1^i \times \mathbb{R}_2^i \times \mathbb{R}_3 \times \mathbb{R}_4^i \times ... \times \mathbb{R}_n) \cup \\ &(\mathbb{R}_1^i \times \mathbb{R}_2^i \times \mathbb{R}_3 \times \mathbb{R}_4 \times \mathbb{R}_5^i \times ... \times \mathbb{R}_n) \cup ... \cup (\mathbb{R}_1^i \times \mathbb{R}_2^i \times \mathbb{R}_3 \times ... \times \mathbb{R}_n^i) \cup \\ &... \cup \\ &(\mathbb{R}_1^i \times \mathbb{R}_2^i \times \mathbb{R}_3^i \times \mathbb{R}_4^i \times \mathbb{R}_5^i \times ... \times \mathbb{R}_n^i))\}, \\ R \neq \emptyset&. \end{aligned}$$

We shall offer a generic definition for imaginary numbers by reflecting Lemma 3.9.1.

3.10 A Generic Definition for Imaginary Numbers

Proposition 3.10.1. *Let R be a relation on some arbitrary subsets of real numbers, $\mathbb{R}_1, \mathbb{R}_2, ..., \mathbb{R}_n$ represent the sets containing all possible numbers within those subsets, and n be a positive integer greater than 1 where*

$$R = \{(q_1, q_2, ..., q_n) \in (R_1 \times R_2 \times ... \times R_n)\},$$

such that relation R does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. Then some hypothetical object instances are imagined to exist to satisfy the conditions in updated relation R:

$$\begin{aligned} R = \{&(a_1, a_2, ..., a_n) \in \\ &((\mathbb{R}_1^i \times \mathbb{R}_2 \times ... \times \mathbb{R}_n) \cup (\mathbb{R}_1 \times \mathbb{R}_2^i \times ... \times \mathbb{R}_n) \cup ... \cup (\mathbb{R}_1 \times \mathbb{R}_2 \times ... \times \mathbb{R}_n^i) \cup \\ &(\mathbb{R}_1^i \times \mathbb{R}_2^i \times ... \times \mathbb{R}_n) \cup (\mathbb{R}_1^i \times \mathbb{R}_2 \times \mathbb{R}_3^i \times ... \times \mathbb{R}_n) \cup ... \cup (\mathbb{R}_1^i \times \mathbb{R}_2 \times ... \times \mathbb{R}_n^i) \cup \\ &(\mathbb{R}_1 \times \mathbb{R}_2^i \times \mathbb{R}_3^i \times ... \times \mathbb{R}_n) \cup (\mathbb{R}_1 \times \mathbb{R}_2^i \times \mathbb{R}_3 \times \mathbb{R}_4^i \times ... \times \mathbb{R}_n) \cup ... \cup (\mathbb{R}_1 \times \mathbb{R}_2^i \times ... \times \mathbb{R}_n^i) \cup \\ &... \cup \\ &(\mathbb{R}_1^i \times \mathbb{R}_2^i \times \mathbb{R}_3^i \times ... \times \mathbb{R}_n) \cup (\mathbb{R}_1^i \times \mathbb{R}_2^i \times \mathbb{R}_3 \times \mathbb{R}_4^i \times ... \times \mathbb{R}_n) \cup \\ &(\mathbb{R}_1^i \times \mathbb{R}_2^i \times \mathbb{R}_3 \times \mathbb{R}_4 \times \mathbb{R}_5^i \times ... \times \mathbb{R}_n) \cup ... \cup (\mathbb{R}_1^i \times \mathbb{R}_2^i \times \mathbb{R}_3 \times ... \times \mathbb{R}_n^i) \cup \\ &... \cup \\ &(\mathbb{R}_1^i \times \mathbb{R}_2^i \times \mathbb{R}_3^i \times \mathbb{R}_4^i \times \mathbb{R}_5^i \times ... \times \mathbb{R}_n^i))\}, \\ R \neq \emptyset&. \end{aligned}$$

Where x is a positive integer and relation R comes to map some hypothetical object instances that are *imagined to exist, a_x, among real numbers:*

$$a_1 \in \mathbb{R}_1, a_2 \in \mathbb{R}_2, ..., a_x \in \mathbb{R}^i_x, ..., a_n \in \mathbb{R}_n,$$
$$(a_1, a_2, ..., a_x, ..., a_n) \in R,$$

if relation R does not include any condition that is against the fundamental properties of numbers, such a_x are called imaginary numbers;

else if relation R does include at least one condition that is against the fundamental properties of numbers, such a_x are called instances of a non-predefined mathematical object type.

Where x and y are positive integers, $x + y \leq n$, and relation R comes to map some hypothetical object instances that are imagined to exist, $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}\}$, among themselves or among themselves and also real numbers:

$$a_1 \in \mathbb{R}_1, a_2 \in \mathbb{R}_2, ..., a_x \in \mathbb{R}^i_x, a_{x+1} \in \mathbb{R}^i_{x+1}, a_{x+2} \in \mathbb{R}^i_{x+2}, ..., a_{x+y} \in \mathbb{R}^i_{x+y}, ..., a_n \in \mathbb{R}_n,$$
$$(a_1, a_2, ..., a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}, ..., a_n) \in R,$$

if relation R does not include any condition that is against the fundamental properties of numbers,

and if each one of the object instances in the set $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}\}$ except one of the object instances, a_{x+t}, were already identified as imaginary numbers, such that t is an arbitrary positive integer and $t \leq y$, then such a_{x+t} are called imaginary numbers;

otherwise if none of the object instances in the set $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}\}$ were already identified as imaginary numbers, then a tuple containing these object instances, $[a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}]$, is called a tuple of imaginary numbers;

otherwise if some of the object instances, $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+w}\}$, in the set $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}\}$ were already identified as imaginary numbers, while some other object instances, $\{a_{x+w+1}, a_{x+w+2}, a_{x+w+3}, ..., a_{x+y}\}$, were not already identified as imaginary numbers, such that w is an arbitrary positive integer and $w < y$, then a tuple containing the object instances, $[a_{x+w+1}, a_{x+w+2}, a_{x+w+3}, ..., a_{x+y}]$, is called a tuple of imaginary numbers;

else if relation R does include at least one condition that is against the fundamental properties of numbers,

and if each one of the object instances in the set $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}\}$ except one of the object instances, a_{x+t}, were already identified as imaginary numbers or instances of some non-predefined mathematical object types, such that t is an arbitrary positive integer and $t \leq y$, then such a_{x+t} are called instances of a non-predefined mathematical object type;

otherwise if none of the object instances in the set $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}\}$ were already identified as imaginary numbers or instances of some non-predefined mathematical object types, then a tuple containing these object instances, $[a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}]$, is called a tuple of instances of some non-predefined mathematical object types;

otherwise if some of the object instances, $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+w}\}$, in the set $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}\}$ were already identified as imaginary numbers or instances of some non-predefined mathematical object types, while some other object instances, $\{a_{x+w+1}, a_{x+w+2}, a_{x+w+3}, ..., a_{x+y}\}$, were not already identified as imaginary numbers or instances of some non-predefined mathematical object types, such that w is an arbitrary positive *integer and $w < y$, then a tuple containing the object instances, $[a_{x+w+1}, a_{x+w+2}, a_{x+w+3}, ..., a_{x+y}]$, is called a tuple of instances of some non-predefined mathematical object types.*

3.11 Understanding Quaternions and Noting Octonions

Let us consider quaternions, defined by William Rowan Hamilton [Hamilton (1844)], in the light of the generic definition for imaginary numbers. We could formulate the definition of quaternions as in Definition 3.11.1.

Definition 3.11.1. *Let H be a number defined as*

$$H = a + bi + cj + dk$$

where a, b, c and d are real numbers and i, j and k are specifically defined numbers such that

$$\begin{aligned} &i^2 = j^2 = k^2 = ijk = -1, \\ &ij = k, ji = -k, \\ &jk = i, kj = -i, \\ &ki = j, ik = -j. \end{aligned}$$

Such H is defined as a quaternion and i, j and k are defined as quaternion units.

Let us bring the definition of the quaternion units into the relation format to examine them.

