

# NON-INERTIAL FRAMES IN SPECIAL RELATIVITY

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This article presents a new formulation of special relativity which is invariant under transformations between inertial and non-inertial ( non-rotating ) frames. Additionally, a simple solution to the twin paradox is presented and a new universal force is proposed.

## Introduction

The intrinsic mass ( $m$ ) and the frequency factor ( $f$ ) of a massive particle are given by:

$$m \doteq m_o$$

$$f \doteq \left(1 - \frac{\mathbf{v} \cdot \mathbf{v}}{c^2}\right)^{-1/2}$$

where ( $m_o$ ) is the rest mass of the massive particle, ( $\mathbf{v}$ ) is the relational velocity of the massive particle and ( $c$ ) is the speed of light in vacuum.

The intrinsic mass ( $m$ ) and the frequency factor ( $f$ ) of a non-massive particle are given by:

$$m \doteq \frac{h \kappa}{c^2}$$

$$f \doteq \frac{\nu}{\kappa}$$

where ( $h$ ) is the Planck constant, ( $\nu$ ) is the relational frequency of the non-massive particle, ( $\kappa$ ) is a positive universal constant with dimension of frequency and ( $c$ ) is the speed of light in vacuum.

In this article, a massive particle is a particle with non-zero rest mass and a non-massive particle is a particle with zero rest mass.

## The Invariant Kinematics

The special position (  $\bar{\mathbf{r}}$  ), the special velocity (  $\bar{\mathbf{v}}$  ) and the special acceleration (  $\bar{\mathbf{a}}$  ) of a ( massive or non-massive ) particle are given by:

$$\bar{\mathbf{r}} \doteq \int f \mathbf{v} dt$$

$$\bar{\mathbf{v}} \doteq \frac{d\bar{\mathbf{r}}}{dt} = f \mathbf{v}$$

$$\bar{\mathbf{a}} \doteq \frac{d\bar{\mathbf{v}}}{dt} = f \frac{d\mathbf{v}}{dt} + \frac{df}{dt} \mathbf{v}$$

where (  $f$  ) is the frequency factor of the particle, (  $\mathbf{v}$  ) is the relational velocity of the particle and (  $t$  ) is the relational time of the particle.

## The Invariant Dynamics

If we consider a ( massive or non-massive ) particle with intrinsic mass (  $m$  ) then the linear momentum (  $\mathbf{P}$  ) of the particle, the angular momentum (  $\mathbf{L}$  ) of the particle, the net force (  $\mathbf{F}$  ) acting on the particle, the work (  $W$  ) done by the net force acting on the particle, and the kinetic energy (  $K$  ) of the particle are given by:

$$\mathbf{P} \doteq m \bar{\mathbf{v}} = m f \mathbf{v}$$

$$\mathbf{L} \doteq \mathbf{P} \dot{\times} \mathbf{r} = m \bar{\mathbf{v}} \dot{\times} \mathbf{r} = m f \mathbf{v} \dot{\times} \mathbf{r}$$

$$\mathbf{F} = \frac{d\mathbf{P}}{dt} = m \bar{\mathbf{a}} = m \left[ f \frac{d\mathbf{v}}{dt} + \frac{df}{dt} \mathbf{v} \right]$$

$$W \doteq \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 \frac{d\mathbf{P}}{dt} \cdot d\mathbf{r} = \Delta K$$

$$K \doteq m f c^2$$

where (  $f$ ,  $\mathbf{r}$ ,  $\mathbf{v}$ ,  $t$ ,  $\bar{\mathbf{v}}$ ,  $\bar{\mathbf{a}}$  ) are the frequency factor, the relational position, the relational velocity, the relational time, the special velocity and the special acceleration of the particle and (  $c$  ) is the speed of light in vacuum. The kinetic energy (  $K_o$  ) of a massive particle at relational rest is (  $m_o c^2$  )

## Relational Quantities

From an auxiliary massive particle ( called auxiliary-point ) some kinematic quantities ( called relational quantities ) can be obtained. These are invariant under transformations between inertial and non-inertial (non-rotating) frames.

