Disproof of the Riemann hypothesis

Igor Hrnčić
Ludbreška 1b
42000 Varaždin
Croatia
ihrncic1@yahoo.com

Abstract

This paper disproves the Riemann hypothesis by analyzing the integral representation of the Riemann zeta function that converges absolutely in the root-free region. The analysis is performed upon the well known inverse Mellin transform of zeta, that defines the Mertens function. The contour of integration is taken arbitrarily close to the nontrivial roots, and then only arbitrarily small parts of the integrand that are infinitely close to the nontrivial roots on such contour are analyzed. The convergence of the integral at hand then implies a result that a series over the derivative of zeta and over nontrivial roots closest to the roots free region converges. This result is in a contradiction with the well known result that the very same series, when taken over the critical line and under the truth of the Riemann hypothesis, diverges. This disproves the Riemann hypothesis.

1 Introduction

The starting point is the definition of the Riemann zeta function $\zeta(s)$ by the use of the Euler product over all primes p, as well as by the use of the classical Möbius function $\mu(n)$:

$$\frac{1}{\zeta(s)} = \prod_{p=2}^{\infty} 1 - \frac{1}{p^s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \qquad , \Re(s) > 1 \tag{1}$$

The series in eq. (1) converges absolutely on $\Re(s) > 1$. However, it converges only conditionally on $\Re(s) \leq 1$.

On the other hand, the series $\sum_{n=1}^{\infty} \mu(n)/n^s$ from eq. (1) can be rewritten as the Riemann-Stieltjes integral:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = s \int_{1}^{\infty} \frac{M(a)}{a^{s+1}} da \qquad , \Re(s) > R \tag{2}$$

Here, in eq. (2), $M(a)$ is the Mertens function $M(a) = \sum_{n=1}^{a} \mu(n)$, and R is the largest real part of zeta roots: $R = \max\{\Re(\rho) : \zeta(\rho) = 0\}$.

The interesting feature of the integral in eq. (2) is that it converges absolutely on $\Re(s) > R$ as soon as it converges, or in other words, as soon as the Mertens function behaves asymptotically as $M(a) = O(a^{R+\delta})$ for every $\delta > 0$. Since integral in eq. (2) converges absolutely in the root-free region of the critical strip, it represents an analytic function in that region. And so, by the uniqueness of the analytic continuation, we find that eq. (2) stands true on the half-plane $\Re(s) > R$.

Since integral in eq. (2) converges absolutely on $\Re(s) > R$, it can be Mellin inverted in that region:

$$M(a) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{a^s}{s\zeta(s)} ds \qquad , \Re(s) > R \tag{3}$$

Here, b is any real number from the region $\Re(s) > R$ of course.

Since we know that the Mellin inverse (3) exists, the theory of Mellin transforms tells us that the function

$$\frac{1}{s\zeta(s)} \qquad , \Re(s) > R \tag{4}$$

must be absolutely integrable along any line parallel to the imaginary axis on its region of convergence $\Re(s) > R$.

The results listed so far are all very well known. We need two more well known results. Namely, we shall make an implicit use of the Prime Number Theorem: *zeta has no roots on the line* $\Re(s) = 1$.

Finally, we shall also use the fact [1, ch.14.27, p.374] that, under the truth of the Riemann hypothesis, and under the assumption that all nontrivial roots are simple, the series $\sum_\rho 1/|\rho\zeta'(\rho)|$, summed over all nontrivial zeta roots ρ, all located on the critical line $\Re(s) = 1/2$, diverges.

2 A short sketch of this disproof

One considers the Mellin transform

$$M(a) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{a^s}{s\zeta(s)} ds \qquad , \Re(s) > R \tag{5}$$

As with all Mellin transforms, it's of the form

$$f(a) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} a^s F(s) ds \qquad , \Re(s) > R \tag{6}$$

We know from the theory of Mellin transforms that $F(s)$ is absolutely integrable along the contour of integration on the fundamental strip $\Re(s) > R$. Hence, $1/s\zeta(s)$ is absolutely integrable along the contour of integration on $\Re(s) > R$. This condition of absolute integrability of $1/s\zeta(s)$ reads

$$-i \int_{b-i\infty}^{b+i\infty} \frac{ds}{|s||\zeta(s)|} < \infty \tag{7}$$

We now define the contour of integration to be arbitrarily close to the nontrivial zeta root. And then we only pay attention to arbitrarily small parts of the contour of integration that are next to nontrivial roots. These arbitrarily small parts of the contour of integration are in an ε-neighborhood of nontrivial roots, and hence we know the zeta function along such parts of the contour of integration behaves as

$$\zeta(\rho+\varepsilon) = \zeta^{(n)}(\rho)\varepsilon^n \qquad , 0 < \zeta^{(n)}(\rho) < \infty \tag{8}$$