Example 3.11.1. *Let R be a relation on real numbers such that*

$$\begin{aligned} R =& \{(q, r, s) \in (\mathbb{R} \times \mathbb{R} \times \mathbb{R}) \mid (q^2 = r^2 = s^2 = qrs = -1) \\ &\wedge (qr = s) \wedge (rq = -s) \wedge (rs = q) \wedge (sr = -q) \wedge (sq = r) \wedge (qs = -r)\}. \end{aligned}$$

Hence relation R does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. For the updated relation R as

$$\begin{aligned} R = \{&(i, j, k) \in \\ &((\mathbb{R}^i \times \mathbb{R} \times \mathbb{R}) \cup \\ &(\mathbb{R} \times \mathbb{R}^i \times \mathbb{R}) \cup \\ &(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^i) \cup \\ &(\mathbb{R}^i \times \mathbb{R}^i \times \mathbb{R}) \cup \\ &(\mathbb{R}^i \times \mathbb{R} \times \mathbb{R}^i) \cup \\ &(\mathbb{R} \times \mathbb{R}^i \times \mathbb{R}^i) \cup \\ &(\mathbb{R}^i \times \mathbb{R}^i \times \mathbb{R}^i))\}, \\ &\mid (i^2 = j^2 = k^2 = ijk = -1) \\ &\wedge (ij = k) \wedge (ji = -k) \wedge (jk = i) \\ &\wedge (kj = -i) \wedge (ki = j) \wedge (ik = -j)\}, \end{aligned}$$

quaternion units i, j and k, are identified to satisfy the conditions. Since the relation conditions include at least one condition that is against the fundamental properties of numbers, to be specific the commutative *property of multiplication, such that*

$$i \in \mathbb{R}^i, j \in \mathbb{R}^i, k \in \mathbb{R}^i,$$
$$ij = k, ji = -k,$$
$$jk = i, kj = -i,$$
$$ki = j, ik = -j,$$

we identify a tuple containing i, j, and k, $[i, j, k]$, as a tuple of instances of a non-predefined mathematical object type. In other words, the quaternian units i, j and k are each instances of a number-like object type but they are not numbers themselves. Collectively as a tuple, we imagine them to satisfy the conditions in updated relation R, however we could not identify any one of them explicitly.

Quaternions are not actual extensions of imaginary numbers, and they are not taking complex numbers into a multi-dimensional space on their own. Quaternion units are instances of some number-like object type, identified collectively, but they are not numbers (be it real or imaginary). In other words, they form a closed, internally consistent set of object instances; they can of course be plotted visually on a multi-dimensional space but this only is a visualization within their own definition.

Leaving the exact definition of octonions, initially defined by John Thomas Graves [Baez (2002)], out of scope for this article, it is noteworthy to mention that in a similar fashion to the examination of quaternion units in Example 3.11.1, we could show that the definition of octonion units is also based on some conditions against the fundamental properties of numbers, namely the commutative property of multiplication and the associative property of multiplication. Hence octonion units as well are instances of a number-like object type, but they are not numbers. Octonion units are multiple instances of a non-predefined mathematical object type that get identified collectively, rather than explicitly.

4 Defining Imaginary Objects

4.1 Imaginary Instances of Predefined Mathematical Object Types

Despite that they might be the most commonly used ones, numbers comprise only one type of mathematical objects among other predefined mathematical object types such as matrices, sets, permutations and partitions. So far we have not considered the imagination of instances of these other predefined object types.

Until now, we have been identifying imaginary instances of numbers since we always started off by looking to satisfy given conditions in relations that are defined over numbers but not other mathematical object types.

With an example, we shall explore what we could imagine if we didn't start off by setting our conditions for some numbers but for another type of mathematical object, such as matrices.

Example 4.1.1. *Let R be a relation, M be the set containing all possible matrices, n be an arbitrary positive integer, Q and T be n by n square matrices and I be the n by n identity matrix*

$$R = \{(Q, T) \in (M \times M) \mid \forall Q \mid (\det Q = 0) \wedge (Q \times T = I)\},$$

hence relation R does not map any matrices: $R = \emptyset$, and does not include at least one condition that is *against the integrity of any mathematical notation. For the updated relation R as*

$$R = \{(A, B) \in ((M \times M^i) \cup (M^i \times M) \cup (M^i \times M^i)) \mid \forall A \mid (\det A = 0) \wedge (A \times B = I)\},$$

we identify such B as an imaginary matrix to satisfy the conditions

$$A \in M, B \in M^i \mid \forall A \mid (\det A = 0) \wedge (A \times B = I).$$

We come to identify an imaginary matrix in Example 4.1.1. Conventionally, a complex matrix would be defined as a matrix that can hold complex numbers as its elements, in addition to real numbers as its elements. For the sake of simplifying this argument, an imaginary matrix conventionally would be a matrix that can hold imaginary numbers as its elements, in addition to real numbers as its elements. The definition for an imaginary matrix we offer here, however, is different than that conventional definition. In our arguments, an imaginary matrix is a hypothetical matrix that is imagined to exist.

We could expand our examples by considering sets as a type of objects to be imagined.

Example 4.1.2. *Let R be a relation, S be the set containing all possible sets,*

$$R = \{(q, r) \in (S \times S) \mid \forall r \mid q \cup r = \emptyset\},$$

such that relation R does not map any sets: $R = \emptyset$, and does not include at least one condition that is against the integrity of any mathematical notation. For the updated relation R as

$$R = \{(u, v) \in ((S^i \times S) \cup (S \times S^i) \cup (S^i \times S^i)) \mid \forall v \mid u \cup v = \emptyset\},$$

we identify such u as an imaginary set to satisfy the conditions

$$u \in S^i, v \in S \mid \forall v \mid u \cup v = \emptyset.$$

This time, we come to identify an imaginary set in Example 4.1.2. This is a set that is imagined to exist which is different than a set of imaginary objects which would be a set that holds imaginary (and real at the same time) objects as elements. This is analogous to the argument we had before about imaginary matrices versus matrices of imaginary numbers.

In the light of our imagination examples for objects other than numbers, we shall expand our understanding of imagination to encapsulate all types of mathematical objects, as offered in Lemma 4.1.1.

Lemma 4.1.1. *Let R be a relation on some arbitrary subsets of real mathematical objects of a certain type, $O_1, O_2, ..., O_n$ represent the sets containing all possible instances within those subsets, and n be a positive integer greater than 1 where*

$$R = \{(q_1, q_2, ..., q_n) \in (O_1 \times O_2 \times ... \times O_n)\},$$

such that relation R does not map any object instances: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. Then some hypothetical object instances are *imagined to exist to satisfy the conditions in updated relation R:*

$$
\begin{aligned}
R = \{&(a_1, a_2, ..., a_n) \in \\
&((O_1^i \times O_2 \times ... \times O_n) \cup (O_1 \times O_2^i \times ... \times O_n) \cup ... \cup (O_1 \times O_2 \times ... \times O_n^i) \cup \\
&(O_1^i \times O_2^i \times ... \times O_n) \cup (O_1^i \times O_2 \times O_3^i \times ... \times O_n) \cup ... \cup (O_1^i \times O_2 \times ... \times O_n^i) \cup \\
&(O_1 \times O_2^i \times O_3^i \times ... \times O_n) \cup (O_1 \times O_2^i \times O_3 \times O_4^i \times ... \times O_n) \cup ... \cup (O_1 \times O_2^i \times ... \times O_n^i) \cup \\
&... \cup \\
&(O_1^i \times O_2^i \times O_3^i \times ... \times O_n) \cup (O_1^i \times O_2^i \times O_3 \times O_4^i \times ... \times O_n) \cup \\
&(O_1^i \times O_2^i \times O_3 \times O_4 \times O_5^i \times ... \times O_n) \cup ... \cup (O_1^i \times O_2^i \times O_3 \times ... \times O_n^i) \cup \\
&... \cup \\
&(O_1^i \times O_2^i \times O_3^i \times O_4^i \times O_5^i \times ... \times O_n^i))\}, \\
R \neq \emptyset&.
\end{aligned}
$$

4.2 The Importance of the Fundamental Properties of Object Types

Revisiting our arguments in Section 3.7, we have certain fundamental expectations from numbers, be it real or imaginary. That is the reason, why we identify hypothetical object instances imagined to exist that do not comply with the fundamental properties of numbers, as instances of a non-predefined mathematical object types on their own, but not numbers. We discussed the condition

$$x + x \neq 2 \times x$$

to be against the fundamental properties of numbers, we could as well expand our thinking into the condition

$$2 \times x \neq x \times 2$$

and explore an imaginary identification on this condition as in Example 4.2.1.

