Non-Inertial Frames in Special Relativity

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This article presents a new formulation of special relativity which is invariant under transformations between inertial and non-inertial (non-rotating) frames. Additionally, a simple solution to the twin paradox is presented and a new universal force is proposed.

Introduction

The intrinsic mass (m) and the frequency factor (f) of a massive particle are given by:

$$m \doteq m_o$$

$$f \doteq \left(1 - \frac{\mathbf{v} \cdot \mathbf{v}}{c^2}\right)^{-1/2}$$

where (m_o) is the rest mass of the massive particle, ($\mathbf{v}$) is the relational velocity of the massive particle and (c) is the speed of light in vacuum.

The intrinsic mass (m) and the frequency factor (f) of a non-massive particle are given by:

$$m \doteq \frac{h\,\kappa}{c^2}$$

$$f \doteq \frac{\nu}{\kappa}$$

where (h) is the Planck constant, (ν) is the relational frequency of the non-massive particle, (κ) is a positive universal constant with dimension of frequency and (c) is the speed of light in vacuum.

In this article, a massive particle is a particle with non-zero rest mass and a non-massive particle is a particle with zero rest mass.

The Invariant Kinematics

The special position ($\bar{\mathbf{r}}$), the special velocity ($\bar{\mathbf{v}}$) and the special acceleration ($\bar{\mathbf{a}}$) of a (massive or non-massive) particle are given by:

$$\bar{\mathbf{r}} \doteq \int f\,\mathbf{v}\,dt$$

$$\bar{\mathbf{v}} \doteq \frac{d\bar{\mathbf{r}}}{dt} = f\,\mathbf{v}$$

$$\bar{\mathbf{a}} \doteq \frac{d\bar{\mathbf{v}}}{dt} = f\,\frac{d\mathbf{v}}{dt} + \frac{df}{dt}\,\mathbf{v}$$

where (f) is the frequency factor of the particle, ($\mathbf{v}$) is the relational velocity of the particle and (t) is the relational time of the particle.

The Invariant Dynamics

If we consider a (massive or non-massive) particle with intrinsic mass (m) then the linear momentum ($\mathbf{P}$) of the particle, the angular momentum ($\mathbf{L}$) of the particle, the net force ($\mathbf{F}$) acting on the particle, the work (W) done by the net force acting on the particle, and the kinetic energy (K) of the particle are given by:

$$\mathbf{P} \doteq m\,\bar{\mathbf{v}} = m\,f\,\mathbf{v}$$

$$\mathbf{L} \doteq \mathbf{P} \dot{\times} \mathbf{r} = m\,\bar{\mathbf{v}} \dot{\times} \mathbf{r} = m\,f\,\mathbf{v} \dot{\times} \mathbf{r}$$

$$\mathbf{F} = \frac{d\mathbf{P}}{dt} = m\,\bar{\mathbf{a}} = m\left[f\,\frac{d\mathbf{v}}{dt} + \frac{df}{dt}\,\mathbf{v} \right]$$

$$\mathrm{W} \doteq \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 \frac{d\mathbf{P}}{dt} \cdot d\mathbf{r} = \Delta\,\mathrm{K}$$

$$\mathrm{K} \doteq m\,f\,c^2$$

where (f, $\mathbf{r}$, $\mathbf{v}$, t, $\bar{\mathbf{v}}$, $\bar{\mathbf{a}}$) are the frequency factor, the relational position, the relational velocity, the relational time, the special velocity and the special acceleration of the particle and (c) is the speed of light in vacuum. The kinetic energy (K_o) of a massive particle at relational rest is ($m_o\,c^2$)

Relational Quantities

From an auxiliary massive particle (called auxiliary-point) some kinematic quantities (called relational quantities) can be obtained. These are invariant under transformations between inertial and non-inertial (non-rotating) frames.