This simplifies calculations considerably. So, instead of calculating the entire absolutely convergent integral along the entire contour of integration, we just compute the parts of the integral that are in the ε-neighborhoods of nontrivial zeta roots, not really aiming at computing the entire integral. Since the entire integral converges absolutely along the entire contour of integration, we find that the part along the contour lying in ε-neighborhoods of nontrivial zeta roots must converge as well. However, since $\zeta(\rho+\varepsilon) = \zeta^{(n)}(\rho)\varepsilon^n$, the result depends on ε. The arbitrarily small quantity ε is a free parameter, it has no fixed magnitude. Hence, the partial integral could grow arbitrarily large if it was dominated by the $1/\varepsilon$ term. Hence, the result cannot depend on $1/\varepsilon$. This demonstrates that all of the nontrivial zeta roots closest to the line $\Re(s) = 1$ are simple. Finally, we arrive at the fact that the partial absolutely convergent integral of $1/s\zeta(s)$ is proportional to the series $\sum 1/|\rho\zeta'(\rho)|$ that runs over all nontrivial roots ρ that are closest to the root-free region. Hence, since the whole absolutely convergent integral converges, so does its part. This means that $\sum 1/|\rho\zeta'(\rho)|$, taken over all nontrivial roots closest to the root free region, converges. However, the series $\sum 1/|\rho\zeta'(\rho)|$ over all nontrivial zeta roots diverges. Therefore, not all nontrivial zeta roots ρ are located on a single line, some roots closest to the root free region must be off critical line. This disproves the Riemann hypothesis then, since the Riemann hypothesis states that all the nontrivial zeta roots are located on a single line $\Re(s) = 1/2$.

3 Analysis

3.1 Choosing the integration contour

Authors usually start by employing the residue theorem upon the Mellin transform defining the Mertens function 3 or 5, thus forming series over all trivial and nontrivial roots of zeta. In this paper we shall analyze the very same Mellin transform strictly on the contour of integration, not departing away from it. This way, similar sums over nontrivial roots shall be deduced, but this time not necessarily over all the nontrivial zeta roots. Instead, we are only interested in nontrivial roots that are closest to the root free region $\Re(s) \geq 1$.

We start the analysis by defining $b = R + \varepsilon$ in eqs. (3), (4), (5) and (7). In other words, we shift the contour of integration in eqs. (3), (4), (5) and (7) as close to the zeta roots as desired, with ε being an arbitrarily small strictly positive real number, as depicted in Figure 1.

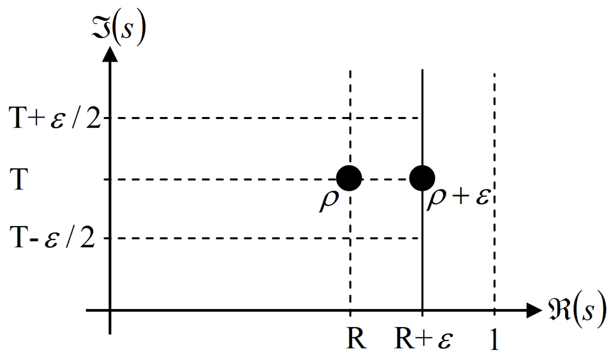

Figure 1: *Root* $\rho = R + iT$ *and the integration contour* $\Re(s) = b = R + \varepsilon$

Since function $1/s\zeta(s)$ from eq. (4) being absolutely integrable on $\Re(s) > R$ along any line parallel to the imaginary axis, we pay attention to this absolutely convergent integral:

$$-i \int_{R+\varepsilon-i\infty}^{R+\varepsilon+i\infty} \frac{ds}{|s||\zeta(s)|} = \int_{-\infty}^{+\infty} \frac{d\tau}{|R+\varepsilon+i\tau||\zeta(R+\varepsilon+i\tau)|} < \infty \tag{9}$$

Next, rewrite the last integral of eq. (9) in the form of its Riemannean sum:

$$\lim_{N\to\infty} \sum_{k=-N}^{k=N} \frac{\Delta\tau_k}{|R+\varepsilon+i\tau_k||\zeta(R+\varepsilon+i\tau_k)|} < \infty \tag{10}$$

We notice here that all the summands in the Riemannean sum (10) are positive.