Example 4.2.1. *Let R be a relation where*

$$R = \{(q, r) \in (\mathbb{R} \times \mathbb{R}) \mid ((2 \times q = r) \wedge (r \neq q \times 2))\},$$

hence relation R does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. For the updated relation R as

$$R = \{(x, y) \in ((\mathbb{R} \times \mathbb{R}^i) \cup (\mathbb{R}^i \times \mathbb{R}) \cup (\mathbb{R}^i \times \mathbb{R}^i)) \mid ((2 \times x = y) \wedge (y \neq x \times 2))\},$$

we identify x as an instance of a non-predefined mathematical object type to satisfy the conditions

$$x \in \mathbb{R}^i, y \in \mathbb{R} \mid (2 \times x = y) \wedge (y \neq x \times 2),$$

since relation R does include at least one condition that is against the fundamental properties of numbers.

We could move Example 4.2.1 to more generic terms of multiplication, rather than multiplication by 2 only, as in Example 4.2.2.

Example 4.2.2. *Let R be a relation on real numbers where*

$$R = \{(q, r) \in (\mathbb{R} \times \mathbb{R}) \mid \forall r \mid q \times r \neq r \times q\},$$

hence relation R does not map any numbers: $R = \emptyset$ *and does not include at least one condition that is against the integrity of any mathematical notation. For the updated relation R as*

$$R = \{(x, y) \in ((\mathbb{R} \times \mathbb{R}^i) \cup (\mathbb{R}^i \times \mathbb{R}) \cup (\mathbb{R}^i \times \mathbb{R}^i)) \mid \forall y \mid x \times y \neq y \times x\},$$

we identify x *as an instance of a non-predefined mathematical object type to satisfy the conditions*

$$x \in \mathbb{R}^i, y \in \mathbb{R} \mid \forall y \mid x \times y \neq y \times x,$$

since relation R does include at least one condition that is against the fundamental properties of numbers.

While violation of the commutative property of the multiplication operation is a behavior outside the fundamental properties of numbers, the same behavior would not be a violation of the fundamental properties when the objects in question are matrices, instead of numbers.

Example 4.2.3. *Let R be a relation on real matrices and M be the set containing all possible matrices*

$$R = \{(Q, T) \in (M \times M) \mid (Q \times T \neq T \times Q)\},$$

hence relation R is mapping many matrices: $R \neq \emptyset$*, so we do not come to identify any imaginary matrices with the conditions of relation R.*

Having established in Section 3.7 the need for specifying the fundamental properties of numbers hence it could be judged whether a given relation condition is in violation of those fundamental properties or not, the same holds per each one of the mathematical object types. So that, there could be a structured way of determining what conditions fall out of the fundamental properties of the mathematical object types. However, such definition per each object type is beyond the scope of this article.

We shall offer an initial definition for imaginary objects in Proposition 4.3.1 to expand the imaginary number definition into all types of mathematical objects.

4.3 An Initial Definition for Imaginary Objects

Proposition 4.3.1. *Let R be a relation on some arbitrary subsets of real instances of a certain mathematical object type,* $O_1, O_2, ..., O_n$ *represent the sets containing all possible instances within those subsets, and n be a positive integer greater than* 1 *where*

$$R = \{(a_1, a_2, ..., a_n) \in (O_1 \times O_2 \times ... \times O_n)\},$$

such that relation R does not map any object instances: $R = \emptyset$ *and does not include at least one condition that is against the integrity of any mathematical notation. Then some hypothetical object instances are* *imagined to exist to satisfy the conditions in updated relation R:*

$$
\begin{aligned}
R = \{&(a_1, a_2, ..., a_n) \in \\
&((O_1^i \times O_2 \times ... \times O_n) \cup (O_1 \times O_2^i \times ... \times O_n) \cup ... \cup (O_1 \times O_2 \times ... \times O_n^i) \cup \\
&(O_1^i \times O_2^i \times ... \times O_n) \cup (O_1^i \times O_2 \times O_3^i \times ... \times O_n) \cup ... \cup (O_1^i \times O_2 \times ... \times O_n^i) \cup \\
&(O_1 \times O_2^i \times O_3^i \times ... \times O_n) \cup (O_1 \times O_2^i \times O_3 \times O_4^i \times ... \times O_n) \cup ... \cup (O_1 \times O_2^i \times ... \times O_n^i) \cup \\
&... \cup \\
&(O_1^i \times O_2^i \times O_3^i \times ... \times O_n) \cup (O_1^i \times O_2^i \times O_3 \times O_4^i \times ... \times O_n) \cup \\
&(O_1^i \times O_2^i \times O_3 \times O_4 \times O_5^i \times ... \times O_n) \cup ... \cup (O_1^i \times O_2^i \times O_3 \times ... \times O_n^i) \cup \\
&... \cup \\
&(O_1^i \times O_2^i \times O_3^i \times O_4^i \times O_5^i \times ... \times O_n^i))\}, \\
R \neq \emptyset&.
\end{aligned}
$$

Where x is a positive integer and relation R comes to map some hypothetical object instances that are imagined to exist, a_x, among some real object instances:

$$
\begin{aligned}
&a_1 \in O_1, a_2 \in O_2, ..., a_x \in O_x^i, ..., a_n \in O_n, \\
&(a_1, a_2, ..., a_x, ..., a_n) \in R,
\end{aligned}
$$

if relation R does not include any condition that is against the fundamental properties of that certain mathematical object type, such a_x are called imaginary instances of that object type;

else if relation R does include at least one condition that is against the fundamental properties of that certain mathematical object type, such a_x are called instances of a non-predefined mathematical object type.

Where x and y are positive integers, $x + y \leq n$, and relation R comes to map some hypothetical object instances that are imagined to exist, $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}\}$, among themselves or among themselves and also real object instances:

$$
\begin{aligned}
&a_1 \in O_1, a_2 \in O_2, ..., a_x \in O_x^i, a_{x+1} \in O_{x+1}^i, a_{x+2} \in O_{x+2}^i, ..., a_{x+y} \in O_{x+y}^i, ..., a_n \in O_n, \\
&(a_1, a_2, ..., a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}, ..., a_n) \in R,
\end{aligned}
$$

if relation R does not include any condition that is against the fundamental properties of that certain mathematical object type,

and if each one of the object instances in the set $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}\}$ except one of the object instances, a_{x+t}, were already identified as imaginary instances of that object type, such that t is an arbitrary positive integer and $t \leq y$, then such a_{x+t} are called imaginary instances of that object type;

otherwise if none of the object instances in the set $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}\}$ were already identified as imaginary instances of that object type, then a tuple containing these object instances, $[a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}]$, is called a tuple of imaginary instances of that object type;

otherwise if some of the object instances, $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+w}\}$, in the set $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}\}$ were already identified as imaginary instances of that object type, while some other object instances, $\{a_{x+w+1}, a_{x+w+2}, a_{x+w+3}, ..., a_{x+y}\}$, were not already identified as imaginary instances of that object type, such that w is an arbitrary positive integer *and $w < y$, then a tuple containing the object instances, $[a_{x+w+1}, a_{x+w+2}, a_{x+w+3}, ..., a_{x+y}]$, is called a tuple of imaginary instances of that object type;*

else if relation R does include at least one condition that is against the fundamental properties of that certain mathematical object type,

and if each one of the object instances in the set $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}\}$ except one of the object instances, a_{x+t}, were already identified as imaginary instances of that object type or instances of some non-predefined mathematical object types, such that t is an arbitrary positive integer and $t \leq y$, then such a_{x+t} are called instances of a non-predefined mathematical object type;

otherwise if none of the object instances in the set $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}\}$ were already identified as imaginary instances of that object type or instances of some non-predefined mathematical object types, then a tuple containing these object instances, $[a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}]$, is called a tuple of instances of some non-predefined mathematical object types;

otherwise if some of the object instances, $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+w}\}$, in the set $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}\}$ were already identified as imaginary instances of that object type or instances of some non-predefined mathematical object types, while some other object instances, $\{a_{x+w+1}, a_{x+w+2}, a_{x+w+3}, ..., a_{x+y}\}$, were not already identified as imaginary instances of that object type or instances of some non-predefined mathematical object types, such that w is an arbitrary positive integer and $w < y$, then a tuple containing the object instances, $[a_{x+w+1}, a_{x+w+2}, a_{x+w+3}, ..., a_{x+y}]$, is called a tuple of instances of some non-predefined mathematical object types.

