On the division of planar graphs in consecutive prime parts

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"Entia non sunt multiplicanda praeter necessitatem" (Ockam, W.)

Abstract

In this paper it is discussed the following problem: “A mathematician wants to divide is garden into consecutive prime parts (first in two parts, after in three parts, and so on), only making straight paths, in a simple way (without retracing his own steps), and without going out of his plot of land. In how many parts can the mathematician divide his garden? ”

Keywords. *Planar graph, prime number, vertex, edge, simple path.*

1 Problem solution

The garden can be mathematically described as a planar graph, and the gardener path as a simple path over some graph $G = (V, E)$ inscribed in the planar graph, with each vertex on the planar graph perimeter and straight edges joining the vertices. In how many parts can a planar graph be divided into consecutive prime parts drawing some graph $G = (V, E)$ through a simple path inscribed in the planar graph?

Theorem. *A convex planar graph divided consecutively in prime parts with a simple path inscribed in its perimeter, can at most be divided in seven parts.*

Proof.

Let us call the starting vertex of the graph over which the simple path goesv_0.

To divide some convex planar graph in two parts (the first prime number), v_0 must be at some point of the planar graph's perimeter, and it must be done a path from v_0 to another point v_1 of the planar graph's perimeter, such that $v_0 \neq v_1$. The path goes over the edge $e_1 = (v_0, v_1)$.

Then, starting from v_1, the path must divide the planar graph in three parts (the second prime number), going over some edge $e_2 = (v_1, v_2)$, where v_2 is whichever point of the perimeter such that $v_2 \neq v_1 \neq v_0$.

Note that, at this point, the planar graph's perimeter is divided by the vertices in three different sections, which can be denominated (with the vertices as subindexes) as $s_{0,1}$, $s_{0,2}$, and $s_{1,2}$.

After that, starting from v_2, the path must divide the planar graph in five parts (the third prime number), going over some edge $e_3 = (v_2, v_3)$. We can easily see that, for the planar graph to be divided in five parts, e_3 must cross over e_1 at some cross point c_1; therefore, $v_3 \subset s_{0,1}$, and it transforms $s_{0,1}$ in two different sections, $s_{0,3}$ and $s_{3,1}$. The cross point c_1 divides both e_1 and e_3 in two subedges, which can be denominated with the vertices as subindexes.

From this point v_3, the path must divide the planar graph in seven parts (the fourth prime number), going over some edge $e_4 = (v_3, v_4)$. This can only be achieved if e_4 crosses over the subedge e_{v_0,c_1} at some cross point c_2, so therefore $v_4 \subset s_{0,2}$, and it transforms $s_{0,2}$ in two different sections, $s_{0,4}$ and $s_{4,2}$.

Finally, the path must divide the planar graph in eleven parts (the fifth prime number), going over some edge $e_5 = (v_4, v_5)$. However, this cannot be achieved. To divide the planar graph in eleven parts, the edge e_5 must cross over three subedges, and that could only be done if $v_4 \subset s_{0,4}$ and $v_5 \subset s_{1,2}$. As $v_4 \notin s_{0,4}$, we can not continue dividing the planar graph consecutively in prime parts with a simple path inscribed in its perimeter, and the theorem is demonstrated.

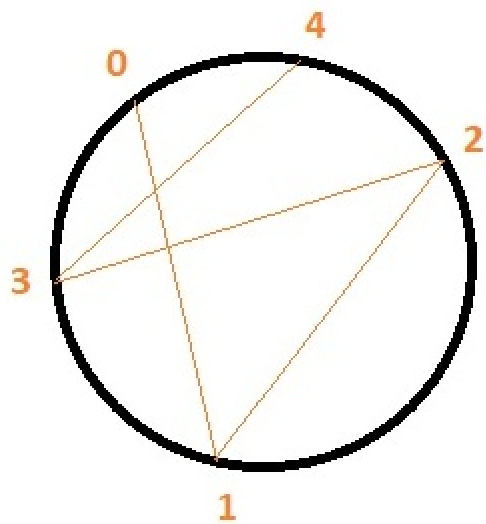

Figure 1:

A circle is a convex planar graph of infinite vertex, and thus can only be divided consecutively in prime parts until it is divided in seven parts.

Note that, if the planar graph is concave (at least has one mouth), the mouth can be used as a tangency point for edge e_5, and then divide the planar graph in elevent parts. An example is showed in Figure 2. In fact, as showed in Figure 3, it seems that the more mouths a concave planar graph has, the more consecutive divisions in prime parts we can achieve.

It is not the purpose of this paper to study this relationship, but we point it as interesting for further investigation.

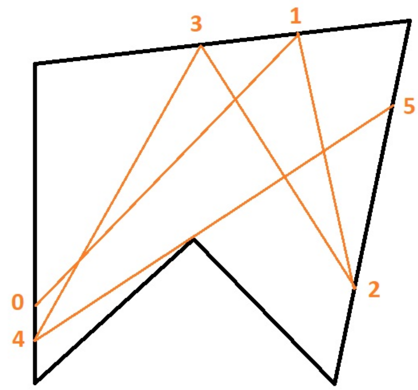

Figure 2:

This concave planar graph has one mouth, and that mouth can be used to trace a path from 4 to 5 that achieves its division from seven parts to eleven parts.

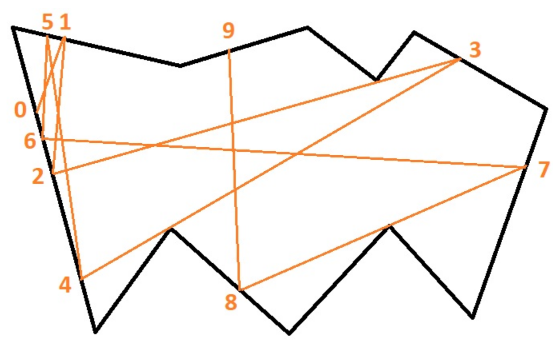

Figure 3:

This concave planar graph has four mouths, and properly used, they can be employed to trace paths that achieve further divisions in consecutive prime parts. In this case, the planar graph has been divided consecutively until twenty three parts.