An auxiliary-point is an arbitrary massive particle that is free of forces (that is, the net force acting on it is zero)

The relational time $(\, t\,)$, the relational position $(\,\mathbf{r}\,)$, the relational velocity $(\,\mathbf{v}\,)$ and the relational acceleration $(\,\mathbf{a}\,)$ of a (massive or non-massive) particle relative to an inertial or non-inertial (non-rotating) frame S are given by:

$$t \doteq \int_{0}^{\mathtt{t}} \gamma \, \mathtt{dt} - \gamma \, \frac{\vec{r} \cdot \vec{\varphi}}{c^2}$$

$$\mathbf{r} \doteq \vec{r} + \frac{\gamma^2}{\gamma + 1} \, \frac{(\vec{r} \cdot \vec{\varphi}) \, \vec{\varphi}}{c^2} - \int_{0}^{\mathtt{t}} \gamma \, \vec{\varphi} \, \mathtt{dt}$$

$$\mathbf{v} \doteq \frac{d\mathbf{r}}{dt} \quad , \quad \mathbf{a} \doteq \frac{d\mathbf{v}}{dt}$$

where $(\,\mathtt{t},\, \vec{r}\,)$ are the time and the position of the particle relative to the frame S, $(\,\vec{\varphi}\,)$ is the velocity of the auxiliary-point relative to the frame S, $(\,c\,)$ is the speed of light in vacuum, and $\gamma \doteq (\,1 - \vec{\varphi} \cdot \vec{\varphi} / c^2\,)^{-1/2}$

The velocity of the auxiliary-point $(\,\vec{\varphi}\,)$ is a constant in inertial frames and the factor $(\,\gamma\,)$ is a constant in non-inertial (uniform circular motion) frames.

The relational frequency $(\,\nu\,)$ of a non-massive particle relative to an inertial or non-inertial (non-rotating) frame S is given by:

$$\nu \doteq \mathtt{v} \; \gamma \left(1 - \frac{\vec{c} \cdot \vec{\varphi}}{c^2} \right)$$

where $(\,\mathtt{v}\,)$ is the frequency of the non-massive particle relative to the frame S, $(\,\vec{c}\,)$ is the velocity of the non-massive particle relative to the frame S, $(\,\vec{\varphi}\,)$ is the velocity of the auxiliary-point relative to the frame S, $(\,c\,)$ is the speed of light in vacuum, and $\gamma \doteq (\,1 - \vec{\varphi} \cdot \vec{\varphi} / c^2\,)^{-1/2}$

§ In arbitrary frames ($t_\alpha \neq \tau_\alpha$ or $\mathbf{r}_\alpha \neq 0$) (α = auxiliary-point) a constant must be add in the definition of relational time such that the relational time and the proper time of the auxiliary-point are the same ($t_\alpha = \tau_\alpha$) and another constant must be add in the definition of relational position such that the relational position of the auxiliary-point is zero ($\mathbf{r}_\alpha = 0$)

§ In the particular case of an isolated system of (massive or non-massive) particles, all observers should preferably use an auxiliary-point such that the linear momentum of the isolated system of particles is zero ($\sum_z m_z \bar{\mathbf{v}}_z = 0$)

§ It is important to emphasize that any auxiliary-point must be a free massive particle (that is, the net force acting on it must be zero)

Relational Metric

It is known that in inertial frames the local geometry is Euclidean and that in non-inertial frames the local geometry is in general non-Euclidean.

According to this article, in an inertial or non-inertial (non-rotating) frame S the local line element must be obtained from the relational line element.

Therefore, in the frame S the relational line element (in rectilinear coordinates) and the local line element are given by:

$$ds^2 \;=\; c^2 dt^2 - d\mathbf{r}^2$$

$$ds^2 \;=\; \left[\left(1 + \frac{\vec{\mathrm{w}} \cdot \vec{r}}{c^2} \right)^2 - \left(\frac{\vec{\phi} \times \vec{r}}{c} \right)^2 \right] c^2\, \mathtt{dt}^2 \;-\; 2 \left(\vec{\phi} \times \vec{r} \right) d\vec{r}\; \mathtt{dt} \;-\; d\vec{r}^{\,2}$$

$$\vec{\mathrm{w}} \;\doteq\; -\,\gamma^2 \left(\vec{\alpha} \,+\, \frac{\gamma^2}{\gamma + 1}\, \frac{(\vec{\alpha} \cdot \vec{\varphi})\, \vec{\varphi}}{c^2} \right) \quad , \quad \vec{\phi} \;\doteq\; -\,\gamma \left(\frac{\gamma^2}{\gamma + 1}\, \frac{(\vec{\alpha} \times \vec{\varphi})}{c^2} \right)$$