4.4 Imagination Across Multiple Object Types

Let us consider Example 4.4.1 for identifying yet another type of imaginary numbers.

Example 4.4.1. *Let R be a relation and M be the set containing all possible matrices such that*

$$R = \{(q, P) \in (\mathbb{R} \times M) \mid \forall P \mid (q \times P \neq P \times q)\}.$$

Hence relation R does not map any object instances: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. For the updated relation R as

$$R = \{(x, A) \in ((\mathbb{R}^i \times M) \cup (\mathbb{R} \times M^i) \cup (\mathbb{R}^i \times M^i)) \mid \forall A \mid (x \times A \neq A \times x)\},$$

we identify such x as an imaginary number to satisfy the conditions

$$x \in \mathbb{R}^i, A \in M \mid x \times A \neq A \times x.$$

We could make three observations about Example 4.4.1:

i. The domain and the range of the relation does not have to be a single one object type for us to identify imaginary mathematical object instances. Relation R in Example 4.4.1 spans across numbers and matrices:

$$(x, A) \in (\mathbb{R} \times M).$$

ii. As the relation domain and range spans across multiple object types, so does the imagination context based on that relation. The imagination context in Example 4.4.1 is

$$((\mathbb{R}^i \times M) \cup (\mathbb{R} \times M^i) \cup (\mathbb{R}^i \times M^i)).$$

iii. We discussed in Section 3.7 and Section 4.2 that specifications for the fundamental properties per each predefined mathematical object type is needed hence it could be judged whether a given relation condition conforms or violates those fundamental properties. Additionally, since the domain and the range of a relation can span across multiple object types, as in Example 4.4.1, specifications for valid interactions among predefined object types are also needed, hence it could be judged whether a given relation condition, such as

$$q \times P \neq P \times q,$$

conforms or violates the fundamental properties of interactions among those object types that are in question.

We shall formulate these three observations in Lemma 4.4.1.

Lemma 4.4.1. *Let R be a relation on some arbitrary subsets of real mathematical objects of various types, $O_1, O_2, ..., O_n$ represent the sets containing all possible instances within those subsets, and n be a positive integer greater than 1 where*

$$R = \{(a_1, a_2, ..., a_n) \in (O_1 \times O_2 \times ... \times O_n)\}$$

such that relation R does not map any object instances: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. Then some hypothetical object instances are imagined to exist to satisfy the conditions in updated relation R:

$$\begin{aligned}
R = \{&(a_1, a_2, ..., a_n) \in \\
&((O_1^i \times O_2 \times ... \times O_n) \cup (O_1 \times O_2^i \times ... \times O_n) \cup ... \cup (O_1 \times O_2 \times ... \times O_n^i) \cup \\
&(O_1^i \times O_2^i \times ... \times O_n) \cup (O_1^i \times O_2 \times O_3^i \times ... \times O_n) \cup ... \cup (O_1^i \times O_2 \times ... \times O_n^i) \cup \\
&(O_1 \times O_2^i \times O_3^i \times ... \times O_n) \cup (O_1 \times O_2^i \times O_3 \times O_4^i \times ... \times O_n) \cup ... \cup (O_1 \times O_2' \times ... \times O_n^i) \cup \\
&... \cup \\
&(O_1^i \times O_2^i \times O_3^i \times ... \times O_n) \cup (O_1^i \times O_2^i \times O_3 \times O_4^i \times ... \times O_n) \cup \\
&(O_1^i \times O_2^i \times O_3 \times O_4 \times O_5^i \times ... \times O_n) \cup ... \cup (O_1^i \times O_2^i \times O_3 \times ... \times O_n^i) \cup \\
&... \cup \\
&(O_1^i \times O_2^i \times O_3^i \times O_4^i \times O_5^i \times ... \times O_n^i))\}, \\
R \neq \emptyset&.
\end{aligned}$$

If relation R does not include at least one condition that is against the fundamental properties of any object type that is referenced in relation R and at least one condition that is against the fundamental properties of interactions among those object types, such hypothetical object instances that are imagined to exist are called imaginary instances per the subset of their object type;

else if relation R does include at least one condition that is against the fundamental properties of any object type that is referenced in relation R and/or at least one condition that is against the fundamental properties of interactions among those object types, such hypothetical object instances that are imagined to exist are called instances of a non-predefined mathematical object type.

In the light of Lemma 4.4.1, we shall offer the conclusive definition for imaginary objects.

4.5 The Conclusive Definition for Imaginary Objects

Theorem 4.5.1. *Let R be a relation on some arbitrary subsets of real instances of various mathematical object types, $O_1, O_2, ..., O_n$ represent the sets containing all possible instances within those subsets, and n be a positive integer greater than 1 where*

$$R = \{(a_1, a_2, ..., a_n) \in (O_1 \times O_2 \times ... \times O_n)\},$$

such that relation R does not map any object instances: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. Then some hypothetical object instances are imagined to exist to satisfy the conditions in updated relation R:

$$\begin{aligned}
R = \{&(a_1, a_2, ..., a_n) \in \\
&((O_1^i \times O_2 \times ... \times O_n) \cup (O_1 \times O_2^i \times ... \times O_n) \cup ... \cup (O_1 \times O_2 \times ... \times O_n^i) \cup \\
&(O_1^i \times O_2^i \times ... \times O_n) \cup (O_1^i \times O_2 \times O_3^i \times ... \times O_n) \cup ... \cup (O_1^i \times O_2 \times ... \times O_n^i) \cup \\
&(O_1 \times O_2^i \times O_3^i \times ... \times O_n) \cup (O_1 \times O_2^i \times O_3 \times O_4^i \times ... \times O_n) \cup ... \cup (O_1 \times O_2^i \times ... \times O_n^i) \cup \\
&... \cup \\
&(O_1^i \times O_2^i \times O_3^i \times ... \times O_n) \cup (O_1^i \times O_2^i \times O_3 \times O_4^i \times ... \times O_n) \cup \\
&(O_1^i \times O_2^i \times O_3 \times O_4 \times O_5^i \times ... \times O_n) \cup ... \cup (O_1^i \times O_2^i \times O_3 \times ... \times O_n^i) \cup \\
&... \cup \\
&(O_1^i \times O_2^i \times O_3^i \times O_4^i \times O_5^i \times ... \times O_n^i))\}, \\
R \neq \emptyset&.
\end{aligned}$$

Where x is a positive integer and relation R comes to map some hypothetical object instances that are imagined to exist, a_x, among some real object instances:

$$\begin{aligned}
&a_1 \in O_1, a_2 \in O_2, ..., a_x \in O_x^i, ..., a_n \in O_n, \\
&(a_1, a_2, ..., a_x, ..., a_n) \in R,
\end{aligned}$$

if relation R does not include any condition that is against the fundamental properties of any mathematical object type that is referenced in relation R and any condition that is against the fundamental properties of interactions among those object types, such a_x are called imaginary instances per the subset of their object type;

else if relation R does include at least one condition that is against the fundamental properties of any mathematical object type that is referenced in relation R and/or at least one condition that is against the fundamental properties of interactions among those object types, such a_x are called instances of a non-predefined mathematical object type.

