Riemann Zeta function – Constants, approximations, and some related functions

Pedro Caceres
Longmeadow, MA- February 22th, 2018
Email: pcaceres@comcast.net
Phone: +1 (763) 412-8915

Abstract:

The Riemann zeta function or Euler–Riemann zeta function, ζ(s), is a function of a complex variable z that analytically continues the sum of the Dirichlet series:

$$\zeta(z) = \sum_{k=1}^{\infty} k^{-z}$$

The Riemann zeta function is a meromorphic function on the whole complex z-plane, which is holomorphic everywhere except for a simple pole at z = 1 with residue 1.

One of the most important advance in the study of Prime numbers was the paper by Bernhard Riemann in November 1859 called "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" (On the number of primes less than a given quantity). In this paper, Riemann gave a formula for the number of primes less than x in terms the integral of 1/log(x), and also provided insights into the roots (zeros) of the zeta function, formulating a conjecture about the location of the zeros of $\zeta(z)$ in the critical line Re(z)=1/2.

The Riemann Zeta function is one of the most studied and well known mathematical functions in history. In this paper, we will formulate new propositions to advance in the knowledge of the Riemann Zeta function.

Proposition Page

1. Functions used in this paper

1.1. Generalized Harmonic Function $H_n^{(k)}$:

$$H_n^{(k)} = \sum_{j=1}^{n} j^{-k} = \left(\frac{1}{1^k} + \frac{1}{2^k} + \cdots + \frac{1}{n^k}\right) \qquad \text{converges for } k > 1$$

1.2. Gamma Function $\Gamma(n+1)$:

$$\Gamma(n+1) = n!$$

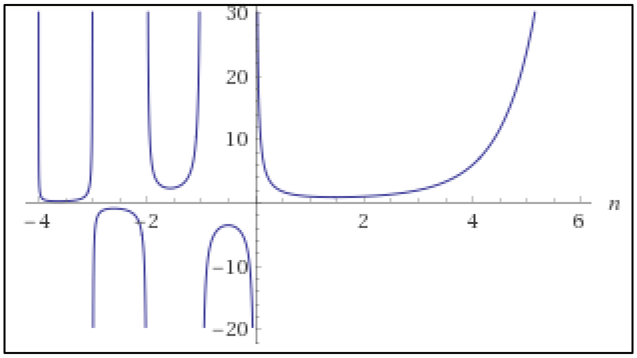

Figure 1. Gamma function in R

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^{z}}{1+\frac{z}{n}}$$

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi)}$$

If z is positive even integer, then:

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi}$$

Integral representation:

$$\Gamma(z) = \int_0^{\infty} e^{-x} x^{z} \frac{dx}{x}$$

The derivative of the Gamma function can be given by:

$$\Gamma'(n+1) = n!\,(-\gamma + \sum_{k=1}^{n}\frac{1}{k})$$

Where γ, is the Euler-Mascheroni constant.

1.3. K function K(n):

$$K(n) = \prod_{j=1}^{n-1} j^{j} = 1^1 \times 2^2 \times \ldots \times 1(n-1)^{(n-1)}$$

1.4. Hyperfactorial function H_n:

$$H_n = K(n+1)$$

1.5. Digamma function $\psi^{(0)}(z)$:

$$\psi^{(0)}(z) = \frac{d}{dz}\ln\big(\Gamma(z)\big) = \frac{\Gamma'(z)}{\Gamma(z)}$$

For integers n:

$\psi^{(0)}(n) = -\gamma + H_{n-1}$ where γ is the Euler-Mascheroni constant

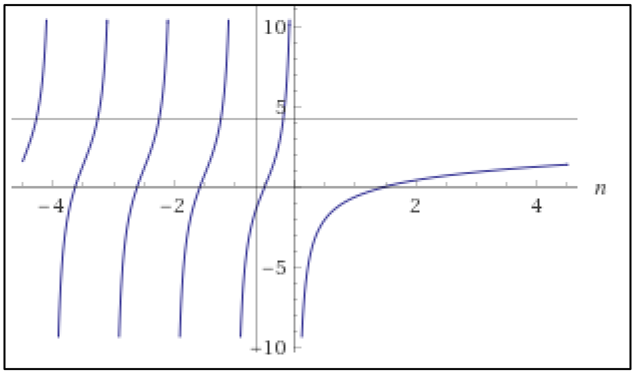

Figure 2. Digamma function in R

1.6. Polygamma function $\psi^{(0)}(z)$:

$$\psi_m(z) = (-1)^{m+1}\ m!\ \zeta(1+m, z)$$

1.7. Pochhammer symbol:

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\ldots(x+n-1)$$

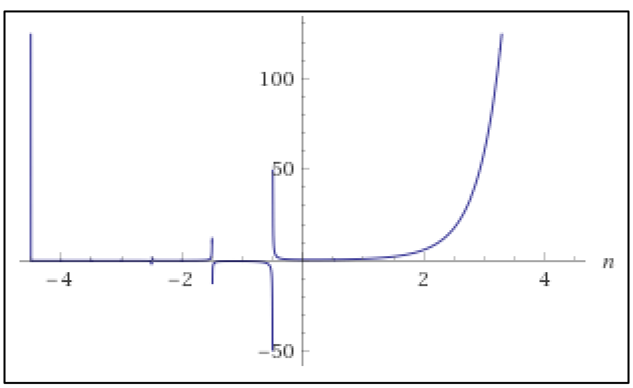

Figure 3. Pochhammer symbols in R

1.8. Hypergeometric function:

$$_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n}\ \frac{z^n}{n!} = 1 + \frac{ab}{1!\,c} z + \frac{a(a+1)b(b+1)}{2!\,c\,(c+1)} z^2 + \cdots$$

Where $(a)_n$ is a Pochhammer symbol. The solution exists if c is not a negative integer (1) for all of |z|<1 and (2) on the unit circle |z|=1 if Re(c-a-b)>0.