An auxiliary-point is an arbitrary massive particle that is free of forces ( that is, the net force acting on it is zero )

The relational time (  $t$  ), the relational position (  $\mathbf{r}$  ), the relational velocity (  $\mathbf{v}$  ) and the relational acceleration (  $\mathbf{a}$  ) of a (massive or non-massive) particle relative to an inertial or non-inertial (non-rotating) frame S are given by:

$$t \doteq \int_0^{\mathbf{t}} \gamma \, d\mathbf{t} - \gamma \frac{\vec{r} \cdot \vec{\varphi}}{c^2}$$

$$\mathbf{r} \doteq \vec{r} + \frac{\gamma^2}{\gamma + 1} \frac{(\vec{r} \cdot \vec{\varphi}) \vec{\varphi}}{c^2} - \int_0^{\mathbf{t}} \gamma \vec{\varphi} \, d\mathbf{t}$$

$$\mathbf{v} \doteq \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{dt} \frac{d\mathbf{t}}{d\mathbf{t}} = \frac{d\mathbf{r}}{d\mathbf{t}} \frac{d\mathbf{t}}{dt} = \left( \frac{d\mathbf{r}}{d\mathbf{t}} \right) \left( \frac{1}{dt/d\mathbf{t}} \right)$$

$$\mathbf{a} \doteq \frac{d\mathbf{v}}{dt} = \frac{d\mathbf{v}}{dt} \frac{d\mathbf{t}}{d\mathbf{t}} = \frac{d\mathbf{v}}{d\mathbf{t}} \frac{d\mathbf{t}}{dt} = \left( \frac{d\mathbf{v}}{d\mathbf{t}} \right) \left( \frac{1}{dt/d\mathbf{t}} \right)$$

where (  $\mathbf{t}$ ,  $\vec{r}$  ) are the time and the position of the particle relative to the frame S, (  $\vec{\varphi}$  ) is the velocity of the auxiliary-point relative to the frame S and (  $c$  ) is the speed of light in vacuum. (  $\vec{\varphi}$  ) is a constant in inertial frames, (  $\gamma$  ) is a constant in non-inertial ( uniform circular motion ) frames, and  $\gamma \doteq (1 - \vec{\varphi} \cdot \vec{\varphi} / c^2)^{-1/2}$

The relational frequency (  $\nu$  ) of a non-massive particle relative to an inertial or non-inertial (non-rotating) frame S is given by:

$$\nu \doteq \mathbf{v} \frac{\left( 1 - \frac{\vec{c} \cdot \vec{\varphi}}{c^2} \right)}{\sqrt{1 - \frac{\vec{\varphi} \cdot \vec{\varphi}}{c^2}}}$$

where (  $\mathbf{v}$  ) is the frequency of the non-massive particle relative to the frame S, (  $\vec{c}$  ) is the velocity of the non-massive particle relative to the frame S, (  $\vec{\varphi}$  ) is the velocity of the auxiliary-point relative to the frame S and (  $c$  ) is the speed of light in vacuum.

§ In arbitrary frames ( $t_\alpha \neq \tau_\alpha$  or  $\mathbf{r}_\alpha \neq 0$ ) ( $\alpha$  = auxiliary-point ) a constant must be add in the definition of relational time such that the relational time and the proper time of the auxiliary-point are the same ( $t_\alpha = \tau_\alpha$ ) and another constant must be add in the definition of relational position such that the relational position of the auxiliary-point is zero ( $\mathbf{r}_\alpha = 0$ )

§ In the particular case of an isolated system of ( massive or non-massive ) particles, all observers should preferably use an auxiliary-point such that the linear momentum of the isolated system of particles is zero ( $\sum_z m_z \bar{\mathbf{v}}_z = 0$ )

§ It is important to emphasize that any auxiliary-point must be a free massive particle ( that is, the net force acting on it must be zero )

## Relational Metric

It is known that in inertial frames the local geometry is Euclidean and that in non-inertial (non-rotating) frames the local geometry is non-Euclidean.

According to this article, in an inertial or non-inertial (non-rotating) frame S the local line element must be obtained from the relational line element.

