# Non-Inertial Frames in Special Relativity

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This article presents a new formulation of special relativity which is invariant under transformations between inertial and non-inertial ( non-rotating ) frames. Additionally, a simple solution to the twin paradox is presented and a new universal force is proposed.

## Introduction

The intrinsic mass ( $m$ ) and the frequency factor ( $f$ ) of a massive particle are given by:

$$m \doteq m_o$$

$$f \doteq \left(1 - \frac{\mathbf{v} \cdot \mathbf{v}}{c^2}\right)^{-1/2}$$

where ( $m_o$ ) is the rest mass of the massive particle, ( $\mathbf{v}$ ) is the relational velocity of the massive particle and ( $c$ ) is the speed of light in vacuum.

The intrinsic mass ( $m$ ) and the frequency factor ( $f$ ) of a non-massive particle are given by:

$$m \doteq \frac{h\,\kappa}{c^2}$$

$$f \doteq \frac{\nu}{\kappa}$$

where ( $h$ ) is the Planck constant, ( $\nu$ ) is the relational frequency of the non-massive particle, ( $\kappa$ ) is a positive universal constant with dimension of frequency and ( $c$ ) is the speed of light in vacuum.

In this article, a massive particle is a particle with non-zero rest mass and a non-massive particle is a particle with zero rest mass.

## The Invariant Kinematics

The special position ( $\bar{\mathbf{r}}$ ), the special velocity ( $\bar{\mathbf{v}}$ ) and the special acceleration ( $\bar{\mathbf{a}}$ ) of a ( massive or non-massive ) particle are given by:

$$\bar{\mathbf{r}} \doteq \int f\,\mathbf{v}\,dt$$

$$\bar{\mathbf{v}} \doteq \frac{d\bar{\mathbf{r}}}{dt} = f\,\mathbf{v}$$

$$\bar{\mathbf{a}} \doteq \frac{d\bar{\mathbf{v}}}{dt} = f\,\frac{d\mathbf{v}}{dt} + \frac{df}{dt}\,\mathbf{v}$$

where ( $f$ ) is the frequency factor of the particle, ( $\mathbf{v}$ ) is the relational velocity of the particle and ( $t$ ) is the relational time of the particle.

## The Invariant Dynamics

If we consider a ( massive or non-massive ) particle with intrinsic mass ( $m$ ) then the linear momentum ( $\mathbf{P}$ ) of the particle, the angular momentum ( $\mathbf{L}$ ) of the particle, the net force ( $\mathbf{F}$ ) acting on the particle, the work ( $\mathrm{W}$ ) done by the net force acting on the particle, and the kinetic energy ( $\mathrm{K}$ ) of the particle are given by:

$$\mathbf{P} \doteq m\,\bar{\mathbf{v}} = m\,f\,\mathbf{v}$$

$$\mathbf{L} \doteq \mathbf{P} \dot{\times} \mathbf{r} = m\,\bar{\mathbf{v}} \dot{\times} \mathbf{r} = m\,f\,\mathbf{v} \dot{\times} \mathbf{r}$$

$$\mathbf{F} = \frac{d\mathbf{P}}{dt} = m\,\bar{\mathbf{a}} = m \left[ f\,\frac{d\mathbf{v}}{dt} + \frac{df}{dt}\,\mathbf{v} \right]$$

$$\mathrm{W} \doteq \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 \frac{d\mathbf{P}}{dt} \cdot d\mathbf{r} = \Delta\,\mathrm{K}$$

$$\mathrm{K} \doteq m\,f\,c^2$$

where ( $f$, $\mathbf{r}$, $\mathbf{v}$, $t$, $\bar{\mathbf{v}}$, $\bar{\mathbf{a}}$ ) are the frequency factor, the relational position, the relational velocity, the relational time, the special velocity and the special acceleration of the particle and ( $c$ ) is the speed of light in vacuum. The kinetic energy ( $\mathrm{K}_o$ ) of a massive particle at relational rest is ( $m_o\,c^2$ )

**Relational Quantities**

From an auxiliary massive particle ( called auxiliary-point ) some kinematic quantities ( called relational quantities ) can be obtained. These are invariant under transformations between inertial and non-inertial (non-rotating) frames.