This function is related to the Hypergeometric Differential equation:

$$z(1-z)y'' + [c-(a+b+1)z]\,y' - aby = 0$$

Some particular values:

$$_2F_1(1,1;1;z) = \frac{1}{1-z}$$

$$_2F_1(1,1;2;z) = \frac{-\ln(1-z)}{z}$$

$$_2F_1(1,2;1;z) = \frac{1}{(1-z)^2}$$

$$_2F_1(1,2;2;z) = \frac{1}{1-z}$$

1.9. Polylogarithm $\mathrm{Li}_s(z)$ defined by a power series in z, which is also a Dirichlet series in s:

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^s} \qquad \text{converges for } s>1 \text{ and } |z|<1$$

Particular values:

$\mathrm{Li}_1(z)$	$\mathrm{Li}_0(z)$	$\mathrm{Li}_1(1/2)$	$\mathrm{Li}_2(1/2)$	$\mathrm{Li}_s(\pm i)$
$-\ln(1-z)$	z/(1-z)	ln(2)	$\frac{1}{2}(\zeta(2)-\ln(2)^2)$	$-2^{-s}\eta(s) \pm i\beta(s)$

Table 1

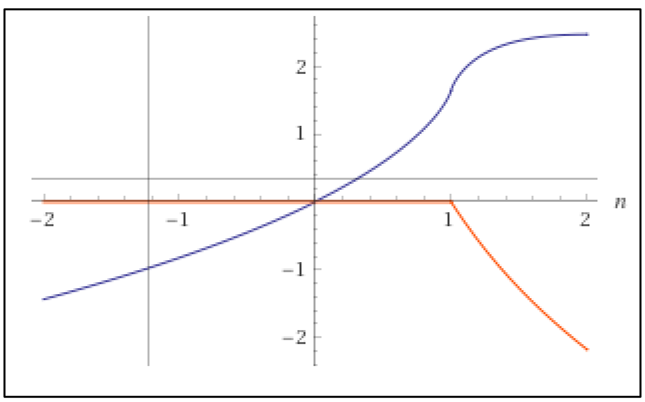

Figure 4. Polylogarithm function in R

Where:

$$\eta(s) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} \quad \text{is the } \eta \text{ (eta)} - \text{Dirichlet function}$$

$$\beta(s) = \sum_{n=0}^{\infty}\frac{(-1)^{n}}{(2n+1)^s} \quad \text{is the } \beta \text{ (beta)} - \text{Dirichlet function}$$

1.10. The Lerch Transcendent function $\Phi(z,s,q)$:

$$\Phi(z,s,q) = \sum_{k=0}^{\infty}\frac{z^k}{(k+q)^s}$$

2. Constants used in this paper

2.1. Glaisher-Kinkelin constant

$$A = e^{\frac{1}{12} - \zeta'(-1)} = 1.28242712 \ldots$$

2.2. Euler-Mascheroni constant:

$$\gamma = \lim_{m\to\infty} \left(\sum_{k=1}^{m} \frac{1}{k} - \int \frac{dm}{m} = \lim_{m\to\infty} \left(\sum_{k=1}^{m} \frac{1}{k} - \ln(m) \right. \right. = 0.57721566490 \ldots$$

2.3. Stieltjes constants:

$$\gamma_n = \lim_{m\to\infty} \left(\sum_{k=1}^{m} \frac{(\ln k)^n}{k} - \int \frac{(\ln m)^n}{m} dk \right) = \lim_{m\to\infty} \left(\sum_{k=1}^{m} \frac{(\ln k)^n}{k} - \frac{(\ln m)^{n+1}}{n+1} \right)$$

n=0	n=1	n=2	n=3	n=4
0.5772456…	-0.072815…	-0.00969…	0.00205…	0.0007933…

Table 2

3. The Riemann Zeta function $\zeta(s)$ in R

As defined in literature (Sondow et al, Ellinor et al, Andrews)

3.1. $\zeta(k) = \sum_{j=1}^{\infty} j^{-k}$ converges for $k \neq 1$

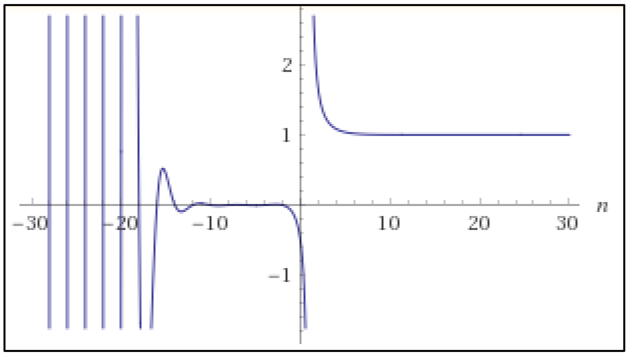

Figure 5. Riemann Zeta function in R

3.2. Euler Product Formula that ties $\zeta(k)$ with the distribution of prime numbers

$$\zeta(s) = \sum_{j=1}^{\infty} j^{-s} = \prod_{p\ \text{prime}} \frac{1}{1 - p^{-s}}$$

Example for k=2

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{1}{1 - 2^{-2}} x \frac{1}{1 - 3^{-2}} x \frac{1}{1 - 5^{-2}} x \frac{1}{1 - 7^{-2}} x \ldots$$

3.3. Integral definition:

$$\zeta(s)=\sum_{j=1}^{\infty} j^{-s}=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{1}{e^x-1}x^s\frac{dx}{x}$$

Where $\Gamma(s)$, is the Gamma function

3.4. Analytical continuation for:

Re(s)>0: [Dirichlet]

$$\zeta(s)=\frac{1}{s-1}\sum_{k=1}^{\infty}(\frac{n}{(n+1)^s}-\frac{n-s}{n^\wedge s}$$

0<Re(s)<1:

$$\zeta(s)=\frac{1}{1-2^{1-s}}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^s}$$

-k<Re(s) [Kopp, Konrad. 1945]:

$$\zeta(s)=\frac{1}{s-1}\sum_{k=1}^{\infty}\frac{k(k+1)}{2}(\frac{2k+3+s)}{(k+1)^{s+2}}-\frac{2k-1-s}{k^{s+2}})$$

3.5. Laurent series at s=1:

$$\zeta(s)=\frac{1}{s-1}+\sum_{k=}^{\infty}\frac{(-1)^k\gamma_n}{k!}(s-1)^k)$$

where $\gamma_n$are the Stieltjes constants, defined in 4.3.

3.6. Hurwitz function $\zeta(k,z)$:

$$\zeta(k,z)=\sum_{j=0}^{\infty}(j+z)^{-k}=\sum_{j=z}^{\infty}j^{-k}\qquad \text{converges for } k>1$$

3.7. Generalized Harmonic Function $H_n^{(k)}$:

$$H_n^{(k)}=\sum_{j=1}^{n}j^{-k}=\left(\frac{1}{1^k}+\frac{1}{2^k}+\cdots+\frac{1}{n^k}\right)\qquad \text{converges for } k>1$$

3.8. $\zeta(s)$ converges for s>1 to the following values:

s	ζ(s)	Known ζ(s) representations over π
2	1.6449179	$\pi^2/6$
4	1.0823232	$\pi^4/90$
6	1.0173431	$\pi^6/945$
8	1.0040774	$\pi^8/9450$

Table 3. Values of ζ(s)

What happens with the odd values of s? Do they have also a representation in a close form?

$$\zeta(2n+1)=\frac{\pi^{2n+1}}{k}$$

We are going to propose two different approaches to answer the question.