where $(\, t,\, \mathbf{r} \,)$ are relational time and relational position relative to the frame S, $(\, \mathtt{t},\, \vec{r} \,)$ are time and position relative to the frame S, $(\, \vec{\varphi},\, \vec{\alpha} \,)$ are the velocity and the acceleration of the auxiliary-point relative to the frame S, $(\, c \,)$ is the speed of light in vacuum, and $\gamma \;\doteq\; (\, 1 - \vec{\varphi} \cdot \vec{\varphi} / c^2 \,)^{-1/2}$

The frame S is inertial when ($\vec{\alpha} \,=\, 0$) the frame S is non-inertial (rectilinear accelerated motion) when ($\vec{\alpha} \,\neq\, 0$) & ($\vec{\alpha} \times \vec{\varphi} \,=\, 0$) and the frame S is non-inertial (uniform circular motion) when ($\vec{\alpha} \neq 0$) & ($\vec{\alpha} \cdot \vec{\varphi} \,=\, 0$)

General Observations

§ Forces and fields must be expressed with relational quantities (the Lorentz force must be expressed with the relational velocity $\mathbf{v}$, the electric field must be expressed with the relational position $\mathbf{r}$, etc.)

§ The operator ($\dot{\times}$) must be replaced by the operator ($\times$) or the operator ($\wedge$) as follows: ($\mathbf{a} \,\dot{\times}\, \mathbf{b} = \mathbf{b} \times \mathbf{a}$) or ($\mathbf{a} \,\dot{\times}\, \mathbf{b} = \mathbf{b} \wedge \mathbf{a}$)

§ Inertial and non-inertial observers must not introduce fictitious forces into $\mathbf{F}$.

§ According to this article and special relativity, intrinsic mass is not additive.

§ The intrinsic mass quantity (m) is invariant under transformations between inertial and non-inertial (all) frames.

§ The relational quantities ($\nu, t, \mathbf{r}, \mathbf{v}, \mathbf{a}$) are invariant under transformations between inertial and non-inertial (non-rotating) frames.

§ Therefore, the kinematic and dynamic quantities ($f, \bar{\mathbf{r}}, \bar{\mathbf{v}}, \bar{\mathbf{a}}, \mathbf{P}, \mathbf{L}, \mathbf{F}, \mathrm{W}, \mathrm{K}$) are invariant under transformations between inertial and non-inertial (non-rotating) frames.

§ However, it is natural to consider the following generalization:

• It would also be possible to obtain relational quantities ($\nu, t, \mathbf{r}, \mathbf{v}, \mathbf{a}$) that would be invariant under transformations between inertial and non-inertial (all) frames.

• The kinematic and dynamic quantities ($f, \bar{\mathbf{r}}, \bar{\mathbf{v}}, \bar{\mathbf{a}}, \mathbf{P}, \mathbf{L}, \mathbf{F}, \mathrm{W}, \mathrm{K}$) would also be given by the equations of this article.

• Therefore, the kinematic and dynamic quantities ($f, \bar{\mathbf{r}}, \bar{\mathbf{v}}, \bar{\mathbf{a}}, \mathbf{P}, \mathbf{L}, \mathbf{F}, \mathrm{W}, \mathrm{K}$) would be invariant under transformations between inertial and non-inertial (all) frames.

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The Twin Paradox

If a clock A is at rest at the origin O of an inertial or non-inertial (uniform circular motion) frame S and another clock B is at rest at the origin O' of a non-inertial (uniform circular motion) frame S' then the relational time t_A of clock A and the relational time t_B of clock B are given by:

$$t_A \;=\; \int_0^{\mathtt{t}_A} \gamma_{(\,\vec{\varphi}\,)}\; \mathtt{dt}_A \;-\; \gamma_{(\,\vec{\varphi}\,)}\; \frac{\vec{r}_A \cdot \vec{\varphi}}{c^2}$$

$$t_B \;=\; \int_0^{\mathtt{t}_B} \gamma_{(\,\vec{\varphi}\,')}\; \mathtt{dt}_B \;-\; \gamma_{(\,\vec{\varphi}\,')}\; \frac{\vec{r}_B \cdot \vec{\varphi}\,'}{c^2}$$