An auxiliary-point is an arbitrary massive particle that is free of forces ( that is, the net force acting on it is zero )

The relational time $(\, t\,)$, the relational position $(\, \mathbf{r}\,)$, the relational velocity $(\, \mathbf{v}\,)$ and the relational acceleration $(\, \mathbf{a}\,)$ of a (massive or non-massive) particle relative to an inertial or non-inertial (non-rotating) frame S are given by:

$$t \doteq \int_{\mathtt{0}}^{\mathtt{t}} \gamma \, \mathtt{dt} - \gamma \, \frac{\vec{r} \cdot \vec{\varphi}}{c^2}$$

$$\mathbf{r} \doteq \vec{r} + \frac{\gamma^2}{\gamma + 1} \frac{(\vec{r} \cdot \vec{\varphi})\, \vec{\varphi}}{c^2} - \int_{\mathtt{0}}^{\mathtt{t}} \gamma \, \vec{\varphi} \, \mathtt{dt}$$

$$\mathbf{v} \doteq \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{dt} \frac{\mathtt{dt}}{\mathtt{dt}} = \frac{d\mathbf{r}}{\mathtt{dt}} \frac{\mathtt{dt}}{dt} = \left( \frac{d\mathbf{r}}{\mathtt{dt}} \right) \left( \frac{1}{dt \,/\, \mathtt{dt}} \right)$$

$$\mathbf{a} \doteq \frac{d\mathbf{v}}{dt} = \frac{d\mathbf{v}}{dt} \frac{\mathtt{dt}}{\mathtt{dt}} = \frac{d\mathbf{v}}{\mathtt{dt}} \frac{\mathtt{dt}}{dt} = \left( \frac{d\mathbf{v}}{\mathtt{dt}} \right) \left( \frac{1}{dt \,/\, \mathtt{dt}} \right)$$

where $(\, \mathtt{t},\, \vec{r}\,)$ are the time and the position of the particle relative to the frame S, $(\, \vec{\varphi}\,)$ is the velocity of the auxiliary-point relative to the frame S and $(\, c\,)$ is the speed of light in vacuum. $(\, \vec{\varphi}\,)$ is a constant in inertial frames, $(\, \gamma\,)$ is a constant in non-inertial ( uniform circular motion ) frames, and $\gamma \doteq (\, 1 - \vec{\varphi} \cdot \vec{\varphi} / c^2\,)^{-1/2}$

The relational frequency $(\, \nu\,)$ of a non-massive particle relative to an inertial or non-inertial (non-rotating) frame S is given by:

$$\nu \doteq \mathtt{v} \, \frac{\left( 1 - \dfrac{\vec{c} \cdot \vec{\varphi}}{c^2} \right)}{\sqrt{1 - \dfrac{\vec{\varphi} \cdot \vec{\varphi}}{c^2}}}$$

where $(\, \mathtt{v}\,)$ is the frequency of the non-massive particle relative to the frame S, $(\, \vec{c}\,)$ is the velocity of the non-massive particle relative to the frame S, $(\, \vec{\varphi}\,)$ is the velocity of the auxiliary-point relative to the frame S and $(\, c\,)$ is the speed of light in vacuum.

§ In arbitrary frames ( $t_\alpha \neq \tau_\alpha$ or $\mathbf{r}_\alpha \neq 0$ ) ( $\alpha$ = auxiliary-point ) a constant must be add in the definition of relational time such that the relational time and the proper time of the auxiliary-point are the same ( $t_\alpha = \tau_\alpha$ ) and another constant must be add in the definition of relational position such that the relational position of the auxiliary-point is zero ( $\mathbf{r}_\alpha = 0$ )

§ In the particular case of an isolated system of ( massive or non-massive ) particles, all observers should preferably use an auxiliary-point such that the linear momentum of the isolated system of particles is zero ( $\sum_z m_z \bar{\mathbf{v}}_z = 0$ )

§ It is important to emphasize that any auxiliary-point must be a free massive particle ( that is, the net force acting on it must be zero )

## Relational Metric

It is known that in inertial frames the local geometry is Euclidean and that in non-inertial (non-rotating) frames the local geometry is non-Euclidean.

According to this article, in an inertial or non-inertial (non-rotating) frame S the local line element must be obtained from the relational line element.