3.8.1. Solution 1: A close form for $\zeta(2n+1) = \frac{a}{b}\, C^{2n+1}$

The problem is proposed as an optimization problem to minimize the function:

$$\sum_{n=1}^{\infty} \left|\zeta(2n+1) - \frac{a}{b}\, C^{2n+1}\right|^2$$

This function is the aggregated quadratic error of the approximations of $\zeta(2n+1)$ to $\left(\frac{a}{b}\, C^{2n+1}\right)$.

The result of this calculation is:

C=3.0677724318720092274489185949583964933828786704320291807 73…

And the values for $\zeta(2n+1)$:

(2n+1)	a	b	Error $\zeta(2n+1) - \frac{a}{b} C^{2n+1}$
ζ (3)	21635641	519653864	$2.9\, x\, 10^{-16}$
ζ (5)	18604295	4875065106	$3.8\, x\, 10^{-16}$
ζ (7)	9001873	22828841506	$3.8\, x\, 10^{-16}$
ζ (9)	759823	18249431499	$3.6\, x\, 10^{-16}$
ζ (11)	1	226381	0
ζ (13)	13555	28889967723	$6.6x\, 10^{-16}$

Table 4

As we can see, C can be defined as:

$C = [\zeta(11) * 226381]^{\frac{1}{11}}$ **[Caceres Proposition 1]**

3.8.2. Solution 2: An approximation for the values of ζ(s) in R

We can calculate that:

$$\lim_{s\to\infty}\left(\frac{\zeta(s)}{\zeta(s)+1}\right)^{1/s} = 1$$

And:

$$\lim_{s\to\infty}\left(\frac{\zeta(s)}{\zeta(s)-1}\right)^{1/s} = 2$$

Based on this expression, we can say that for s sufficiently large, we can represent ζ(s) as a multiple of π^s:

$\zeta(s) = \frac{\pi^s}{K_s}$ with $K_s = (2^s - 1) * \frac{\pi^s}{2^s}$

with a very good approximation given by:

$K_s^* = int\left((2^s - 1) * \frac{\pi^s}{2^s}\right) - 1$ where int(k) is the integer part of k.

The error between the K_s^* calculated and K_s actual is very small for s>4.

Some calculated values of K_s^* calculated and K_s actual:

s	Calculated	Actual
2	6.0	6.0
3	26.0	25.8
4	90.0	90.0
5	295.0	295.1
6	945.0	945.0
7	2,995.0	2,995.3
8	9,450.0	9,450.0
9	29,749.0	29,749.4
10	93,555.0	93,555.0
11	294,059.0	294,058.7

Table 5. Values of K_s^* calculated and K_s actual

3.9. Approximation for ζ(s) [Caceres, Pedro. 2017]:

$\zeta(s) \approx \frac{1}{1-\pi^{-s}-2^{-s}}$ **[Caceres Proposition 2]**

Graphically:

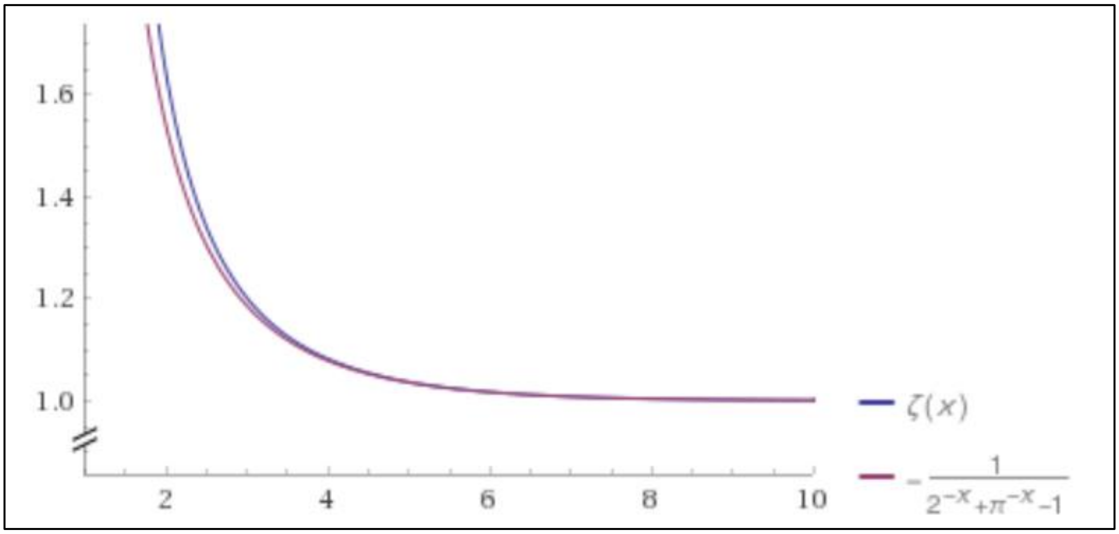

Figure 6. Caceres' approximation for the Riemann Zeta function in R

	s=3	s=4	s=10	s=14
ζ(s)Actual	1.20206	1.0823	1.000994	1.0000612
ζ(s)Approx	1.18659	1.0784	1.000988	1.0000611

Table 6

4. A set of Constants involving $\zeta(s)$

4.1. Lemma 1:

$$\sum_{j=2}^{\infty}\sum_{k=2}^{\infty} j^{-k} = 1$$

Proof:

$$\sum_{j=2}^{\infty}\sum_{k=2}^{\infty} j^{-k} = \lim_{n,m\to\infty}\sum_{j=2}^{n}\sum_{k=2}^{m} j^{-k} =$$

$$= \lim_{n,m\to\infty}\sum_{j=2}^{n}\frac{j^{-m-1}(j^m - j)}{j-1} = \lim_{n\to\infty}\sum_{j=2}^{n}\frac{j^{-1}}{j-1} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right) = 1$$

4.2. Lemma 2:

$$\sum_{j=2}^{\infty}(\zeta(j)-1) = 1$$

Proof:

$$\sum_{j=2}^{\infty}(\zeta(j)-1) = \sum_{j=2}^{\infty}(\sum_{k=1}^{\infty} j^{-k} - 1) = \sum_{j=2}^{\infty}\sum_{k=2}^{\infty} j^{-k} = 1$$

Per lemma 1

4.3. Lemma 3:

$$\lim_{k\to\infty}\zeta(k) = 1$$

Proof:

$$\zeta(k) = \sum_{j=1}^{\infty} j^{-k} = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \frac{1}{5^k} + \cdots$$

$$\lim_{k\to\infty}\zeta(k) = \lim_{k\to\infty} 1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \frac{1}{5^k} + \cdots = 1$$