The position of the origin O relative to the frame S is always zero ($\vec{r}_A = 0$) and since $\gamma_{(\,\vec{\varphi}\,)}$ is a constant in the frame S, we obtain:

$$t_A \;=\; \int_0^{\mathtt{t}_A} \gamma_{(\,\vec{\varphi}\,)}\; \mathtt{dt}_A$$

$$t_A \;=\; \gamma_{(\,\vec{\varphi}\,)}\; \mathtt{t}_A$$

The position of the origin O' relative to the frame S' is always zero ($\vec{r}_B = 0$) and since $\gamma_{(\,\vec{\varphi}\,')}$ is a constant in the frame S', we obtain:

$$t_B \;=\; \int_0^{\mathtt{t}_B} \gamma_{(\,\vec{\varphi}\,')}\; \mathtt{dt}_B$$

$$t_B \;=\; \gamma_{(\,\vec{\varphi}\,')}\; \mathtt{t}_B$$

The clocks A and B spatially coincide at the relational time ($t_0 = t_{0A} = t_{0B}$) and at the relational time ($t = t_A = t_B$) Since ($t_A = t_B$) then we have:

$$\gamma_{(\,\vec{\varphi}\,)}\; \mathtt{t}_A \;=\; \gamma_{(\,\vec{\varphi}\,')}\; \mathtt{t}_B$$

Therefore, if $\gamma_{(\,\vec{\varphi}\,)} > \gamma_{(\,\vec{\varphi}\,')}$ then ($\mathtt{t}_A < \mathtt{t}_B$) if $\gamma_{(\,\vec{\varphi}\,)} = \gamma_{(\,\vec{\varphi}\,')}$ then ($\mathtt{t}_A = \mathtt{t}_B$) and if $\gamma_{(\,\vec{\varphi}\,)} < \gamma_{(\,\vec{\varphi}\,')}$ then ($\mathtt{t}_A > \mathtt{t}_B$)

Where ($\vec{\varphi}$) is the velocity of the auxiliary-point relative to the frame S and ($\vec{\varphi}\,'$) is the velocity of the auxiliary-point relative to the frame S'.

The Kinetic Force

The kinetic force $\mathbf{K}^a_{ij}$ exerted on a particle i with intrinsic mass m_i by another particle j with intrinsic mass m_j is given by:

$$\mathbf{K}^a_{ij} \;=\; -\left[\;\frac{m_i\,m_j}{\mathbb{M}}\,(\,\bar{\mathbf{a}}_i - \bar{\mathbf{a}}_j\,)\;\right]$$

where $\bar{\mathbf{a}}_i$ is the special acceleration of particle i, $\bar{\mathbf{a}}_j$ is the special acceleration of particle j and $\mathbb{M}$ ($= \sum_z m_z$) is the sum of the intrinsic masses of all the particles of the Universe.

The kinetic force $\mathbf{K}^u_i$ exerted on a particle i with intrinsic mass m_i by the Universe is given by:

$$\mathbf{K}^u_i \;=\; -\,m_i\,\frac{\sum_z m_z\,\bar{\mathbf{a}}_z}{\sum_z m_z}$$

where m_z and $\bar{\mathbf{a}}_z$ are the intrinsic mass and the special acceleration of the *z*-th particle of the Universe.

From the above equations it follows that the net kinetic force $\mathbf{K}_i$ ($= \sum_j\ \mathbf{K}^a_{ij}$ $+\ \mathbf{K}^u_i$) acting on a particle i with intrinsic mass m_i is given by:

$$\mathbf{K}_i \;=\; -\,m_i\,\bar{\mathbf{a}}_i$$

where $\bar{\mathbf{a}}_i$ is the special acceleration of particle i.

Now, substituting ($\mathbf{F}_i = m_i\,\bar{\mathbf{a}}_i$) and rearranging, we obtain:

$$\mathbf{K}_i + \mathbf{F}_i \;=\; 0$$

If we define $\mathbf{T}_i$ ($\doteq\ \mathbf{K}_i + \mathbf{F}_i$) as the total force acting on the particle i then:

$$\mathbf{T}_i \;=\; 0$$

Therefore, the total force $\mathbf{T}_i$ acting on any particle i is always zero.