Therefore, in the frame S the relational line element ( in rectilinear coordinates ) and the local line element are given by:

$$ds^2 = c^2 dt^2 - d\mathbf{r}^2$$

$$ds^2 = \left[ \left( 1 - \frac{\vec{\mathbf{w}} \cdot \vec{\mathbf{r}}}{c^2} \right)^2 - \left( \frac{\vec{\mathbf{\varnothing}} \times \vec{\mathbf{r}}}{c} \right)^2 \right] c^2 d\mathbf{t}^2 - 2 \left( \vec{\mathbf{\varnothing}} \times \vec{\mathbf{r}} \right) d\vec{\mathbf{r}} d\mathbf{t} - d\vec{\mathbf{r}}^2$$

$$\vec{\mathbf{w}} \doteq \gamma^2 \vec{\mathbf{\alpha}} + \frac{\gamma^4}{\gamma + 1} \frac{(\vec{\mathbf{\varphi}} \cdot \vec{\mathbf{\alpha}}) \vec{\mathbf{\varphi}}}{c^2}$$

$$\vec{\mathbf{\varnothing}} \doteq \frac{\gamma^3}{\gamma + 1} \frac{(\vec{\mathbf{\varphi}} \times \vec{\mathbf{\alpha}})}{c^2}$$

where ( $t, \mathbf{r}$ ) are relational time and relational position relative to the frame S, ( $\mathbf{t}, \vec{\mathbf{r}}$ ) are time and position relative to the frame S, ( $\vec{\mathbf{\varphi}}, \vec{\mathbf{\alpha}}$ ) are the velocity and the acceleration of the auxiliary-point relative to the frame S and ( $c$ ) is the speed of light in vacuum. ( $\vec{\mathbf{\alpha}}$ ) is zero in inertial frames, ( $\vec{\mathbf{\varphi}} \times \vec{\mathbf{\alpha}}$ ) is zero in non-inertial ( linear accelerated motion ) frames, and  $\gamma \doteq (1 - \vec{\mathbf{\varphi}} \cdot \vec{\mathbf{\varphi}}/c^2)^{-1/2}$

## General Observations

§ Forces and fields must be expressed with relational quantities ( the Lorentz force must be expressed with the relational velocity  $\mathbf{v}$ , the electric field must be expressed with the relational position  $\mathbf{r}$ , etc. )

§ The operator (  $\dot{\times}$  ) must be replaced by the operator (  $\times$  ) or the operator (  $\wedge$  ) as follows: (  $\mathbf{a} \dot{\times} \mathbf{b} = \mathbf{b} \times \mathbf{a}$  ) or (  $\mathbf{a} \dot{\times} \mathbf{b} = \mathbf{b} \wedge \mathbf{a}$  )

§ Inertial and non-inertial observers must not introduce fictitious forces into  $\mathbf{F}$ .

§ According to this article and special relativity, intrinsic mass is not additive.

§ The intrinsic mass quantity (  $m$  ) is invariant under transformations between inertial and non-inertial (all) frames.

§ The relational quantities (  $\nu, t, \mathbf{r}, \mathbf{v}, \mathbf{a}$  ) are invariant under transformations between inertial and non-inertial (non-rotating) frames.

§ Therefore, the kinematic and dynamic quantities (  $f, \bar{\mathbf{r}}, \bar{\mathbf{v}}, \bar{\mathbf{a}}, \mathbf{P}, \mathbf{L}, \mathbf{F}, W, K$  ) are invariant under transformations between inertial and non-inertial (non-rotating) frames.

§ However, it is natural to consider the following generalization:

- It would also be possible to obtain relational quantities (  $\nu, t, \mathbf{r}, \mathbf{v}, \mathbf{a}$  ) that would be invariant under transformations between inertial and non-inertial (all) frames.
- The kinematic and dynamic quantities (  $f, \bar{\mathbf{r}}, \bar{\mathbf{v}}, \bar{\mathbf{a}}, \mathbf{P}, \mathbf{L}, \mathbf{F}, W, K$  ) would also be given by the equations of this article.
- Therefore, the kinematic and dynamic quantities (  $f, \bar{\mathbf{r}}, \bar{\mathbf{v}}, \bar{\mathbf{a}}, \mathbf{P}, \mathbf{L}, \mathbf{F}, W, K$  ) would be invariant under transformations between inertial and non-inertial (all) frames.