Where x and y are positive integers, $x + y \leq n$, and relation R comes to map some hypothetical object instances that are imagined to exist, $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}\}$, among themselves or among themselves and also real object instances:

$$\begin{aligned}
&a_1 \in O_1, a_2 \in O_2, ..., a_x \in O_x^i, a_{x+1} \in O_{x+1}^i, a_{x+2} \in O_{x+2}^i, ..., a_{x+y} \in O_{x+y}^i, ..., a_n \in O_n, \\
&(a_1, a_2, ..., a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}, ..., a_n) \in R,
\end{aligned}$$

if relation R does not include any condition that is against the fundamental properties of any mathematical object type that is referenced in relation R and any condition that is against the fundamental properties of interactions among those object types, *and if each one of the object instances in the set* $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}\}$ *except one of the object instances,* a_{x+t}*, were already identified as imaginary instances of their object types, such that* t *is an arbitrary positive integer and* $t \leq y$*, then such* a_{x+t} *are called imaginary instances per the subset of their object type;*

otherwise if none of the object instances in the set $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}\}$ *were already identified as imaginary instances of their object types, then a tuple containing these object instances,* $[a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}]$*, is called a tuple of imaginary instances per the subset of each corresponding object type;*

otherwise if some of the object instances, $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+w}\}$*, in the set* $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}\}$ *were already identified as imaginary instances of that object types, while some other object instances,* $\{a_{x+w+1}, a_{x+w+2}, a_{x+w+3}, ..., a_{x+y}\}$*, were not already identified as imaginary instances of their object types, such that* w *is an arbitrary positive integer and* $w < y$*, then a tuple containing the object instances,* $[a_{x+w+1}, a_{x+w+2}, a_{x+w+3}, ..., a_{x+y}]$*, is called a tuple of imaginary instances per the subset of each corresponding object type;*

else if relation R *does include at least one condition that is against the fundamental properties of any mathematical object type that is referenced in relation* R *and/or at least one condition that is against the fundamental properties of interactions among those object types,*

and if each one of the object instances in the set $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}\}$ *except one of the object instances,* a_{x+t}*, were already identified as imaginary instances of their object types or instances of some non-predefined mathematical object types, such that* t *is an arbitrary positive integer and* $t \leq y$*, then such* a_{x+t} *are called instances of a non-predefined mathematical object type;*

otherwise if none of the object instances in the set $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}\}$ *were already identified as imaginary instances of their object types or instances of some non-predefined mathematical object types, then a tuple containing these object instances,* $[a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}]$*, is called a tuple of instances of some non-predefined mathematical object types;*

otherwise if some of the object instances, $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+w}\}$*, in the set* $\{a_x, a_{x+1}, a_{x+2}, ..., a_{x+y}\}$ *were already identified as imaginary instances of their object types or instances of some non-predefined mathematical object types, while some other object instances,* $\{a_{x+w+1}, a_{x+w+2}, a_{x+w+3}, ..., a_{x+y}\}$*, were not already identified as imaginary instances of their object types or instances of some non-predefined mathematical object types, such that* w *is an arbitrary positive integer and* $w < y$*, then a tuple containing the object instances,* $[a_{x+w+1}, a_{x+w+2}, a_{x+w+3}, ..., a_{x+y}]$*, is called a tuple of instances of some non-predefined mathematical object types.*

5 Redefining Complex Numbers and Defining Complex Objects

5.1 Observations on the Conventional Definition of Complex Numbers

Revisiting the existing definition of complex numbers as presented in Definition 1.0.3, there are two observations we could make:

i. Since suggested by Caspar Wessel [Nahin (1998)], the form for complex numbers (the $a + bi$ form, composed of a real component and an imaginary component combined by addition) seems to be useful in many practical applications and seems to work well for plotting such numbers on a two-dimensional plane.

ii. Any given number seem indeed to conform with the $a + bi$ form, even if they were expressed in various other initial forms. To exemplify some:

2 could be written as $2 + 0 \times i$,

$5/i + 7$ could be written as $7 - 5 \times i$,

$(2 + 3i)^{(5-4i)}$ could be written as $169\ \sqrt{13}\ e^{4\arctan(3/2)}\ cos(5\arctan(3/2)\ -\ 2\ log(13))\ +\ 169\ i\ \sqrt{13}\ e^{4\arctan(3/2)}\ sin(5\arctan(3/2)\ -\ 2\ log(13))$ following Euler's formula,

expression $(5i - (2^i))!$ could be approximated into its numerical value in the $a + bi$ form using the Gamma function as $(5i - (2^i))! = \Gamma(1 + 5i - 2^i)$.

Since in Sections 3 and 4 we have identified further types of imaginary numbers than the only one that is conventionally recognized, we shall offer an updated definition for complex numbers that simply combines a real number and all possible types of imaginary numbers with addition to each other.

5.2 An Updated Definition for Complex Numbers

Proposition 5.2.1. *Let $a, b_1, b_2, ..., b_n$ be real numbers and $i_1, i_2, ..., i_n$ be imaginary numbers,, then c is defined as a complex number such that*

$$c = a + b_1 i_1 + b_2 i_2 + ... + b_n i_n$$

where n is a positive integer.

5.3 Observations on the Updated Definition of Complex Numbers

Regarding the updated complex number definition (Proposition 5.2.1), it is worth emphasizing that the imaginary components contained, i.e. $i_1, i_2, ..., i_n$ are specified as any imaginary numbers, but not necessarily as the unit imaginary numbers per their types. This is since some types of imaginary numbers will not have unit imaginary numbers by their natures.

Revisiting observation **ii** we made in Section 5.1, we recall that $a+bi$ as the generic form for conventional complex numbers works well because it seems that there is no number that could not be expressed in this form. However, as we now have many more types of imaginary numbers in hand, the amount of possible combinations is enormous. The question we now need to consider is whether we can express any given number in the form of $a + b_1 i_1 + b_2 i_2 + ... + b_n i_n$.

Where h, g, f and d are the unit imaginary numbers, we identified in Section 2.2, it will be straightforward to convert $2i - 5f$ into $0 + 2i - 5f$ and $7 + (4/3) \times h + (5^{1/3}) \times f - h$ into $7 + (1/3)h + (5^{1/3})f$; but how about $5 + i^d \times \sqrt{h} \times g^3 \times f$ as an example, could this be expressed in form $a + b_1 i_1 + b_2 i_2 + ... + b_n i_n$?

The initial reaction to this question could be simply to work out a way of converting $i^d \times \sqrt{h} \times g^3 \times f$ into the form of $b_1 i + b_2 h + b_3 f + b_4 g + b_5 d$ or similar. However, such conversion might not be necessary; as with our new definition of imaginary numbers, we are not really bound by a limited set of imaginary number types. We could as well identify $i^d \times \sqrt{h} \times g^3 \times f$ as an imaginary number itself as in Example 5.3.1.

Example 5.3.1. *Let R_1 be a relation on real numbers such that*

$$R_1 = \{(q, r) \in (\mathbb{R} \times \mathbb{R}) \mid (q \times q = r) \wedge (r < 0)\}.$$

Hence relation R_1 *does not map any numbers:* $R = \emptyset$ *and does not include at least one condition that is against the integrity of any mathematical notation. For the updated relation*

$$R_1 = \{(x, y) \in ((\mathbb{R}^i \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}^i) \cup (\mathbb{R}^i \times \mathbb{R}^i)) \mid (x \times x = y) \wedge (y < 0)\},$$

we could identify these mappings as some of the potential mappings to satisfy the conditions:

$$\begin{aligned} R_1 = \{&(i, -1),\\ &(-i, -1),\\ &...\} \end{aligned}$$

Noting that updated relation R_1 *does not include any condition that is against the fundamental properties of numbers:*
Mapping $(i, -1)$ *is from a hypothetical number imagined to exist,* i*, to a real number,* -1*. Hence,* i *is identified as an imaginary number.*
Mapping $(-i, -1)$ *is from a hypothetical number imagined to exist,* $-i$*, to a real number,* -1*. Hence,* $-i$ *is identified as an imaginary number.*
Let R_2 *be a relation on real numbers such that*

$$R_2 = \{(q, r) \in (\mathbb{R} \times \mathbb{R}) \mid (q \ / \ q = r) \wedge (r \neq 1)\}.$$

Hence relation R_2 *does not map any numbers:* $R = \emptyset$ *and does not include at least one condition that is against the integrity of any mathematical notation. For the updated relation*

$$R_2 = \{(x, y) \in ((\mathbb{R}^i \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}^i) \cup (\mathbb{R}^i \times \mathbb{R}^i)) \mid (x \ / \ x = y) \wedge (y \neq 1)\},$$

we could identify these mappings as some of the potential mappings to satisfy the conditions:

$$\begin{aligned} R_2 = \{&(0, 0 \ / \ 0),\\ &(h, -1),\\ &(g, 2),\\ &(\sqrt{g}, \sqrt{2}),\\ &(g^3, 8),\\ &(i^d \times \sqrt{h} \times g^3, 8),\\ &(\infty, \infty \ / \ \infty),\\ &(1 \ / \ 0, 0 \ / \ 0),\\ &(i^f, i),\\ &(i^d, -i),\\ &(\sqrt{h}, i),\\ &(h \times f, -1),\\ &(h \times d, -1),\\ &(g \times f, 2),\\ &(g \times d, 2),\\ &...\} \end{aligned}$$