4.4. Theorem 1.0: **[Caceres Proposition 3]**

The infinite sums $\sum_{j=1}^{\infty}[\zeta(u*k \pm n) - \zeta(v*k \pm m)]$ converge to a value in the interval (-1,1) for all $u \geq 1, v \geq 1, n, m \in N$ such that $(u*k \pm n) > 1$ and $(v*k \pm m) > 1$ for all $j \in N$

Proof:

$$\sum_{j=1}^{\infty}[\zeta(uj \pm n) - \zeta(vj \pm m)] =$$

$$= \zeta(u \pm n) - \zeta(v \pm m) + \zeta(2u \pm n) - \zeta(2v \pm m) + \cdots = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(\frac{1}{j^{uk \pm n}} - \frac{1}{j^{vk \pm m}})$$

The largest differences in value between the parameters $uk \pm n$ and $vk \pm m$ occurs when $u = 1, n = 1,$ and $v =$ infinity.

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(\frac{1}{j^{uk \pm n}} - \frac{1}{j^{vk \pm m}}) \leq \sum_{j=1}^{\infty}(\sum_{k=1}^{\infty}\frac{1}{j^{k+1}}) - 1 = \sum_{j=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{j^{k+1}} = \sum_{j=2}^{\infty}\sum_{k=2}^{\infty}\frac{1}{j^{k}} = 1$$

As per lemmas 1,2,3.

Following the same logic, we can also say that:

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(\frac{1}{j^{uk \pm n}} - \frac{1}{j^{vk \pm m}}) \geq -1$$

Therefore, the infinite sums $\sum_{j=1}^{\infty}[\zeta(uk \pm n) - \zeta(vk \pm m)]$ converge to a value in the interval (-1, 1) for all $u, v, n, m \in N$ such that $(u*k \pm n) > 1$ and $(v*k \pm m) > 1$ for all $j \in N$

Similarly, we can propose and formulate the following Theorem.

4.5. Theorem 2.0: The infinite sums $\sum_{j=1}^{\infty}[\zeta(u*k \pm n) - \zeta(v*k \pm m)]$ converge to a value in the interval (-∞,∞) for all $u \geq 1, v \geq 1, n, m \in R$ such that $(u*k \pm n) > 1$ and $(v*k \pm m) > 1$

Example:

$$\sum_{j=1}^{\infty}\zeta(1.1*j + 2) - \zeta(1.2*j + 0.1) = -3.14132\ldots$$

$$\sum_{j=1}^{\infty} \zeta(1.001 * j) - \zeta(1.1 * j + 1) = 999.733 \ldots$$

4.6. These theorems let us define a set of infinite constants of the type:

$CZ_{u,v,n,m}^{(q)} = \sum_{j=q}^{\infty} [\zeta(uj \pm n) - \zeta(vj \pm m)] = \text{constant}$ **[Caceres Proposition 4]**

(By default, q=1 will not be written)

Some of the CZ constants we will use through the paper are:

$$CZ_{2,0,2,1} = \sum_{j=1}^{\infty} [\zeta(2j) - \zeta(2j+1)] = 0.5$$

$$CZ_{4,0,4,-2} = \sum_{j=1}^{\infty} [\zeta(4j) - \zeta(4j-2))] = -0.5766744746 \ldots$$

$$CZ_{4,1,4,-1} = \sum_{j=1}^{\infty} [\zeta(4j+1) - \zeta(4j-1))] = -0.171865985524 \ldots$$

The following table shows some values of the $CZ_{u,v,1,0}$ constants:

n=1, m=0	v=2	v=3	v=4	v=5	v=200
u=2	-0.500000	0.028310	0.163337	0.212046	0.250000
u=3	-0.658193	-0.129882	0.005144	0.053853	0.091800
u=4	-0.719330	-0.182622	-0.047596	0.001113	0.039067
u=5	-0.732147	-0.203836	-0.068810	-0.020101	0.017853
u=200	-0.750000	-0.221689	-0.086663	-0.037954	0.000000

Table 7

These constants appear in sums and products of infinite Dirichlet-like functions.

4.6.1. Example for $f(z) = [re^{i\theta}]^{-k}$ with $z = re^{i\theta}$

$$\sum_{r=2}^{\infty}\sum_{k=2}^{\infty} [re^{i\theta}]^{-k} = \lim_{n\to\infty} \frac{(\psi^{(0)}(n+1-e^{-i\theta}) - \psi^{(0)}(n+1) - \psi^{(0)}(2-e^{-i\theta}) - \gamma + 1}{e^{i\theta}})$$

Values for some θ:

Θ	*Sum Matrix* $[re^{i\theta}]^{-k}$	Numeric result
π/2	$i\left(-1+\gamma+\psi^{(0)}(2+i)\right)$	$CZ_{4,0,4,-2} - CZ_{4,1,4,-1}\, i$
Π	1/2	$CZ_{2,0,2,1}$
-π/2	$-i\left(-1+\gamma+\psi^{(0)}(2-i)\right)$	$CZ_{4,0,4,-2} + CZ_{4,1,4,-1}\, i$

Table 8

4.6.2. Example for f(z)= $1+re^{i\theta}]^{-k}$, with z=$re^{i\theta}$

$$\sum_{r=2}^{\infty}\sum_{k=2}^{\infty}[1+re^{i\theta}]^{-k} = \lim_{n\to\infty}\frac{(-\psi^{(0)}(n+1-e^{-i\boldsymbol{\theta}})+\psi^{(0)}(n+1)+\psi^{(0)}(2+e^{-i\boldsymbol{\theta}})+\gamma-1}{e^{i\boldsymbol{\theta}}})$$

Θ	*Sum Matrix* $[1+re^{i\theta}]^{-k}$
0	$CZ_{2,0,2,1}=1/2$
π/2	$-i\left(-1+\gamma+\psi^{(0)}(2-i)\right) = CZ_{4,0,4,-2} + CZ_{4,1,4,-1}\, i$
Π	$2 * CZ_{2,0,2,1} = 1$
-π/2	$i\left(-1+\gamma+\psi^{(0)}(2+i)\right) = CZ_{4,0,4,-2} - CZ_{4,1,4,-1}\, i$