## Bibliography

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## The Twin Paradox

The clock A is at rest at the origin O of an inertial or non-inertial (uniform circular motion) frame S.

$$t_A = \int_0^{\tau_A} \gamma(\vec{\varphi}) d\tau_A - \gamma(\vec{\varphi}) \frac{\vec{r}_A \cdot \vec{\varphi}}{c^2}$$

Another clock B is at rest at the origin O' of another non-inertial (uniform circular motion) frame S'.

$$t_B = \int_0^{\tau_B} \gamma(\vec{\varphi}') d\tau_B - \gamma(\vec{\varphi}') \frac{\vec{r}_B \cdot \vec{\varphi}'}{c^2}$$

The origin O relative to the frame S always equals zero (  $\vec{r}_A = 0$  ) and since  $\gamma(\vec{\varphi})$  is a constant in the frame S, then

$$t_A = \int_0^{\tau_A} \gamma(\vec{\varphi}) d\tau_A$$

$$t_A = \gamma(\vec{\varphi}) \tau_A$$

The origin O' relative to the frame S' always equals zero (  $\vec{r}_B = 0$  ) and since  $\gamma(\vec{\varphi}')$  is a constant in the frame S', then

$$t_B = \int_0^{\tau_B} \gamma(\vec{\varphi}') d\tau_B$$

$$t_B = \gamma(\vec{\varphi}') \tau_B$$

The origins O and O' spatially coincide at the relational time (  $t_0 = t_{0A} = t_{0B}$  ) and at the relational time (  $t = t_A = t_B$  ) Since (  $t_A = t_B$  ) then

$$\gamma(\vec{\varphi}) \tau_A = \gamma(\vec{\varphi}') \tau_B$$

Therefore, if (  $\vec{\varphi} > \vec{\varphi}'$  ) then (  $\tau_A < \tau_B$  ), if (  $\vec{\varphi} = \vec{\varphi}'$  ) then (  $\tau_A = \tau_B$  ) and if (  $\vec{\varphi} < \vec{\varphi}'$  ) then (  $\tau_A > \tau_B$  )

Where (  $\vec{\varphi}$  ) is the velocity of the auxiliary-point relative to the frame S and (  $\vec{\varphi}'$  ) is the velocity of the auxiliary-point relative to the frame S'.

## The Kinetic Force

The kinetic force  $\mathbf{K}_{ij}^a$  exerted on a particle  $i$  with intrinsic mass  $m_i$  by another particle  $j$  with intrinsic mass  $m_j$  is given by:

$$\mathbf{K}_{ij}^a = - \left[ \frac{m_i m_j}{\mathbb{M}} (\bar{\mathbf{a}}_i - \bar{\mathbf{a}}_j) \right]$$

where  $\bar{\mathbf{a}}_i$  is the special acceleration of particle  $i$ ,  $\bar{\mathbf{a}}_j$  is the special acceleration of particle  $j$  and  $\mathbb{M} (= \sum_z m_z)$  is the sum of the intrinsic masses of all the particles of the Universe.

The kinetic force  $\mathbf{K}_i^u$  exerted on a particle  $i$  with intrinsic mass  $m_i$  by the Universe is given by:

$$\mathbf{K}_i^u = - m_i \frac{\sum_z m_z \bar{\mathbf{a}}_z}{\sum_z m_z}$$

where  $m_z$  and  $\bar{\mathbf{a}}_z$  are the intrinsic mass and the special acceleration of the  $z$ -th particle of the Universe.

From the above equations it follows that the net kinetic force  $\mathbf{K}_i (= \sum_j \mathbf{K}_{ij}^a + \mathbf{K}_i^u)$  acting on a particle  $i$  with intrinsic mass  $m_i$  is given by:

$$\mathbf{K}_i = - m_i \bar{\mathbf{a}}_i$$

where  $\bar{\mathbf{a}}_i$  is the special acceleration of particle  $i$ .