Noting that updated relation R_2 *does not include any condition that is against the fundamental properties* *of numbers:*

Mapping $(0, 0 \ / \ 0)$ *is from a real number,* 0*, to a hypothetical number imagined to exist,* $0 \ / \ 0$*. Hence,* $0 \ / \ 0$ *is identified as an imaginary number.*

Mapping $(h, -1)$ *is from a hypothetical number imagined to exist,* h*, to a real number,* -1*. Hence,* h *is identified as an imaginary number.*

Mapping $(g, 2)$ *is from a hypothetical number imagined to exist,* g*, to a real number,* 2*. Hence,* g *is identified as an imaginary number.*

Mapping $(\sqrt{g}, \sqrt{2})$ *is from a hypothetical number imagined to exist,* $\sqrt{g}$*, to a real number,* $\sqrt{2}$*. Hence,* $\sqrt{g}$ *is identified as an imaginary number.*

Mapping $(g^3, 8)$ *is from a hypothetical number imagined to exist,* g^3*, to a real number,* 8*. Hence,* g^3 *is identified as an imaginary number.*

Mapping $(i^d \times \sqrt{h} \times g^3, 8)$ *is from a hypothetical number imagined to exist,* $i^d \times \sqrt{h} \times g^3$*, to a real number,* 8*. Hence,* $i^d \times \sqrt{h} \times g^3$ *is identified as an imaginary number.*

Mapping $(\infty, \infty \ / \ \infty)$ *is from a hypothetical number imagined to exist,* ∞*, to a hypothetical number imagined to exist,* $\infty \ / \ \infty$*. As* ∞ *is already identified as an imaginary number explicitly in Example 3.2.1,* $\infty \ / \ \infty$ *is identified as an imaginary number.*

Mapping $(1 \ / \ 0, 0 \ / \ 0)$ *is from a hypothetical number imagined to exist,* $1 \ / \ 0$*, to a hypothetical number imagined to exist,* $0 \ / \ 0$*. As* $0 \ / \ 0$ *is already identified as an imaginary number explicitly in this very example,* $1 \ / \ 0$ *is identified as an imaginary number.*

Mapping (i^f, i) *is from a hypothetical number imagined to exist,* i^f*, to a hypothetical number imagined to exist,* i*. As* i *is already identified as an imaginary number explicitly in this very example and in Definition 3.1.1,* i^f *is identified as an imaginary number.*

Mapping $(i^d, -i)$ *is from a hypothetical number imagined to exist,* i^d*, to a hypothetical number imagined to exist,* $-i$*. As* $-i$ *is already identified as an imaginary number explicitly in this very example,* i^d *is identified as an imaginary number.*

Mapping $(\sqrt{h}, i)$ *is from a hypothetical number imagined to exist,* $\sqrt{h}$*, to a hypothetical number imagined to exist,* i*. As* i *is already identified as an imaginary number explicitly in this very example and in Definition 3.1.1,* $\sqrt{h}$ *is identified as an imaginary number.*

Mapping $(h \times f, -1)$ *is from a hypothetical number imagined to exist,* $h \times f$*, to a real number,* -1*. Hence,* $h \times f$ *is identified as an imaginary number.*

Mapping $(h \times d, -1)$ *is from a hypothetical number imagined to exist,* $h \times d$*, to a real number,* -1*. Hence,* $h \times d$ *is identified as an imaginary number.*

Mapping $(g \times f, 2)$ *is from a hypothetical number imagined to exist,* $g \times f$*, to a real number,* 2*. Hence,* $g \times f$ *is identified as an imaginary number.*

Mapping $(g \times d, 2)$ *is from a hypothetical number imagined to exist,* $g \times d$*, to a real number,* 2*. Hence,* $g \times d$ *is identified as an imaginary number.*

Let R_3 *be a relation on real numbers such that*

$$R_3 = \{(q, r) \in (\mathbb{R} \times \mathbb{R}) \mid (q - q = r) \wedge (r \neq 0)\}.$$

Hence relation R_3 *does not map any numbers:* $R = \emptyset$ *and does not include at least one condition that is against the integrity of any mathematical notation. For the updated relation*

$$R_3 = \{(x, y) \in ((\mathbb{R}^i \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}^i) \cup (\mathbb{R}^i \times \mathbb{R}^i)) \mid (x - x = y) \wedge (y \neq 0)\},$$

we could identify these mappings as some of the potential mappings to satisfy the conditions:

$$\begin{aligned} R_3 = \{ & (f, 1), \\ & (d, -1), \\ & (\infty, \infty - \infty), \\ & (i^d \times \sqrt{h} \times g^3 \times f, i^d \times \sqrt{h} \times g^3), \\ & ...\} \end{aligned}$$

Noting that updated relation R_2 does not include any condition that is against the fundamental properties of numbers:
Mapping $(f, 1)$ is from a hypothetical number imagined to exist, f, to a real number, 1. Hence, f is identified as an imaginary number.
Mapping $(d, -1)$ is from a hypothetical number imagined to exist, d, to a real number, -1. Hence, d is identified as an imaginary number.
Mapping $(\infty, \infty - \infty)$ is from a hypothetical number imagined to exist, ∞, to a hypothetical number imagined to exist, $\infty - \infty$. As ∞ is already identified as an imaginary number explicitly in Example 3.2.1, $\infty - \infty$ is identified as an imaginary number.
Mapping $(i^d \times \sqrt{h} \times g^3 \times f, i^d \times \sqrt{h} \times g^3)$ is from a hypothetical number imagined to exist, $i^d \times \sqrt{h} \times g^3 \times f$, to a hypothetical number imagined to exist, $i^d \times \sqrt{h} \times g^3$. As $i^d \times \sqrt{h} \times g^3$ is already identified as an imaginary number explicitly in this very example, $i^d \times \sqrt{h} \times g^3 \times f$ is identified as an imaginary number.

So, $5 + i^d \times \sqrt{h} \times g^3 \times f$ is a complex number where 5 is the real component of the complex number and $(i^d \times \sqrt{h} \times g^3 \times f)$ is the only imaginary component.

The updated definition for complex numbers (Proposition 5.2.1) works well as our new definition for complex numbers in the form of $a + b_1 i_1 + b_2 i_2 + ... + b_n i_n$ but this is based on the assumption we are making that each imaginary component i_1, i_2, ... , i_n within $a + b_1 i_1 + b_2 i_2 + ... + b_n i_n$ can be identified explicitly as an imaginary number.

The list of the types of imaginary numbers we exemplified in this paper is not exhaustive by any means and analyzing such vast amount of numbers and imaginary number types falls outside the scope of this paper or any single paper of regular volume. However, we shall state our assumption in our definition for complex numbers and offer this definition as a proposed definition rather than a conclusive one.

5.4 A Proposed Definition for Complex Numbers

Proposition 5.4.1. *Let $a, b_1, b_2, ..., b_n$ be real numbers and $i_1, i_2, ..., i_n$ be imaginary numbers, then c is defined as a complex number such that*

$$c = a + b_1 i_1 + b_2 i_2 + ... + b_n i_n$$

where n is a positive integer and each composing element a, i_1, i_2, ..., i_n is called a component of c.
This definition of c is based on the assumption that each imaginary component i_1, i_2, ... , i_n of c can be identified as an imaginary number through the conclusive definition of imaginary objects.

5.5 Dimensions of Complex Numbers

Conventionally defined complex numbers are plotted on a two dimensional plane with a real axis and an imaginary axis. The conclusive definition we offer, however, takes complex numbers to the multi-dimensional space, instead of just two.

A point to observe, however, is that while some types of imaginary numbers might be setting their own dimensions, some others might be sharing a single dimension with each other.

For instance, i shall have its own dimension, as in the conventional definition of complex numbers. However, we could observe that h and g should share a dimension of their own together, and d and f as well should share a dimension of their own together.