Table 9

4.6.3. Example for f(z)= $1-re^{i\theta}]^{-k}$, with z=$re^{i\theta}$

$$\sum_{r=2}^{\infty}\sum_{k=2}^{\infty}[1-re^{i\theta}]^{-k} = \lim_{n\to\infty}\frac{(\psi^{(0)}(n+1-e^{-i\boldsymbol{\theta}})-\psi^{(0)}(n+1)-\psi^{(0)}(2-e^{-i\boldsymbol{\theta}})-\gamma+1}{e^{i\boldsymbol{\theta}}})$$

$$=\frac{\left(\psi^{(0)}(e^{-i\boldsymbol{\theta}}-2\right)-\gamma+1}{e^{i\boldsymbol{\theta}}} \quad \textit{when } e^{i\boldsymbol{\theta}} \neq 0$$

Θ	*Sum Matrix* $[1-re^{i\theta}]^{-k}$
0	$2 * CZ_{2,0,2,1} = 1$
π/2	$i\left(-1+\gamma+\psi^{(0)}(2+i)\right) = CZ_{4,0,4,-2} - CZ_{4,1,4,-1}\, i$
Π	$CZ_{2,0,2,1} = 1/2$
-π/2	-i $\left(-1+\gamma+\psi^{(0)}(2-i)\right) = CZ_{4,0,4,-2} + CZ_{4,1,4,-1}\, i$

Table 10

6. Function in $C_2(x, a, b)$ →R with zeros at a=1/2 and b=Im(z) such that ζ(z)=0.

Let's define the function $C_2(x, a, b)$ in R such that:

$$C_2(x, a, b) = 2 * x^{-a} * \left(\sum_{j=1}^{x-1} j^{-a} * \cos\left(b * (\ln\left(\frac{x}{j}\right))\right)\right)$$

With the following wave representation:

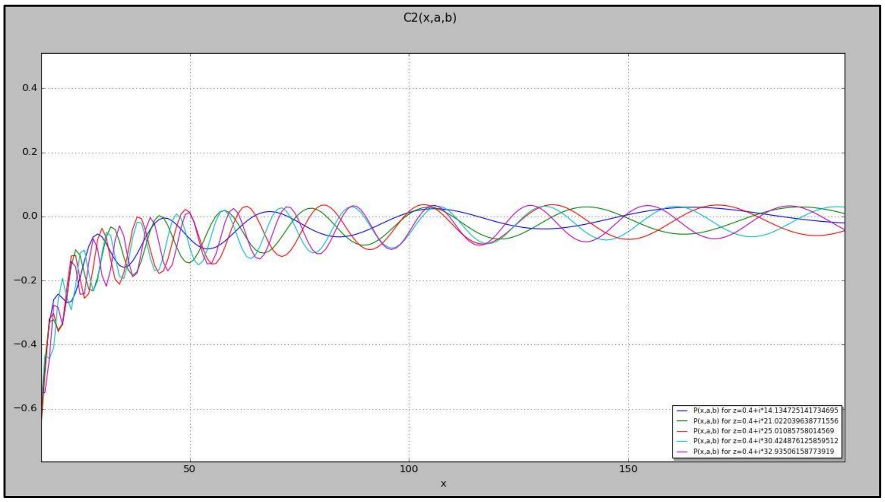

Figure 7. $C_2(x, a, b)$ for a=0.4 and several b

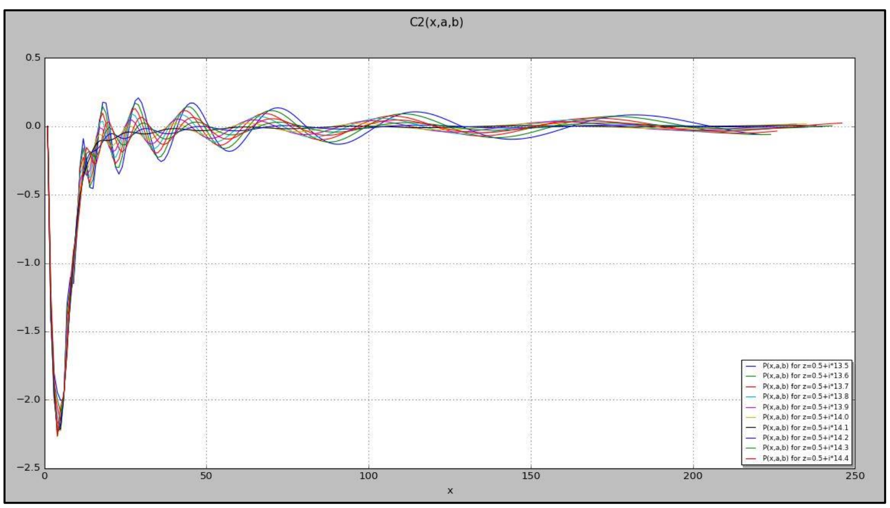

Figure 8. $C_2(x, a, b)$ for a=0.5 and several b

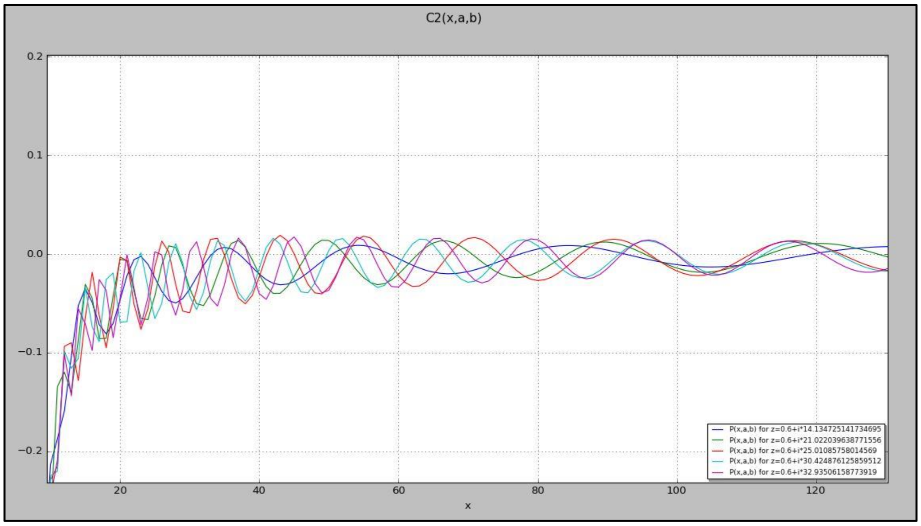

Figure 9. $C_2(x,a,b)$ *for a=0.6 and several b*

As a wave, $C_2(x,a,b)$ has one or more zeros. For $C_2(x,a,b)$ to have only one zero, it must cross the axis y=0 only once, which means that the wave collapses to a polynomial line. A numeric method has been created and coded to find the values of (x,a,b) such that $C_2(x,a,b)$=0. The following table shows an example of those calculated values:

Alfa	Beta	Number of Zeros	Zero at X=
0.4	14.1	5	
0.4	14.2	5	
0.4	14.3	5	
0.4	14.4	5	
0.5	14.07	5	
0.5	14.08	5	
0.5	14.09	5	
0.5	14.1	4	
0.5	14.11	4	
0.5	14.12	3	
0.5	14.13	1	200
0.5	14.14	3	
0.5	20.97	11	
0.5	20.98	11	
0.5	20.99	11	
0.5	21	9	
0.5	21.01	5	
0.5	21.02	1	442
0.5	21.03	3	
0.5	24.96	16	
0.5	24.97	16	
0.5	24.98	15	
0.5	24.99	11	
0.5	25	7	
0.5	25.01	1	626
0.5	25.02	6	
0.5	25.03	10	

Table 11. Number of Zeros of $C_2(x,a,b)$ *for different values of a,b*

The calculations for $a \in (0,1)$ and $b \in [1,100]$ only found single zeros for $C_2(x,a,b)$ for values of $a = 0.5$ as shown in the following table that summarizes the single zeros found in those intervals:

Values (x,a,b) | C2(x,a,b)=0 SINGLE		
x*	a*	b*
200.1000	0.5000	14.1368
442.2000	0.5000	21.0226
625.8000	0.5000	25.0110
926.0000	0.5000	30.4261
1085.0000	0.5000	32.9355
1413.0000	0.5000	37.5866
1674.6000	0.5000	40.9188
1877.5000	0.5000	43.3272
2304.8000	0.5000	48.0057

Table 12. Showing only the first single Zeros of $C_2(x,a,b)$

It can be observed that:

$$if\ \ C_2(x,a,b) = 0 \rightarrow$$

$$a = 1/2$$

$$b = \ Im(z) \qquad with\ \zeta(z) = 0$$

$$\text{(a, b) are the Non} - \text{Trivial Zeros of } \zeta(z) \text{ in the critical line.}$$

$$x = b^2 + (1-a)^2$$

And the calculated values of $\lim_{x\to\infty} C_2(x,a,b)$ for the values of (a,b) from Table 1 are:

Values (x,a,b) | C2)x,a,b=0			Limit (C2(x,a,b))
x	a	b	when x->∞
200.1000	0.5000	14.1368	0.0050
442.2000	0.5000	21.0226	0.0023
625.8000	0.5000	25.0110	0.0016
926.0000	0.5000	30.4261	0.0011
1085.0000	0.5000	32.9355	0.0009
1413.0000	0.5000	37.5866	0.0007
1674.6000	0.5000	40.9188	0.0006
1877.5000	0.5000	43.3272	0.0005

Table 13. Limit of $C_2(x,a,b)$ *for b in Table 1 and x->∞*

Values (x,a,b) | C2(x,a,b)=0			Limit (C2(x,a,b))	
x	a	b	when x->∞	Known Zero
200.1000	0.5000	14.1368	0.0050	14.1347
442.2000	0.5000	21.0226	0.0023	21.0220
625.8000	0.5000	25.0110	0.0016	25.0109
926.0000	0.5000	30.4261	0.0011	30.4249
1085.0000	0.5000	32.9355	0.0009	32.9351
1413.0000	0.5000	37.5866	0.0007	37.5862
1674.6000	0.5000	40.9188	0.0006	40.9187
1877.5000	0.5000	43.3272	0.0005	43.3271
2304.8000	0.5000	48.0057	0.0004	48.0052
2477.7000	0.5000	49.7740	0.0004	49.7738

Table 14. Comparing "b" calculated with known zeros of ζ(z)

[Caceres Proposition 5] $C_2(x, a, b)$ has the following special properties for all (a,b) such that ζ(1/2+bi)=0.

if $S = \frac{1}{b^2+1/4}$

$$C_2(x, 1/2, b) = 0 \; when \; x = 1/S$$

$$\lim_{n \to \infty} C_2(x, 1/2, b) = S$$

Graphically:

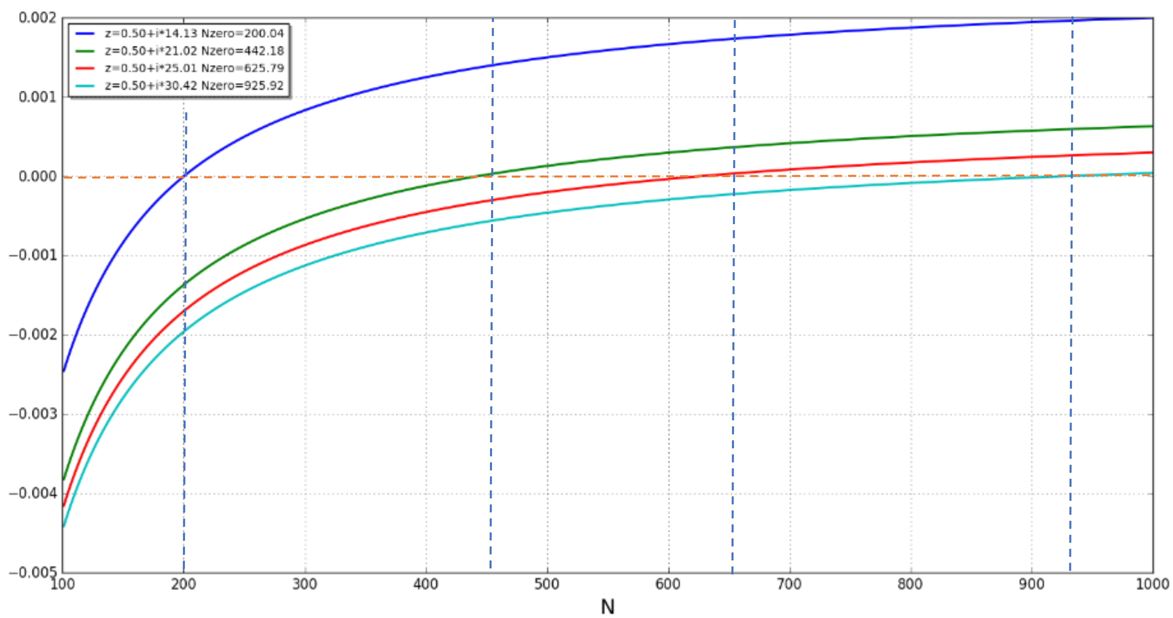

Figure 10. $C_2(x, 1/2, b)$ *such that ζ(1/2+bi)=0*

7. C-transformation of *ζ(z), z*∈ *C*

Let's define the C-transformation of a function f(x) as:

$$C_n\{f\} = \sum_{k=1}^{n} f(k) - \int_0^n f(k)dk$$

And let's call $C\{f\} = \lim_{n\to\infty} C_n\{f\}$ the C-transformation values.