On the other hand, if an observer uses an auxiliary-point such that the linear momentum of the Universe (that is, an isolated system of particles) is zero ($\sum_z m_z\,\bar{\mathbf{v}}_z = 0$) then for this observer the kinetic force $\mathbf{K}^u_i$ exerted on any particle i by the Universe is also zero, since ($\sum_z m_z\,\bar{\mathbf{a}}_z = 0$)

Appendix I

System of Equations I

$$
\begin{tikzcd}
 & \boxed{[1]} \arrow[d, "dt"] & \\
\boxed{[4]} \arrow[d, "dt"] & \boxed{[2]} \arrow[l, "\dot{\times}\,\mathbf{r}"'] \arrow[d, "dt"] & \\
\boxed{[5]} & \boxed{[3]} \arrow[l, "\dot{\times}\,\mathbf{r}"'] \arrow[r, "\int d\mathbf{r}"] & \boxed{[6]}
\end{tikzcd}
$$

$$[1] \qquad \frac{1}{\mu}\left[\int \mathbf{P}\,dt - \int\!\!\int \mathbf{F}\,dt\,dt\right] = 0$$

$$[2] \qquad \frac{1}{\mu}\left[\mathbf{P} - \int \mathbf{F}\,dt\right] = 0$$

$$[3] \qquad \frac{1}{\mu}\left[\frac{d\mathbf{P}}{dt} - \mathbf{F}\right] = 0$$

$$[4] \qquad \frac{1}{\mu}\left[\mathbf{P} - \int \mathbf{F}\,dt\right] \dot{\times}\,\mathbf{r} = 0$$

$$[5] \qquad \frac{1}{\mu}\left[\frac{d\mathbf{P}}{dt} - \mathbf{F}\right] \dot{\times}\,\mathbf{r} = 0$$

$$[6] \qquad \frac{1}{\mu}\left[\int \frac{d\mathbf{P}}{dt}\cdot d\mathbf{r} - \int \mathbf{F}\cdot d\mathbf{r}\right] = 0$$

$[\,\mu\,]$ is an arbitrary constant with dimension of mass (M)

Appendix II

System of Equations II

$$\begin{array}{ccccc}
 & & \boxed{[1]} & & \\
 & & \downarrow dt \downarrow & & \\
\boxed{[4]} & \leftarrow \dot{\times}\, \mathbf{r} \leftarrow & \boxed{[2]} & & \\
\downarrow dt \downarrow & & \downarrow dt \downarrow & & \\
\boxed{[5]} & \leftarrow \dot{\times}\, \mathbf{r} \leftarrow & \boxed{[3]} & \rightarrow \int d\mathbf{r} \rightarrow & \boxed{[6]}
\end{array}$$

$$[\,1\,] \qquad \frac{1}{\mu}\left[\, m\,\bar{\mathbf{r}} - \int\!\!\int \mathbf{F}\, dt\, dt \,\right] = 0$$

$$[\,2\,] \qquad \frac{1}{\mu}\left[\, m\,\bar{\mathbf{v}} - \int \mathbf{F}\, dt \,\right] = 0$$

$$[\,3\,] \qquad \frac{1}{\mu}\left[\, m\,\bar{\mathbf{a}} - \mathbf{F} \,\right] = 0$$

$$[\,4\,] \qquad \frac{1}{\mu}\left[\, m\,\bar{\mathbf{v}} - \int \mathbf{F}\, dt \,\right] \dot{\times}\, \mathbf{r} = 0$$

$$[\,5\,] \qquad \frac{1}{\mu}\left[\, m\,\bar{\mathbf{a}} - \mathbf{F} \,\right] \dot{\times}\, \mathbf{r} = 0$$

$$[\,6\,] \qquad \frac{1}{\mu}\left[\, m\, f\, c^2 - \int \mathbf{F} \cdot d\mathbf{r} \,\right] = 0$$

$[\,\mu\,]$ is an arbitrary constant with dimension of mass $(\,\mathrm{M}\,)$