Additionally, we could observe that the imaginary numbers we identified in Section 3.2, ∞, $-\infty$, $1/\infty$, and $-1/\infty$ do not set their own dimensions at all, but they all share the same dimension with real numbers.

So, we need to define a structured methodology to understand which imaginary number types set their own dimensions and which ones share a dimension with each other and with real numbers.

A good test for figuring out such dimensional overlaps is checking which fundamental properties of real numbers a given imaginary number violates. Imaginary numbers violating the same set of properties shall be plotted on the same dimension. Hence, it is straight-forward to observe that h and g together, and d and f together indeed each share one dimension of their own.

Given that ∞, $-\infty$, $1/\infty$, and $-1/\infty$ do not really violate any fundamental properties of real numbers, but rather they are imaginary based on the fact that their magnitudes are unquantifiable, they do share the same dimension as the real numbers.

Additionally, certain imaginary numbers could be further identified to share a dimension with other imaginary numbers. For instance, observing the fact that the value of the imaginary number, $1/0$, approaches ∞ in limit calculations help us determine that $1/0$ should share the same dimension as ∞, hence the same dimension with real numbers. This, in fact, was the underlying observation behind Riemann Sphere [Riemann (1857)] that models an extended complex plane.

5.6 An Initial Definition for Complex Objects

Based on the similar way of thinking behind the form for complex numbers, we shall define complex objects as objects composed of real object components and imaginary object components, combined by addition.

Proposition 5.6.1. *Let $o_1, o_2, ..., o_n$ and $r_1, r_2, ..., r_m$ be real instances of some object types, $i_1, i_2, ..., i_m$ be imaginary instances of some object types, then c is defined as an instance of a complex object such that*

$$c = o_1 + o_2 + ... + o_n + r_1 i_1 + r_2 i_2 + ... + r_m i_m$$

where n and m are positive integers.

5.7 Observations on the Initial Definition of Complex Objects

There are certain observations we could make about the initial definition of complex numbers (Proposition 5.6.1):

Firstly, this initial definition disregards the fact that each imaginary component could be composed of multiple imaginary objects instead of just one. However, the main shortcoming of the initial definition seem to lie under the fact that not all mathematical object types could be combined by addition and multiplication.

Addition and multiplication of numbers could work for composing complex numbers; e.g. c is a complex number such that

$$c = a + b_1 i_1 + b_2 i_2 + ... + b_n i_n,$$

where $a, b_1, b_2, ..., b_n$ are real numbers and $i_1, i_2, ..., i_n$ are imaginary numbers.

However the same will not work for many other types of mathematical objects. Matrices for instance, can only be added onto and multiplied by each other if they are of compatible sizes; so we could think that M is a complex matrix such that

$$M = A + B_1 T_1 + B_2 T_2 + ... + B_n T_n,$$

where $A, B_1, B_2, ..., B_n$ are real matrices and $T_1, T_2, ..., T_n$ are imaginary matrices that are of compatible sizes to be added and multiplied together.

Sets, on the other hand, can be joined together with the union operation, not addition; so we could think that S is a complex set such that

$$S = p \cup t_1 \cup t_2 \cup ... \cup t_n,$$

where p is a real set and $t_1, t_2, ..., t_n$ are imaginary sets.

To name more operational limitations: Numbers could be multiplied among each other and they could be multiplied with the instances of some predefined and non-predefined other mathematical object types. Matrices, could be multiplied by numbers and they could only be multiplied by each other if their sizes are compatible. Sets, on the other hand, could not be multiplied with another object.

As our philosophy is based on not limiting human imagination, one could as well imagine such additions and multiplications that we consider not possible. The resulting object instance from such operations would be of a non-predefined type but it is possible to imagine such cases nonetheless. However, such cases would not be limited to just addition and multiplication. For instance, one could as well think of a case where c is a complex object such that $c = S^M$ where S is a set and M is a matrix. Again, since the result of such exponential operation is undefined, the resulting object instance, c, would be of a non-predefined type.

Our conclusion from our observations is that, in construction of complex objects, we need to allow any type of interaction among various types of objects. We then just need to judge if the resulting object is of a predefined type or not. Lemma 5.7.1 presents this conclusion.

Lemma 5.7.1. *Let f be a function on real and imaginary instances of arbitrary mathematical object types, n be a positive integer, and C be the resulting mathematical object instance:*

$$C = f(O_1, O_2, ..., O_n).$$

Where function f does not contain any operation against the fundamental properties of the object types involved and does not include any operation against the fundamental properties of interactions among *those involved objects types, C is of a predefined object type; otherwise C is of a non-predefined object type.*

In the light of Lemma 5.7.1, we shall offer a more encapsulating definition for complex objects in Proposition 5.8.1.

5.8 A Proposed Definition for Complex Objects

Proposition 5.8.1. *Let f be a function on real and imaginary instances of arbitrary mathematical object types and n be a positive integer such that the resulting mathematical object instance is C:*

$$C = f(O_1, O_2, ..., O_n).$$

If function f does not contain any operation against the fundamental properties of the object types involved and does not include any operation against the fundamental properties of interactions among those involved objects types, C is a complex object of a predefined object type;
else C is a complex object of a non-predefined object type.

6 Conclusions and Further Directions

The only one type of imaginary numbers that is conventionally recognized, is not the only one that could possibly be imagined. There are further types of numbers not existing in reality but could be imagined to exist. Some types of such imaginary numbers are exemplified here, and many more types could be identified based on the conclusive definition we offer.

Imagination is not a concept limited to numbers, instances of other types of mathematical objects could also be imagined. Additionally, certain mathematical objects that could be imagined would not fit in any predefined type of mathematical objects, hence they would be identified as instances of some non-predefined types of mathematical objects.

While some types of imaginary numbers and imaginary object instances could be identified explicitly, some others could only be identified collectively in dependence to each other. Collectively identified imaginary instances are best expressed in tuples.

In the light of our conclusive definition for imaginary objects, we come to propose new definitions for complex numbers and we also define complex objects in order to encapsulate all various types of objects. While complex numbers conventionally are visualized in a two-dimensional plane, complex numbers and complex objects can now be visualized in a multiple dimensional space.

Our proposed definitions for imaginary and complex numbers and objects has the potential to help better formulate certain mathematical concepts and better standardize certain mathematical operations. For instance, many values and expressions currently recognized as undefined could be handled in a more structured way under the context of imaginary numbers.

Our arguments in this paper could open a new course for imaginary and/or abstract mathematics. They pose the potential for a new, restructured abstract mathematics as a field.

References

Baez, J. C. (2002). The octonions. *Bulletin of the American Mathematical Society*, **39**(2), 145–205.

Hamilton, W. R. (1844). On quaternions, or on a new system of imaginaries in algebra. *Philosophical Magazine*, **25**(3), 489–495.

Nahin, P. J. (1998). *An Imaginary Tale.* Princeton University Press, New Jersey.

Riemann, B. (1857). Theorie der abel'sche funktionen. *Journal fur die reine und angewandte Mathematik (Crelle)*, **54**, 101–155.

A Noting Indeterminate Forms and Undefined Statements

We shall devote this section to examining indeterminate forms and undefined values as imaginary numbers. We will explore identifying the commonly recognized indeterminate forms, $0/0$, ∞/∞, $0 \times \infty$, 1^∞, $\infty - \infty$, 0^0, and ∞^0 as imaginary numbers. Additionally, we will explore identifying $1/0$ as an imaginary number since division of a real number by zero is one of the most commonly referenced undefined values. Throughout our analysis here, we will be making use of the infinity imaginary numbers $\infty, -\infty, 1/\ \infty, -1/\ \infty$ that we already identified as imaginary numbers.