The C-values for $f(x) = \frac{1}{x^z}$ for z≠1, are equal to $\zeta(z)$:

$$C_n\{f\} = \sum_{\mathrm{k=1}}^{\mathrm{n}} \frac{1}{\mathrm{k^z}} - \int_0^n \frac{dk}{\mathrm{k^z}}$$

Applying the exponential expression to the power of $k \in R$ to a complex number $z \in C$:

$$k^{-z} = k^{\alpha} \left[\cos(ß * \ln(k)) + i\ (\sin(ß * \ln(k))\right]$$

And:

$$\int_1^n \frac{1}{k^z}\,\mathrm{dk} = \frac{1}{(1-\alpha)-iß}\left(n^{(1-\alpha)-ßi}\right)$$

Or:

$$\int n^{-z}\ \mathrm{dn} = \left(n^{(1-\alpha)}\left[\cos(ß * ln(n) - i\, sin(ß * ln(n)\right]\right) * \frac{[(1-\alpha)+iß]}{[(1-\alpha)^2+ß^2]}$$

We can now express the real and imaginary components of $C_n\{f\}$ as:

$$Re(C_n\{f\}) = \sum_{k=1}^{n} k^{-\alpha}\ (\cos(ß * \ln(k)) - \\ - \frac{1}{[(1-\alpha)^2+ß^2]}\left(n^{(1-\alpha)}\ [(1-\alpha)*\cos(ß*\ln(n))+ß*\sin(ß*\ln(n))]\right)$$

$$Im(C_n\{f\}) = \sum_{k=1}^{n} k^{-\alpha}\ (\sin(ß * \ln(k)) + \\ + \frac{1}{[(1-\alpha)^2+ß^2]}\left(n^{(1-\alpha)}\ [ß*\cos(ß*\ln(n))-(1-\alpha)*\sin(ß*\ln(n))]\right)$$

We can calculate the following table:

z= α+iß	$\lim_{n\to\infty} C_n\{f\}$	ζ(z)
(2,0)	2.644934 + i*0	ζ(2,0)
(3,0)	1.702057 + i*0	ζ(3,0)
(1, 1)	0.582096 + i* (0.9269-1)	ζ(1,1)
(1/2, 14.134725…)	0 + i*0	Zero of the ζ function

Table 15. Values of $C_n\{f(n) = k^{-z}\}$

We can see that if $z = \alpha + iß \in C$ with α>0, *then* $\lim_{n\to\infty} C_n\{f\} = \zeta(z)$ when Re(z)= α≥0, z≠ 1

Let's define:

$$x1(z) \;=\; \sum_{k=1}^{n} k^{-\alpha}\,(cos(ß * ln(k))$$

$$y1(z) \;=\; [n^{(1-\alpha)}\,\frac{1}{[(1-\alpha)^2+ß^2]}\,[(1-\alpha) * cos(ß * ln(n)) + ß * sin(ß * ln(n))]]$$

$$x2(z) \;=\; \sum_{k=1}^{n} k^{-\alpha}\,(sin(ß * ln(k))$$

$$y2(z) \;=\; -n^{(1-\alpha)}\,\frac{1}{[(1-\alpha)^2+ß^2]}\,[ß * cos(ß * ln(n)) - (1-\alpha) * sin(ß * ln(n))]$$

And let's call:

x(z) = x1(z) +i* x2(z)

y(z) = y1(z) +i* y2(z)

In general, we can now express that any solution in C of ζ(z) as:

ζ(z) = [x1(z)-y1(z)] + i * [x2(z)-y2(z)]

and: **[Caceres Proposition 6]**

ζ(z) = x(z) - y(z)

$$x(z) = \sum_{k=1}^{n} k^{-\alpha}\,(\cos(ß * \ln(k)) \;+\; i^* \;\sum_{k=1}^{n} k^{-\alpha}\,(\sin(ß * \ln(k))$$

$$y(z)= [(\,n^{(1-\alpha)}\,\frac{1}{[(1-\alpha)^2+ß^2]}\,[(1-\alpha)*\cos(ß*\ln(n))+ß*\sin(ß*\ln(n))]) + i\,(n^{(1-\alpha)}\,\frac{1}{[(1-\alpha)^2+ß^2]}\,[ß*\cos(ß*\ln(n))-(1-\alpha)*\sin(ß*\ln(n))]])]$$

We can calculate the modulus of this complex expressions for x(z) and y(z):

$$|y(z)|2 = \;[(\,n^{(1-\alpha)}\,\frac{1}{[(1-\alpha)^2+ß^2]}\,[(1-\alpha) * cos(ß * ln(n)) + ß * sin(ß * ln(n))])2 + \;(n^{(1-\alpha)}\,\frac{1}{[(1-\alpha)^2+ß^2]}\,[ß * cos(ß * ln(n)) - (1-\alpha) * sin(ß * ln(n))])2]$$

$$|y(z)|^2 \;=\; n^{2(1-\alpha)} \;*\; \frac{1}{[ß^2+(1-\alpha)^2]}$$

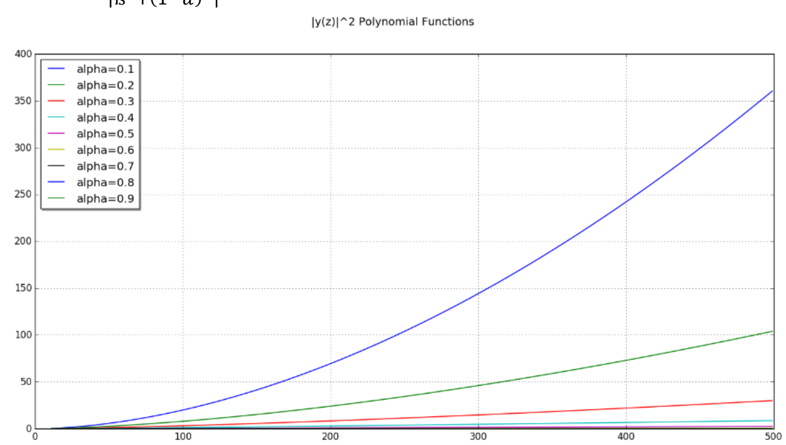

Figure 11: $|y(z)|^2$ *has a polynomial representation*

And:

$$|x(z)|^2 \; = \; (\sum k^{-\alpha} \, cos(ß \, ln(n))^2 + (\sum k^{-\alpha} \, sin(ß \, ln(n))^2$$

$$|x(z)|^2 \; = \; \sum_{k=1}^{n} k^{-2\alpha} + \sum_{k=1}^{n} \sum_{j \neq k}^{n} k^{-\alpha} * j^{-\alpha} * cos\left(ß \left(ln\left(\frac{k}{j}\right)\right)\right)$$