Therefore, in the frame S the relational line element ( in rectilinear coordinates ) and the local line element are given by:

$$ds^2 \;=\; dt^2 - d\mathbf{r}^2$$

$$ds^2 \;=\; \Big[\, \Big( 1 - \frac{\vec{\mathrm{w}} \cdot \vec{r}}{c^2} \Big)^2 - \Big( \frac{\vec{\varnothing} \times \vec{r}}{c} \Big)^2 \,\Big]\, c^2\, \mathtt{dt}^2 \;-\; 2\, \Big( \vec{\varnothing} \times \vec{r} \Big)\, d\vec{r}\; \mathtt{dt} \;-\; d\vec{r}^{\,2}$$

$$\vec{\mathrm{w}} \;\doteq\; \gamma^2\, \vec{\alpha} \,+\, \frac{\gamma^4}{\gamma+1}\, \frac{(\vec{\varphi} \cdot \vec{\alpha})\, \vec{\varphi}}{c^2}$$

$$\vec{\varnothing} \;\doteq\; \frac{\gamma^3}{\gamma+1}\, \frac{(\vec{\varphi} \times \vec{\alpha})}{c^2}$$

where $(\, t,\, \mathbf{r}\,)$ are relational time and relational position relative to the frame S, $(\, \mathtt{t},\, \vec{r}\,)$ are time and position relative to the frame S, $(\, \vec{\varphi},\, \vec{\alpha}\,)$ are the velocity and the acceleration of the auxiliary-point relative to the frame S and $(\, c\,)$ is the speed of light in vacuum. $(\, \vec{\alpha}\,)$ is zero in inertial frames, $(\, \vec{\varphi} \times \vec{\alpha}\,)$ is zero in non-inertial ( linear accelerated motion ) frames, and $\gamma \,\doteq\, (\, 1 - \vec{\varphi} \cdot \vec{\varphi} / c^2\,)^{-1/2}$

## General Observations

§ Forces and fields must be expressed with relational quantities ( the Lorentz force must be expressed with the relational velocity $\mathbf{v}$, the electric field must be expressed with the relational position $\mathbf{r}$, etc. )

§ The operator ( $\dot{\times}$ ) must be replaced by the operator ( $\times$ ) or the operator ( $\wedge$ ) as follows: ( $\mathbf{a} \dot{\times} \mathbf{b} = \mathbf{b} \times \mathbf{a}$ ) or ( $\mathbf{a} \dot{\times} \mathbf{b} = \mathbf{b} \wedge \mathbf{a}$ )

§ Inertial and non-inertial observers must not introduce fictitious forces into $\mathbf{F}$.

§ According to this article and special relativity, intrinsic mass is not additive.

§ The intrinsic mass quantity ( $m$ ) is invariant under transformations between inertial and non-inertial (all) frames.

§ The relational quantities ( $\nu, t, \mathbf{r}, \mathbf{v}, \mathbf{a}$ ) are invariant under transformations between inertial and non-inertial (non-rotating) frames.

§ Therefore, the kinematic and dynamic quantities ($f, \bar{\mathbf{r}}, \bar{\mathbf{v}}, \bar{\mathbf{a}}, \mathbf{P}, \mathbf{L}, \mathbf{F}, \mathrm{W}, \mathrm{K}$) are invariant under transformations between inertial and non-inertial (non-rotating) frames.

§ However, it is natural to consider the following generalization:

• It would also be possible to obtain relational quantities ( $\nu, t, \mathbf{r}, \mathbf{v}, \mathbf{a}$ ) that would be invariant under transformations between inertial and non-inertial (all) frames.

• The kinematic and dynamic quantities ( $f, \bar{\mathbf{r}}, \bar{\mathbf{v}}, \bar{\mathbf{a}}, \mathbf{P}, \mathbf{L}, \mathbf{F}, \mathrm{W}, \mathrm{K}$ ) would also be given by the equations of this article.

• Therefore, the kinematic and dynamic quantities ( $f, \bar{\mathbf{r}}, \bar{\mathbf{v}}, \bar{\mathbf{a}}, \mathbf{P}, \mathbf{L}, \mathbf{F}, \mathrm{W}, \mathrm{K}$ ) would be invariant under transformations between inertial and non-inertial (all) frames.