3.2 One small part of the contour near a root

Next, consider only the small part of the contour of integration in the ε-neighborhood of an arbitrary zeta root $\rho = R + iT$, not the entire contour of integration $\Re(s) = b$, as depicted in Figure 1. In other words, consider just one summand of the Riemannean sum (10), the one evaluated at the point of the contour of integration $\rho+\varepsilon$ closest to the zeta root $\rho = R + i\tau_k$, letting two arbitrarily small quantities $\Delta\tau_k$ and ε being of the same magnitude, $\Delta\tau_k = \varepsilon$, without the loss of generality because eqs. (9) and (10) hold for any arbitrary sufficiently small $\Delta\tau_k$ and ε, so it must hold for $\Delta\tau_k = \varepsilon$ as well:

$$\frac{\varepsilon}{|\rho+\varepsilon||\zeta(\rho+\varepsilon)|} \tag{11}$$

There's a zeta root at ρ by assumption. So, zeta behaves on the contour as $\zeta(\rho+\varepsilon) = \zeta^{(n)}(\rho)\varepsilon^n$, with $\zeta^{(n)}(\rho) \neq 0$ and with $n \in \mathbb{N}$ being the order of the root ρ. Hence, the term (11) reads

$$\frac{\varepsilon^{1-n}}{|\rho\zeta^{(n)}(\rho)|} \tag{12}$$

We have neglected the arbitrarily small quantity ε in $|\rho+\varepsilon|$ because the impact of ε is arbitrarily small and therefore negligible in it. Namely, for arbitrarily small ε, one finds by employing the Taylor series: $1/(\rho+\varepsilon) \approx 1/\rho - \varepsilon/\rho^2 \approx 1/\rho$ as soon as $|\rho|$ is sufficiently large, say, $|\rho| > 1$, which is certainly true for nontrivial roots ρ, since the first nontrivial roots appear approximately at $1/2 \pm i14$. Hence, the error in replacing $1/|\rho+\varepsilon|$ by $1/|\rho|$ is of the lesser order than the result.

3.3 Nearby roots are simple

If $n \neq 1$ in eq. (12), but instead $n \geq 2$, then the term (12) can be arbitrarily large, because ε is arbitrarily small. However, (12) cannot be arbitrarily large, because then the integrals (3) and (10) would be arbitrarily large. However, the integral (10) converges to a value that is not arbitrarily large, and it consists of strictly positive Riemannean summands, so no other part of the Riemannean sum could possibly cancel the arbitrarily large part out, to make the sum bounded. Therefore, one concludes $n = 1$, and hence all the nontrivial zeta roots closest to the line $\Re(s) = 1$ are simple.

3.4 All small parts near roots on contour

Thus, with $n = 1$, the term (12) becomes

$$\frac{1}{|\rho\zeta'(\rho)|} \tag{13}$$

This analysis holds for any nontrivial zeta root ρ_R with $\Re(\rho_R) = R$. Hence, the sum

$$\sum_{\Re(\rho)=R} \frac{1}{|\rho\zeta'(\rho)|} < \infty \tag{14}$$

must converge, because this sum is a part of the Riemannean sum given by the convergent integral (10) of positive terms.

4 Disproof of the Riemann hypothesis

We now make use of the result [1, ch.14.27, p.374], already referred to in Introduction, that, under the truth of the Riemann hypothesis, the series $\sum_\rho 1/|\rho\zeta'(\rho)|$, summed over all nontrivial zeta roots ρ, all located on the critical line $\Re(s) = 1/2$, diverges.

So, let us assume the Riemann hypothesis holds true. Then, all the nontrivial roots are located on the critical line $R = \Re(s) = 1/2$, and the series $\sum_{\Re(\rho)=R=1/2} 1/|\rho\zeta'(\rho)|$ diverges. However, by eq. 14, this cannot be so. Hence, the original assumption about the truth of the Riemann hypothesis does not stand true. There are roots off critical line with $\Re(\rho) = R > 1/2$, such that the sum $\sum_{\Re(\rho)=R} 1/|\rho\zeta'(\rho)|$ converges.

This disproves the Riemann hypothesis.

5 Conclusion

This paper disproves the Riemann hypothesis by analyzing the integral representation of the Riemann zeta function that converges absolutely in the root-free region. The analysis is performed upon the well known inverse Mellin transform of zeta, that defines the Mertens function. The contour of integration is taken arbitrarily close to the nontrivial roots, and then only arbitrarily small parts of the integrand that are infinitely close to the nontrivial roots on such contour are analyzed. The convergence of the integral at hand then implies a result that a series over the derivative of zeta and over nontrivial roots closest to the roots free region converges. This result is in a contradiction with the well known result that the very same series, when taken over the critical line and under the truth of the Riemann hypothesis, diverges. This disproves the Riemann hypothesis.

References

[1] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Second Edition; Revised by D. R. Heath-Brown. Oxford University Press, Oxford, 1986.