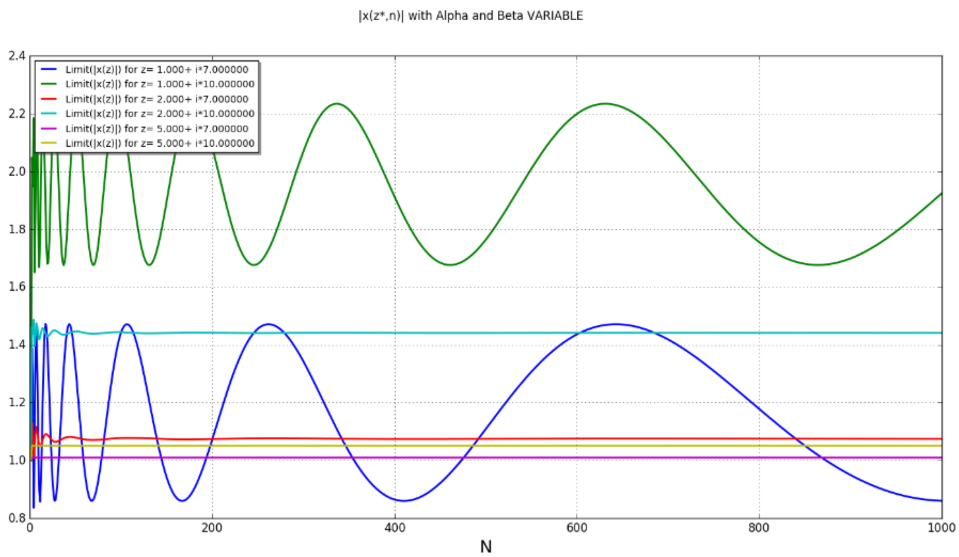

Figure 11. |x(n)|² is a wave that converges when n → ∞ and α>1

In 2017, the author proposed a potential proof for the Riemann Hypothesis based on:

- For ζ(z*)=0, these two modulus must be equal $|x(z)|^2 = |y(z)|^2$
- These two wave modulus functions in R can only be equal if they collapse to an equal form as n->∞
- The only equal form that these two modulus functions share is a straight line when n->∞ with slope equal to $\frac{1}{[ß^2+\frac{1}{4})}$
- These two modulus wave forms can only collapse to a line if and only if Re(z*)=1/2

We can also derive the following propositions from the decomposition of ζ(z).

[Caceres Proposition 7]. A linearization of the Harmonic series using zeros of ζ(z).

The precedent formulations describe also a way to approximate the Harmonic function to a straight line with slope $\frac{1}{[ß^2+(1-\alpha)^2]}$ where α=1/2 and ß=R(n):

$$Hn \; = \; \frac{n}{[ß^2+(1-\alpha)^2]} - \sum_{k=1}^{n} \sum_{j \neq k}^{n} k^{-1/2} * j^{-1/2} * cos(ß \left(ln\left(\frac{k}{j}\right)\right) \qquad \text{when } n \rightarrow \infty$$

We can see this graphically for $ß_1$=14.134725... with O(n) given by:

$$O(n) \; = \sum_{k=1}^{n} \sum_{j \neq k}^{n} k^{-1/2} * j^{-1/2} * cos(ß \left(ln\left(\frac{k}{j}\right)\right)$$

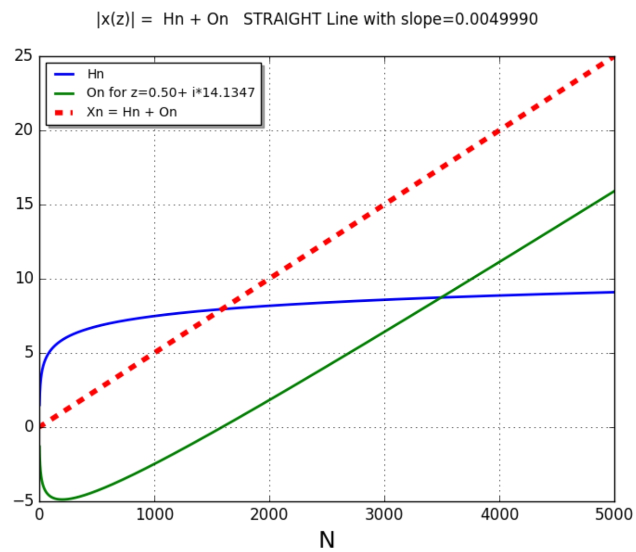

Figure 17. H_n and $|x(z)|^2$

[Caceres Proposition 8] All ß zeros of ζ(z) are related algebraically.

The fact that the same H_n can be expressed in an infinite number of ways as a function of ß for every ß imaginary part of a non-trivial solution of ζ(z), provides an algorithm to calculate all non-trivial zeros from any known zero through the expression. If $ß_1$ and $ß_2$ are imaginary part of a non-trivial solution of ζ(z), z=α+ßi, where α=1/2, then:

$$\frac{n}{[ß_2^2 + (1-\alpha)^2]} - \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * cos(ß_2\,(ln\left(\frac{k}{j}\right)) =$$

$$\frac{n}{[ß_1^2 + (1-\alpha)^2]} - \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * cos(ß_1\,(ln\left(\frac{k}{j}\right)) =$$

when $n \to \infty$, where the size of n will determine the degree of accuracy of the solution.

{_end_}

REFERENCES

(1) B. Riemann, “Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse”, Monat. der Königl. Preuss. Akad. der Wissen. zu Berlin aus der Jahre 1859 (1860), 671–680; also, Gesammelte math. Werke und wissensch. Nachlass, 2. Aufl. 1892, 145–155.

(2) Wolfram Mathematica, [https://www.wolframalfa.com]

(3) Caceres, Pedro. “An Engineer’s approach to the Riemann Hypothesis and why it is true”. 2017

(4) Caceres, Pedro. “The Riemann Hypothesis”. 2018

(5) Edwards, H. M. Riemann's Zeta Function. New York: Dover, 2001.

(6) Editors of Encyclopedia Britannica. “Riemann Zeta Function”. https://www.britannica.com/topic/Riemann-zeta-function

(7) Ellinor, Andrew et al. “Riemann Zeta Function.” https://brilliant.org/wiki/riemann-zeta-function/

(8) Kopp, Konrad. “Theory and Application of Infinite Series.” Blackie & Son Limited, London,

(9) OEIS wiki foundation. “The Riemann Zeta Function.” http://oeis.org/wiki/Riemann_zeta_function

(10) Sondow, Jonathan and Weisstein, Eric W. "Riemann Zeta Function." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/RiemannZetaFunction.html