## Bibliography


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**The Twin Paradox**

The clock A is at rest at the origin O of an inertial or non-inertial (uniform circular motion) frame S.

$$t_A \;=\; \int_0^{\mathtt{t}_A} \gamma_{(\,\vec{\varphi}\,)}\;\mathtt{dt}_A \;-\; \gamma_{(\,\vec{\varphi}\,)}\;\frac{\vec{r}_A\cdot\vec{\varphi}}{c^2}$$

Another clock B is at rest at the origin O' of another non-inertial (uniform circular motion) frame S'.

$$t_B \;=\; \int_0^{\mathtt{t}_B} \gamma_{(\,\vec{\varphi}\,')}\;\mathtt{dt}_B \;-\; \gamma_{(\,\vec{\varphi}\,')}\;\frac{\vec{r}_B\cdot\vec{\varphi}\,'}{c^2}$$

The origin O relative to the frame S always equals zero ( $\vec{r}_A = 0$ ) and since $\gamma_{(\,\vec{\varphi}\,)}$ is a constant in the frame S, then

$$t_A \;=\; \int_0^{\mathtt{t}_A} \gamma_{(\,\vec{\varphi}\,)}\;\mathtt{dt}_A$$

$$t_A \;=\; \gamma_{(\,\vec{\varphi}\,)}\;\mathtt{t}_A$$

The origin O' relative to the frame S' always equals zero ( $\vec{r}_B = 0$ ) and since $\gamma_{(\,\vec{\varphi}\,')}$ is a constant in the frame S', then

$$t_B \;=\; \int_0^{\mathtt{t}_B} \gamma_{(\,\vec{\varphi}\,')}\;\mathtt{dt}_B$$

$$t_B \;=\; \gamma_{(\,\vec{\varphi}\,')}\;\mathtt{t}_B$$

The origins O and O' spatially coincide at the relational time ( $t_0 = t_{0A} = t_{0B}$ ) and at the relational time ( $t = t_A = t_B$ ) Since ( $t_A = t_B$ ) then

$$\gamma_{(\,\vec{\varphi}\,)}\;\mathtt{t}_A \;=\; \gamma_{(\,\vec{\varphi}\,')}\;\mathtt{t}_B$$

Therefore, if ( $\vec{\varphi} > \vec{\varphi}\,'$ ) then ( $\mathtt{t}_A < \mathtt{t}_B$ ), if ( $\vec{\varphi} = \vec{\varphi}\,'$ ) then ( $\mathtt{t}_A = \mathtt{t}_B$ ) and if ( $\vec{\varphi} < \vec{\varphi}\,'$ ) then ( $\mathtt{t}_A > \mathtt{t}_B$ )

Where ( $\vec{\varphi}$ ) is the velocity of the auxiliary-point relative to the frame S and ( $\vec{\varphi}\,'$ ) is the velocity of the auxiliary-point relative to the frame S'.

## The Kinetic Force

The kinetic force $\mathbf{K}^a_{ij}$ exerted on a particle $i$ with intrinsic mass $m_i$ by another particle $j$ with intrinsic mass $m_j$ is given by:

$$\mathbf{K}^a_{ij} \; = \; - \left[ \; \frac{m_i \, m_j}{\mathbb{M}} \, ( \, \bar{\mathbf{a}}_i - \bar{\mathbf{a}}_j \, ) \; \right]$$

where $\bar{\mathbf{a}}_i$ is the special acceleration of particle $i$, $\bar{\mathbf{a}}_j$ is the special acceleration of particle $j$ and $\mathbb{M}$ ( $= \sum_z m_z$ ) is the sum of the intrinsic masses of all the particles of the Universe.

The kinetic force $\mathbf{K}^u_i$ exerted on a particle $i$ with intrinsic mass $m_i$ by the Universe is given by:

$$\mathbf{K}^u_i \; = \; - \, m_i \; \frac{\sum_z m_z \, \bar{\mathbf{a}}_z}{\sum_z m_z}$$

where $m_z$ and $\bar{\mathbf{a}}_z$ are the intrinsic mass and the special acceleration of the *z*-th particle of the Universe.

From the above equations it follows that the net kinetic force $\mathbf{K}_i$ ( $= \sum_j \mathbf{K}^a_{ij}$ $+ \mathbf{K}^u_i$ ) acting on a particle $i$ with intrinsic mass $m_i$ is given by:

$$\mathbf{K}_i \; = \; - \, m_i \, \bar{\mathbf{a}}_i$$

where $\bar{\mathbf{a}}_i$ is the special acceleration of particle $i$.