Let us consider raising of a number to the zero power, division of a number by itself, multiplication of a number by zero, subtraction of a number from itself, and raising of a number to the power of infinity through Examples A.0.1 - A.0.5:

Example A.0.1. *Let R be a relation on real numbers such that*

$$R = \{(q, r) \in (\mathbb{R} \times \mathbb{R}) \mid (q^0 = r) \wedge (r \neq 1)\}.$$

Hence relation R does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. For the updated relation

$$R = \{(x, y) \in ((\mathbb{R}' \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}') \cup (\mathbb{R}' \times \mathbb{R}')) \mid (x^0 = y) \wedge (y \neq 1)\},$$

we could identify these mappings as some of the potential mappings to satisfy the conditions:

$$\begin{aligned} R = \{&(0, 0^0), \\ &(\infty, \infty^0), \\ &...\} \end{aligned}$$

Noting that updated relation R does not include any condition that is against the fundamental properties of numbers:
Mapping $(0, 0^0)$ is from a real number, 0, to a hypothetical number imagined to exist, 0^0. Hence, 0^0 is identified as an imaginary number.
Mapping (∞, ∞^0) is from a hypothetical number imagined to exist, ∞, to a hypothetical number imagined to exist, ∞^0. As ∞ is already identified as an imaginary number explicitly in Example 3.2.1, ∞^0 is identified as an imaginary number.

Example A.0.2. *Let R be a relation on real numbers such that*

$$R = \{(q, r) \in (\mathbb{R} \times \mathbb{R}) \mid (q\ /\ q = r) \wedge (r \neq 1)\}.$$

Hence relation R does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. For the updated relation

$$R = \{(x, y) \in ((\mathbb{R}' \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}') \cup (\mathbb{R}' \times \mathbb{R}')) \mid (x\ /\ x = y) \wedge (y \neq 1)\},$$

we could identify these mappings as some of the potential mappings to satisfy the conditions:

$$\begin{aligned} R = \{&(0, 0\ /\ 0), \\ &(\infty, \infty\ /\ \infty), \\ &...\} \end{aligned}$$

Noting that updated relation R does not include any condition that is against the fundamental properties of numbers:
Mapping $(0, 0\ /\ 0)$ *is from a real number,* 0*, to a hypothetical number imagined to exist,* $0\ /\ 0$*. Hence,* $0\ /\ 0$ *is identified as an imaginary number.*
Mapping $(\infty, \infty\ /\ \infty)$ *is from a hypothetical number imagined to exist,* ∞*, to a hypothetical number imagined to exist,* $\infty\ /\ \infty$*. As* ∞ *is already identified as an imaginary number explicitly in Example 3.2.1,* $\infty\ /\ \infty$ *is identified as an imaginary number.*

Example A.0.3. *Let R be a relation on real numbers such that*

$$R = \{(q, r) \in (\mathbb{R} \times \mathbb{R}) \mid (q \times 0 = r) \wedge (r \neq 0)\}.$$

Hence relation R does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. For the updated relation

$$R = \{(x, y) \in ((\mathbb{R}' \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}') \cup (\mathbb{R}' \times \mathbb{R}')) \mid (x \times 0 = y) \wedge (y \neq 0)\},$$

we could identify these mappings as some of the potential mappings to satisfy the conditions:

$$\begin{aligned} R = \{&(\infty, \infty \times 0), \\ &(1\ /\ 0, 0\ /\ 0), \\ &...\} \end{aligned}$$

Noting that updated relation R does not include any condition that is against the fundamental properties of numbers:
Mapping $(\infty, \infty \times 0)$ *is from a hypothetical number imagined to exist,* ∞*, to a hypothetical number imagined to exist,* $\infty \times 0$*. As* ∞ *is already identified as an imaginary number explicitly in Example 3.2.1,* $\infty \times 0$ *is identified as an imaginary number.*
Mapping $(1\ /\ 0, 0\ /\ 0)$ *is from a hypothetical number imagined to exist,* $1\ /\ 0$*, to a hypothetical number imagined to exist,* $0\ /\ 0$*. As* $0\ /\ 0$ *is already identified as an imaginary number explicitly in Example A.0.2,* $1\ /\ 0$ *is identified as an imaginary number.*

Example A.0.4. *Let R be a relation on real numbers such that*

$$R = \{(q, r) \in (\mathbb{R} \times \mathbb{R}) \mid (q - q = r) \wedge (r \neq 0)\}.$$

Hence relation R does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. For the updated relation

$$R = \{(x, y) \in ((\mathbb{R}' \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}') \cup (\mathbb{R}' \times \mathbb{R}')) \mid (x - x = y) \wedge (y \neq 0)\},$$

we could identify this mapping as one of the potential mappings to satisfy the conditions:

$$\begin{aligned} R = \{&(\infty, \infty - \infty), \\ &...\} \end{aligned}$$

Noting that updated relation R does not include any condition that is against the fundamental properties of numbers:
Mapping $(\infty, \infty - \infty)$ *is from a hypothetical number imagined to exist,* ∞*, to a hypothetical number imagined to exist,* $\infty - \infty$*. As* ∞ *is already identified as an imaginary number explicitly in Example 3.2.1,* $\infty - \infty$ *is identified as an imaginary number.*

Example A.0.5. *Let R be a relation on real numbers such that*

$$R = \{(q, r) \in (\mathbb{R} \times \mathbb{R}) \mid (q^{\infty} = r) \wedge (r \neq \infty)\}.$$

Hence relation R does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. For the updated relation

$$R = \{(x, y) \in ((\mathbb{R}' \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}') \cup (\mathbb{R}' \times \mathbb{R}')) \mid (x^{\infty} = y) \wedge (y \neq \infty)\},$$

we could identify this mapping as one of the potential mappings to satisfy the conditions:

$$R = \{(1, 1^{\infty}), \\ ...\}$$

Noting that updated relation R does not include any condition that is against the fundamental properties of numbers:
Mapping $(1, 1^{\infty})$ is from a real number, 1, to a hypothetical number imagined to exist, 1^{∞}. Hence 1^{∞} is identified as an imaginary number.

Through Examples A.0.1 - A.0.5, we exemplified cases that help us identify all of the referenced indeterminate forms $0/0, \infty/\infty, 0 \times \infty, 1^{\infty}, \infty - \infty, 0^0, \infty^0$ and the undefined value $1/0$ as imaginary numbers.

However, we note that we could also identify further forms as imaginary numbers. To exemplify one such case, let us use the condition from Example A.0.1 again in Example A.0.6.

Example A.0.6. *Let R be a relation on real numbers such that*

$$R = \{(q, r) \in (\mathbb{R} \times \mathbb{R}) \mid (q^0 = r) \wedge (r \neq 1)\}.$$

Hence relation R does not map any numbers: $R = \emptyset$ and does not include at least one condition that is against the integrity of any mathematical notation. For the updated relation

$$R = \{(x, y) \in ((\mathbb{R}' \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}') \cup (\mathbb{R}' \times \mathbb{R}')) \mid (x^0 = y) \wedge (y \neq 1)\},$$

we could identify this mappings as one of the potential mappings to satisfy the conditions:

$$R = \{(1^{\infty}, 1^{\infty \times 0}), \\ ...\}$$

Noting that updated relation R does not include any condition that is against the fundamental properties of numbers:
Mapping $(1^{\infty}, 1^{\infty \times 0})$ is from a hypothetical number imagined to exist, 1^{∞}, to a hypothetical number imagined to exist, $1^{\infty \times 0}$. As 1^{∞} is already identified as an imaginary number explicitly in Example A.0.5, $1^{\infty \times 0}$ is identified as an imaginary number.

B Consequences of Imaginary Number Identifications on Mathematical Operations

Imaginary number identifications do not only lead us to define the boundaries of mathematical object types by specifying the fundamental properties of these object types, but they also lead us to think of mathematical operations in different ways, hence certain operational rules never considered necessary before might now be a necessity due to the fact that imaginary instances behave differently than real instances.

To exemplify this, let us have a look at the operation x^y/x^y where x and y are real numbers. Such operation would normally be converted into 1 either by the cancellation of the dividend and the divisor or by the actual division operation:

$$x^y/\ x^y \ = \ 1.$$

However the same operation becomes ambiguous when it comes to working out the result of x^f/x^f, where f is the unit imaginary number defined in Example 2.2.3. If fraction cancellation has the precedence over actual division operation, than this expression would convert into the result of 1:

$$x^f/\ x^f \ = \ 1.$$

However if the division operation has the precedence over fraction cancellation, then we would get a different result:

$$x^f/\ x^f = x^{f-f} = x^1 = x.$$

This difference is due to the fact that, while working with real numbers only, a precedence rule between fraction cancellation and division operation would not be necessary. However, the introduction of new types of imaginary numbers, violating the fundamental properties of real numbers, would make such rules a necessity.