Now, substituting ( $\mathbf{F}_i = m_i \bar{\mathbf{a}}_i$ ) and rearranging, we obtain:

$$\mathbf{K}_i + \mathbf{F}_i = 0$$

If we define  $\mathbf{T}_i (= \mathbf{K}_i + \mathbf{F}_i)$  as the total force acting on the particle  $i$  then:

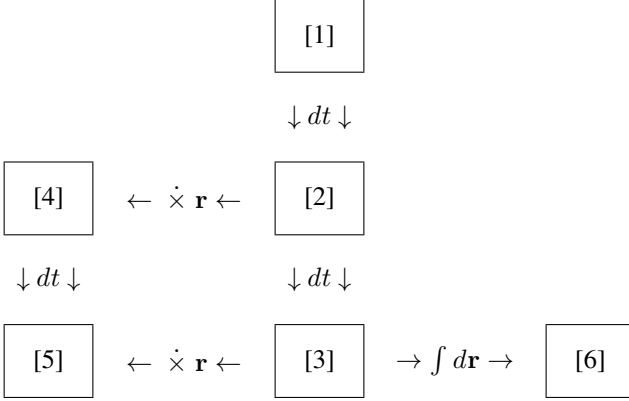
$$\mathbf{T}_i = 0$$

Therefore, the total force  $\mathbf{T}_i$  acting on a particle  $i$  is always zero.

On the other hand, if an observer uses an auxiliary-point such that the linear momentum of the Universe ( that is, an isolated system of particles ) is zero ( $\sum_z m_z \bar{\mathbf{v}}_z = 0$ ) then the kinetic force  $\mathbf{K}_i^u$  exerted on any particle  $i$  by the Universe is zero too, since ( $\sum_z m_z \bar{\mathbf{a}}_z = 0$ )

## Appendix I

### System of Equations I



$$[1] \quad \frac{1}{\mu} \left[ \int \mathbf{P} dt - \iint \mathbf{F} dt dt \right] = 0$$

$$[2] \quad \frac{1}{\mu} \left[ \mathbf{P} - \int \mathbf{F} dt \right] = 0$$

$$[3] \quad \frac{1}{\mu} \left[ \frac{d\mathbf{P}}{dt} - \mathbf{F} \right] = 0$$

$$[4] \quad \frac{1}{\mu} \left[ \mathbf{P} - \int \mathbf{F} dt \right] \dot{\times} \mathbf{r} = 0$$

$$[5] \quad \frac{1}{\mu} \left[ \frac{d\mathbf{P}}{dt} - \mathbf{F} \right] \dot{\times} \mathbf{r} = 0$$

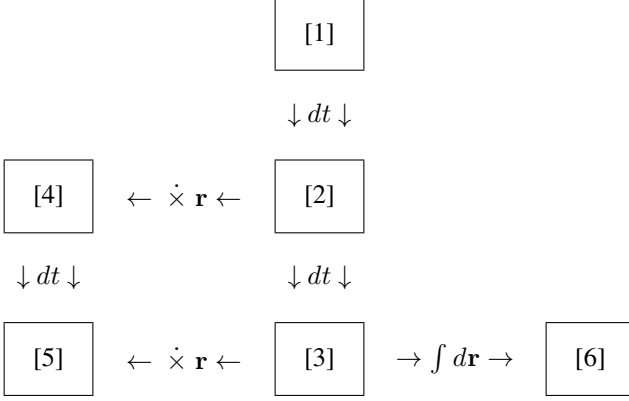
$$[6] \quad \frac{1}{\mu} \left[ \int \frac{d\mathbf{P}}{dt} \cdot d\mathbf{r} - \int \mathbf{F} \cdot d\mathbf{r} \right] = 0$$

$[\mu]$  is an arbitrary constant with dimension of mass (M)



## Appendix II

### System of Equations II



$$[1] \quad \frac{1}{\mu} \left[ m \bar{\mathbf{r}} - \iiint \mathbf{F} dt dt \right] = 0$$

$$[2] \quad \frac{1}{\mu} \left[ m \bar{\mathbf{v}} - \int \mathbf{F} dt \right] = 0$$

$$[3] \quad \frac{1}{\mu} \left[ m \bar{\mathbf{a}} - \mathbf{F} \right] = 0$$

$$[4] \quad \frac{1}{\mu} \left[ m \bar{\mathbf{v}} - \int \mathbf{F} dt \right] \dot{\times} \mathbf{r} = 0$$

$$[5] \quad \frac{1}{\mu} \left[ m \bar{\mathbf{a}} - \mathbf{F} \right] \dot{\times} \mathbf{r} = 0$$

$$[6] \quad \frac{1}{\mu} \left[ m f c^2 - \int \mathbf{F} \cdot d\mathbf{r} \right] = 0$$

$[\mu]$  is an arbitrary constant with dimension of mass (M)