Now, substituting ( $\mathbf{F}_i = m_i \, \bar{\mathbf{a}}_i$ ) and rearranging, we obtain:

$$\mathbf{K}_i + \mathbf{F}_i \; = \; 0$$

If we define $\mathbf{T}_i$ ( $\doteq$ $\mathbf{K}_i + \mathbf{F}_i$ ) as the total force acting on the particle $i$ then:

$$\mathbf{T}_i \; = \; 0$$

Therefore, the total force $\mathbf{T}_i$ acting on a particle $i$ is always zero.

On the other hand, if an observer uses an auxiliary-point such that the linear momentum of the Universe ( that is, an isolated system of particles ) is zero ( $\sum_z m_z \, \bar{\mathbf{v}}_z = 0$ ) then the kinetic force $\mathbf{K}^u_i$ exerted on any particle $i$ by the Universe is zero too, since ( $\sum_z m_z \, \bar{\mathbf{a}}_z = 0$ )

## Appendix I

### System of Equations I

$$\begin{array}{ccccc} & & \boxed{[1]} & & \\ & & \downarrow dt \downarrow & & \\ \boxed{[4]} & \leftarrow \dot{\times}\,\mathbf{r} \leftarrow & \boxed{[2]} & & \\ \downarrow dt \downarrow & & \downarrow dt \downarrow & & \\ \boxed{[5]} & \leftarrow \dot{\times}\,\mathbf{r} \leftarrow & \boxed{[3]} & \rightarrow \int d\mathbf{r} \rightarrow & \boxed{[6]} \end{array}$$

$$[\,1\,] \qquad \frac{1}{\mu}\left[\int \mathbf{P}\,dt - \iint \mathbf{F}\,dt\,dt\right] = 0$$

$$[\,2\,] \qquad \frac{1}{\mu}\left[\mathbf{P} - \int \mathbf{F}\,dt\right] = 0$$

$$[\,3\,] \qquad \frac{1}{\mu}\left[\frac{d\mathbf{P}}{dt} - \mathbf{F}\right] = 0$$

$$[\,4\,] \qquad \frac{1}{\mu}\left[\mathbf{P} - \int \mathbf{F}\,dt\right]\dot{\times}\,\mathbf{r} = 0$$

$$[\,5\,] \qquad \frac{1}{\mu}\left[\frac{d\mathbf{P}}{dt} - \mathbf{F}\right]\dot{\times}\,\mathbf{r} = 0$$

$$[\,6\,] \qquad \frac{1}{\mu}\left[\int \frac{d\mathbf{P}}{dt}\cdot d\mathbf{r} - \int \mathbf{F}\cdot d\mathbf{r}\right] = 0$$

$[\,\mu\,]$ is an arbitrary constant with dimension of mass $(\,\mathrm{M}\,)$

# Appendix II

## System of Equations II

$$\begin{tikzcd} & \boxed{[1]} \arrow[d, "dt"] & \\ \boxed{[4]} \arrow[d, "dt"] & \boxed{[2]} \arrow[l, "\dot{\times}\,\mathbf{r}"'] \arrow[d, "dt"] & \\ \boxed{[5]} & \boxed{[3]} \arrow[l, "\dot{\times}\,\mathbf{r}"'] \arrow[r, "\int d\mathbf{r}"] & \boxed{[6]} \end{tikzcd}$$

$$[\,1\,] \qquad \frac{1}{\mu}\left[\, m\,\bar{\mathbf{r}} - \int\!\!\int \mathbf{F}\, dt\, dt \,\right] = 0$$

$$[\,2\,] \qquad \frac{1}{\mu}\left[\, m\,\bar{\mathbf{v}} - \int \mathbf{F}\, dt \,\right] = 0$$

$$[\,3\,] \qquad \frac{1}{\mu}\left[\, m\,\bar{\mathbf{a}} - \mathbf{F} \,\right] = 0$$

$$[\,4\,] \qquad \frac{1}{\mu}\left[\, m\,\bar{\mathbf{v}} - \int \mathbf{F}\, dt \,\right] \dot{\times}\, \mathbf{r} = 0$$

$$[\,5\,] \qquad \frac{1}{\mu}\left[\, m\,\bar{\mathbf{a}} - \mathbf{F} \,\right] \dot{\times}\, \mathbf{r} = 0$$

$$[\,6\,] \qquad \frac{1}{\mu}\left[\, m\, f\, c^2 - \int \mathbf{F} \cdot d\mathbf{r} \,\right] = 0$$

$[\,\mu\,]$ is an arbitrary constant with dimension of mass $(\,\mathrm{M}\,)$